MATHEMATICAL PROGRAMMING FORMULATION OF STATE CHANGE ANALYSIS OF FINITE ELEMENT MODELS WITH SOME UNILATERAL CONNECTIONS

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Received July 20, 1989

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Abstract

The ideal elastic-plastic and locking-contact behaviour of structures can be handled in a unified manner on the basis of the total duality between the two unilateral phenomena.

The relating variational inequality problems lead to mathematical programming problems.

The mathematical programming problems relating to the finite element realization of the state change analysis of structures with general unilateral plastic-locking connections is presented here.

Introduction

The numerical analysis of structures with unilateral connections representing the locking-contacting or plastic behaviour leads to mathematical programming problems. The elements of a body or structure whose stresses or strains or their combinations are governed by prescribed inequality conditions are termed conditional, subdifferential or unilateral joints. As typical unilateral connections, during a loading process, contacts develop (locking of gaps) or points plastificate, causing the physical nonlinearity of the structure. The ideal elastic-plastic and locking-contact behaviour of structures can be handled in a unified manner on the basis of the relating polygon-type material laws and the concerning nondifferentiable but subdifferentiable energy functionals and the relating variational principles modified by the sign-dependent variables of the inequality subsidiary conditions. This variational inequality problems lead to mathematical programming problems in the case of a numerical approach like a finite element analysis presented here.

Analysing the behaviour of the conditional joints [1, 2, 3, 4], it has become obvious that similarly to the plastic property, also the locking-contact behaviour can be treated as a material characteristic [5, 6]. Consequently, the simultaneous elastic-plastic and locking-contact behaviour can mathematically be handled together [7]. On the basis of the numerical approaches of elasto-plastic analyses [8, 9, 10, 11, 12] and unilateral contact problems [13, 14, 15, 16], by mathematical programming applications, a unified method has been developed to the case of the simultaneous presence of the dual phenomena mentioned above.

1. State variables and inequality conditions of the finite element model

Let $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ be the vectors of the stress and strain functions relating to a single finite element of volume V:

$$\boldsymbol{\sigma}^{\mathrm{T}} = [\sigma_{xx}\ \sigma_{xy}\ \sigma_{xz}\ \sigma_{yy}\ \sigma_{yz}\ \sigma_{zz}],$$

$$\boldsymbol{\epsilon}^{\mathrm{T}} = [\varepsilon_{xx}\ \varepsilon_{xy}\ \varepsilon_{xz}\ \varepsilon_{yy}\ \varepsilon_{yz}\ \varepsilon_{zz}],$$

in which the functions are e.g.:

$$\sigma_{xy} = \sigma_{xy}(\xi, \eta, \zeta) \quad \text{and} \quad \varepsilon_{xy} = \varepsilon_{xy}(\xi, \eta, \zeta),$$

where x, y, z sign the global and ξ, η, ζ note the local coordinate system. Vector $\mathbf{u}$ contains the unknown displacement functions in the element volume V and $\mathbf{v}$ shows the given displacement functions on its surface S_u:

$$\mathbf{u}^{\mathrm{T}} = [u_x\ u_y\ u_z], \qquad \mathbf{v}^{\mathrm{T}} = [v_x\ v_y\ v_z].$$

We denote further the functions of the volume forces by $\mathbf{g}$:

$$\mathbf{g}^{\mathrm{T}} = [g_x\ g_y\ g_z],$$

while vector $\mathbf{p}$ represents the given forces on the surface S_p and vector $\mathbf{r}$ contains the unknown reaction forces on the surface S_