CALCULATION OF CIRCULAR FREQUENCY BY SUBSPACE-ITERATION IN CASE OF FREQUENCY-DEPENDENT DYNAMICAL STIFFNESS MATRIX

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Abstract

We will describe in the paper how the approximate dynamical stiffness matrix can be produced in a significantly simpler way than known from literature in case of displacement functions written as the power series of frequency ω. The displacement functions dependent on ω have to be used only at the production of mass matrix and there only in one of the factors of the matrix series. At big tasks the method of subspace iteration can be suitably used for the calculation of circular frequency in number necessary in practice. We will show how this method can be used for mass matrices dependent on ω making the definition of the sought circular frequencies within one iteration method possible.

1. Producing approximate dynamical stiffness matrix

In course of the dynamical use of the method of finite element the elemental dynamical stiffness matrix should be written with dynamical displacement functions containing the effect of distributing inertia force on the oscillating element. In case of beams there is possibility to calculate beam end force belonging to dynamical displacement of unit beam end. To do it the differential equation of the vibrating beam is to be solved under given boundary conditions and the dynamical stiffness matrix is obtained in the following form [1]

$$\mathbb{K}_{dyn}(\omega) = \mathbb{K}(\omega) - \omega^2 \widetilde{\mathbb{M}}(\omega) \tag{1}$$

Here

$$\mathbb{K}(\omega) = \int_{(V)} \mathbb{B}^*(\omega) \, \mathbb{D} \, \mathbb{B}(\omega) \, dV \tag{2}$$

$$\widetilde{\mathbb{M}}(\omega) = \varrho \int_{(V)} \mathbb{N}^*(\omega) \mathbb{N}(\omega) \, dV \tag{3}$$

The elements of matrix $\mathbb{N}(\omega)$ are displacement functions giving the relation between nodal point displacements and the displacements of inside points, if they are known matrix $\mathbb{B}(\omega)$ describing the relation between nodal point displacements and deformation in inside points can be calculated.

ϱ in (3) is the density of the material of the element, while matrix $\mathbb{D}$ is giving the relation between strain and stress.

The dynamical stiffness matrix can be disintegrated in the following form [2]:

$$\mathbf{K}_{dyn} = \mathbf{K}_{st} - \omega^2 \widehat{\mathbf{M}}(\omega) \tag{4}$$

Here $\mathbf{K}_{st}$ is the stiffness matrix used at statical tests, while

$$\widehat{\mathbf{M}}(\omega) = \varrho \int_{(V)} \mathbf{N}^*(\omega) \mathbf{N}_{st} dV \tag{5}$$

The elements of matrix $\mathbf{N}_{st}$ are the so-called statical displacement functions.

If not beam elements are tested exact production of displacement functions is not possible. If approximation $\mathbf{N}^*(\omega) \approx \mathbf{N}_{st}$ is used at relations (3) and (5) the approximate form of the dynamical stiffness matrix is

$$\mathbb{K}_{dyn} \approx \mathbb{K}_{st} - \omega^2 \mathbb{M} \tag{6}$$

where $\mathbb{M}$ is the so-called consistent mass matrix. While calculating with the consistent mass matrix the circular frequency of the structure is obtained approximately. The above approximation can be improved by concentrating division for elements. Przemieniecki [1] suggested that dynamical displacement functions should be approximated by a power series where the displacement functions appearing as the multipliers of the power of ω are produced starting from statical displacement functions that is

$$\mathbf{N}(\omega) = \mathbf{N}_{st} + \omega^2 \mathbf{N}_2 + \omega^4 \mathbf{N}_4 + \ldots \tag{7}$$

(We can prove that the elements of matrices $\mathbf{N}$ of odd indices are equal with zero.)

The using expressions (2) and (3)

$$\mathbf{K}(\omega) \approx \int_{(V)} (\mathbf{B}_{st}^* + \omega^2 \mathbf{B}_2 + \omega^4 \mathbf{B}_4^* + \ldots) \mathbf{D} (\mathbf{B}_{st} + \omega^2 \mathbf{B}_2 + \omega^4 \mathbf{B}_4 + \ldots) dV \tag{8}$$

$$\widetilde{\mathbf{M}}(\omega) \approx \varrho \int_{(V)} (\mathbf{N}_{st}^* + \omega^2 \mathbf{N}_2^* + \omega^4 \mathbf{N}_4 + \ldots)(\mathbf{N}_{st} + \omega^2 \mathbf{N}_2 + \omega^4 \mathbf{N}_4 + \ldots) dV \tag{9}$$

Gupta [3] used relations (8) and (9) at the dynamical calculations of discs.

It is clear that the approximate dynamical stiffness matrix is obtained in a simple form if the dynamical stiffness matrix is disintegrated in accordance with (4). Matrix $\mathbf{K}_{st}$ is known from statical test while

$$\widehat{\mathbf{M}}(\omega) = \varrho \int_{(V)} (\mathbf{N}_{st}^* + \omega^2 \mathbf{N}_2^* + \omega^4 \mathbf{N}_4 + \ldots) \mathbf{N}_{st} dV \tag{10}$$

Thus

$$\widehat{\mathbf{M}}(\omega) = \mathbf{M}_0 + \omega^2 \mathbf{M}_2 + \omega^4 \mathbf{M}_4 + \omega^6 \mathbf{M}_6 + \ldots \tag{11}$$

Here $\mathbf{M}_0$ is the consistent mass matrix.

Some components of the mass matrix belonging to the bar performing axial oscillation are given as illustration

$$\hat{M} = \frac{\varrho A l}{3} \left\{ \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} + \frac{\psi^2}{15} \begin{bmatrix} 1 & \frac{7}{8} \\ \frac{7}{8} & 1 \end{bmatrix} + \right.$$
$$\left. + \frac{2\psi^4}{315} \begin{bmatrix} 1 & \frac{31}{32} \\ \frac{31}{32} & 1 \end{bmatrix} + \frac{\psi^6}{1425} \begin{bmatrix} 1 & \frac{127}{128} \\ \frac{127}{128} & 1 \end{bmatrix} \right\} . \tag{12}$$

Here A is the cross-section area, l is the length of the beam, $\psi^2 = \frac{l^2 \varrho}{E} \omega^2$ and E is the Young's modulus.

2. Calculation of circular frequency

If there are only two members in expression (11) circular frequencies ω and free vectors v can be obtained from the solution of the homogeneous equation $(K = K_{si})$

$$(K - \omega^2 M_0 - \omega^4 M_2) v = 0. \tag{13}$$

Relation (13) can be reduced to a double size eigenvalue problem

$$\begin{bmatrix} M_2 & \\ & K \end{bmatrix} \begin{bmatrix} \omega^2 v \\ v \end{bmatrix} = \omega^2 \begin{bmatrix} & M_2 \\ M_2 & M_0 \end{bmatrix} \begin{bmatrix} \omega^2 v \\ v \end{bmatrix} \tag{14}$$

Gupta gives an expedient solution of this eigenvalue problem taking the structure of matrices in (14) into consideration in [4].

If more than two members from relation (11) are taken into consideration the homogeneous equation

$$|K - \omega^2 M_0 - \omega^4 M_2 - \omega^6 M_4 - \omega^s M_{s-2}| v = 0 \tag{15}$$

is to be solved.

To calculate the eigenvalues and eigenvectors in [5] the method for definition of the n smallest absolute value generalized eigenvalues and that of the eigenvectors belonging to them of matrix of mth degree and nth order was used. This method gives the eigenvalues and eigenvectors belonging to equation

$$|I\lambda^m + A_1 \lambda^{m-1} + \ldots A_{m-1}\lambda + A_m| v = 0 \tag{16}$$

by the calculation of the eigenvalues and eigenvectors of matrix $\mathbf{Y}$ where matrix $\mathbf{Y}$ can be obtained from the solution of the non-linear matrix equation

$$\mathbf{Y}^m + \mathbf{A}_1 \mathbf{Y}^{m-1} + \ldots \mathbf{A}_{m-1} \mathbf{Y} + \mathbf{A}_m = 0. \tag{17}$$

The disadvantage of the method is that it cannot take the band structure of the stiffness and mass matrix into consideration thus cannot be used for the solution of big tasks.

The method of subspace-iteration for the solution of generalized eigen-vector tasks can be used for the definition of certain number of oscillating forms and oscillating numbers of big systems in a suitable way [7].

Sotiropoulos [6] suggests the method of subspace-iteration in case of mass matrix dependent on ω in a way that in

$$\mathbf{K}\mathbf{V} = \omega^2 \mathbf{M}(\omega)\mathbf{V} \tag{18}$$

task in each iteration step the elements belonging to the given iteration step $\mathbf{M}(\omega)$ are calculated by a previously given approximate value of any frequency ω_r. The method is convergent but only the circular frequency from among the ones belonging to subspace can be accepted as the solution of task (15) that gradually modified matrix $\mathbf{M}(\omega)$. If our task at the structure is to define a certain number of frequency the subspace-iteration is used as many times as many eigenvalues are to be calculated. The question whether the necessary circular frequency and oscillating form belonging to it can be calculated by using subspace-iteration only once is rightful. In the following a method is shown to do it.

Task at (15) can be again reduced to a generalized eigenvalue task by introducing a new unknown:

$$\begin{bmatrix} \mathbf{K} & & & & \\ & \mathbf{M}_2 & & & \\ & & \mathbf{M}_4 & & \\ & & & \ddots & \\ & & & & \mathbf{M}_s \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \omega^2 \mathbf{v} \\ \omega^4 \mathbf{v} \\ \vdots \\ \omega^s \mathbf{v} \end{bmatrix} = \omega^2 \begin{bmatrix} \mathbf{M}_0 & \mathbf{M}_2 & \mathbf{M}_4 & \ldots & \mathbf{M}_s \\ \mathbf{M}_2 & & & & \\ & \mathbf{M}_4 & & & \\ & & \ddots & & \\ & & & & \mathbf{M}_s \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \omega^2 \mathbf{v} \\ \omega^4 \mathbf{v} \\ \vdots \\ \omega^s \mathbf{v} \end{bmatrix} \tag{19}$$

$$\mathbf{A}\mathbf{y} = \lambda \mathbf{B}\mathbf{y}. \tag{20}$$

The subspace-iteration method shown in [7] can be used to define eigenvectors y and eigenvalues belonging to them. Matrices of size $m \times m$ for the definition of the smallest eigenvalue m can be calculated with expressions

$$\mathbf{K}_{k+1} = \overline{\mathbf{X}}^*_{k+1} \mathbf{Y}_k (\overline{\mathbf{X}}^*_{k+1} \mathbf{A} \overline{\mathbf{X}}_{k+1}) \tag{21}$$

$$\mathbf{M}_{k+1} = \overline{\mathbf{X}}^*_{k+1} \overline{\mathbf{Y}}_{k+1} (\overline{\mathbf{X}}^*_{k+1} \mathbf{B} \overline{\mathbf{X}}_{k+1}) \tag{22}$$

where the length of the vector bunch of the right now is $\left(\dfrac{s}{2} + 1\right) n$.

The matrix of $\mathbf{Y}_k$ can be calculated from expression

$$\mathbf{Y}_k = \begin{bmatrix} \mathbf{M}_0 & \mathbf{M}_2 & \mathbf{M}_4 & \cdots & \mathbf{M}_s \\ \mathbf{M}_2 & & & & \\ & \mathbf{M}_4 & & & \\ & & \ddots & & \\ & & & & \mathbf{M}_s \end{bmatrix} \begin{bmatrix} \mathbf{X}_k \\ \mathbf{X}_k\,\omega_k^2 \\ \mathbf{X}_k\,\omega_k^4 \\ \vdots \\ \mathbf{X}_k\,\omega_k^s \end{bmatrix} \qquad (23)$$

where

$\mathbf{X}_k$ contains the kth approximation of vectors v

ω_k^2 is the diagonal matrix containing eigenvalues given in kth approximation.

Matrix $\mathbf{Y}_k$ can be calculated as a block while at symmetrical band matrices $\mathbf{M}_0, \mathbf{M}_2 \ldots \mathbf{M}_s$ it is enough to store the elements in the upper band.

Matrix $\overline{\mathbf{X}}_{k+1}$ can be obtained by solving the equation system

$$\begin{bmatrix} \mathbf{K} & & & & \\ & \mathbf{M}_2 & & & \\ & & \mathbf{M}_4 & & \\ & & & \ddots & \\ & & & & \mathbf{M}_s \end{bmatrix} \begin{bmatrix} \overline{\mathbf{X}}_{k+1}^{(0)} \\ \overline{\mathbf{X}}_{k+1}^{(2)} \\ \overline{\mathbf{X}}_{k+1}^{(4)} \\ \vdots \\ \overline{\mathbf{X}}_{k+1}^{(s)} \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_k^{(0)} \\ \mathbf{M}_2 & \mathbf{X}_2 \\ \mathbf{M}_4 & \mathbf{X}_k & \omega_k^2 \\ \vdots \\ \mathbf{M}_s & \mathbf{X}_k & \omega_k^{s-2} \end{bmatrix}. \qquad (24)$$

We can see that

$$\overline{\mathbf{X}}_{k+1}^{(2)} = \mathbf{X}_k$$

$$\overline{\mathbf{X}}_{k+1}^{(4)} = \mathbf{X}_k\,\omega_k^2$$

$$\vdots$$

$$\overline{\mathbf{X}}_{k+1}^{(s)} = \mathbf{X}_k\,\omega_k^{s-2}$$

and only matrix $\overline{\mathbf{X}}_{k+1}^{(0)}$ is to be calculated by the solution of the equation system $\mathbf{K}\overline{\mathbf{X}}_{k+1}^{(0)} = \mathbf{Y}_k^{(0)}$ of original size.

Matrix $\overline{\mathbf{Y}}_{k+1}$ can be obtained after performing the following matrix

$$\overline{\mathbf{Y}}_{k+1} = \begin{bmatrix} \mathbf{M}_0 & \mathbf{M}_2 & \mathbf{M}_4 & \cdots & \mathbf{M}_s \\ \mathbf{M}_2 & & & & \\ & \mathbf{M}_4 & & & \\ & & \ddots & & \\ & & & & \mathbf{M}_s \end{bmatrix} \begin{bmatrix} \overline{\mathbf{X}}_{k+1}^{(0)} \\ \mathbf{X}_k \\ \mathbf{X}_k\,\omega_k^2 \\ \vdots \\ \mathbf{X}_k\,\omega_k^{s-2} \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{Y}}_{k+1}^{(0)} \\ \overline{\mathbf{Y}}_{k+1}^{(2)} \\ \overline{\mathbf{Y}}_{k+1}^{(4)} \\ \vdots \\ \overline{\mathbf{Y}}_{k+1}^{(s)} \end{bmatrix}.$$

Matrix $\overline{\mathbb{Y}}_{k+1}$ can be calculated again as blocks by multiplication with the vector bunch containing column m of band matrices of original size.

The following expressions can be obtained for the stiffness and mass matrix with ordinal number in accordance with the number of eigenvectors belonging to subspace:

$$\mathbb{K}_{k+1} = \overline{\mathbb{X}}_{k+1}^{(0)} \mathbb{Y}_k^{(0)} + \mathbb{X}_k^{*} \mathbb{M}_2 \mathbb{X}_k + \omega_k^2 \mathbb{X}_k^{*} \mathbb{M}_4 \mathbb{X}_k \omega_k^2 + \omega_k^4 \mathbb{X}_k^{*} \mathbb{M}_6 \mathbb{X}_k \omega_k^4 + \dots,$$

$$\mathbb{M}_{k+1} = \mathbb{X}_{k+1}^{(0)} (\mathbb{M}_0 \overline{\mathbb{X}}_{k+1}^{(0)} + \mathbb{M}_2 \mathbb{X}_k + \mathbb{M}_4 \mathbb{X}_k \omega_k^2 + \mathbb{M}_6 \mathbb{X}_k \omega_k^4 + \dots) +$$
$$+ \mathbb{X}_k^{*} \mathbb{M}_2 \overline{\mathbb{X}}_{k+1}^{(0)} + \omega_k^2 \mathbb{X}_k^{*} \mathbb{M}_4 \mathbb{X}_k + \omega_k^4 \mathbb{X}_k^{*} \mathbb{M}_6 \mathbb{X}_k \omega_k^2 + \dots.$$

While matrix $\mathbb{K}_{k+1}$ is symmetrical (as A in 5.33 is symmetrical) matrix $\mathbb{M}_{k+1}$ is not. Thus eigenvalues and eigenvectors can be both real and complex. If $s = 2$ that is the mass matrix consists of only two members matrix $\mathbb{M}_{k+1}$ will be symmetrical too that is all the eigenvalues and eigenvectors will be real.

3. Numerical experiences

The method introduced takes the band structure of stiffness and mass matrices into account. The coefficient matrix of the equation system necessary in course of subspace-iteration does not change, producing the right side of the equation system requires surplus time. The ordinal number of the eigenvalue problem belonging to the subspace does not increase either but producing the stiffness and mass matrix belonging to the subspace requires surplus time. If only components $\mathbb{M}_0$ and $\mathbb{M}_2$ are taken into account at the mass matrix the matrices of subspace-iteration will be symmetrical and convergence is ensured even if eigenvalues appear. At subspace-iteration the number of iteration steps does not increase if matrix $\mathbb{M}_2$ is taken into account. If further members are taken into account from mass matrix B in (20) will not be symmetrical. In this case transformation QR was used to solve the eigenvalue task belonging to subspace. The appearance of complex eigenvalues and eigenvectors cannot be excluded either. Convergence cannot be ensured for these vectors in course of subspace-iteration. Thus they have to be excluded form the following iteration steps. Our experiences show that the appearance of complex eigenvectors can be expected if the number of sought circular frequencies reach half of the original ordinal number. (It does not appear in course of calculation with real structures.) We have to note that in case of the appearance of complex eigenvectors the number of iteration steps to be used in subspace-iteration can significantly increase. The figure shows a two support beam the oscillation number belonging to it was calculated by neglecting the effect of displacement inertia and shear strain. (For this purpose the exact value of circular frequency is known.) The beam was divided into ten parts and the
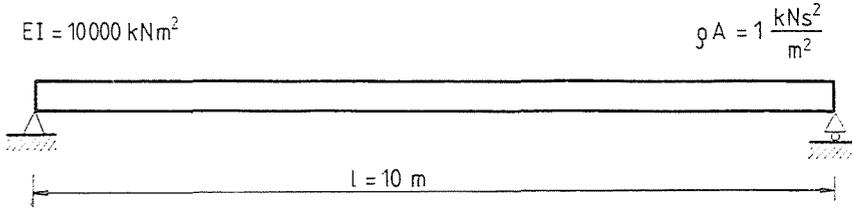
Fig. 1

first ten vibration numbers of the system of twenty degrees of freedom were calculated in case of

$$M = M_0 + \omega^2 M_2 + \omega^4 M_4 + \omega^6 M_6 \tag{28}$$

gradually increasing the number of components taken into account. The table contains the exact circular frequencies (ω_r) and values

$$10^6 \cdot \varepsilon_r = \frac{\bar{\omega}_r - \omega_r}{\omega_r} 10^6 \tag{29}$$

where $\bar{\omega}_r$ is the frequency given by the iteration method. As it is known $\bar{\omega}_r$ approximates the value of ω_r from above. 10^{-6} accuracy is prescribed for the values of eigenvalues following each other in the iteration method thus the approximation of the exact value can be expected with this error. The values in the table give the multiplication factor of 10^{-6} error limit. Where circular frequencies approximates the exact value better than 10^{-6} the error was taken as zero. The results show the improvement of the approximation of circular frequencies well. If matrix $M(\omega)$ consists of four members more than one third

Table 1

Exact circular frequencies and errors in case of different number of mass matrix components

r	ω_r	$10^6 \times \varepsilon_r(M_0)$	$10^6 \times \varepsilon_r(M_2)$	$10^6 \times \varepsilon_r(M_4)$	$10^6 \times \varepsilon_r(M_6)$
1	9.86960	7	0	0	0
2	39.4784	106	0	0	0
3	88.8264	535	1	0	0
4	157.914	1653	6	0	0
5	246.740	3947	45	0	0
6	355.306	7937	189	3	0
7	483.611	14177	610	27	0
8	631.655	23036	1640	128	9
9	799.444	33820	3788	466	53
10	986.960	109923	16211	2951	310

of circular frequencies can be obtained with accuracy prescribed for the so-
lution of eigenvalue task. Accuracy significantly increases even in case of two
components. We have to note that if matrices M_4 and M_6 are taken into account
complex eigenvalues also appeared in course of iteration.

If the beam was divided into twenty parts and our aim was to define 10
circular frequency (the number of vectors in subspace-iteration was 18) only
real eigenvalues were found in course of the iteration method and the neces-
sary eigenvalues were obtained after 4 subspace iteration steps with 10^{-6}
relative error.

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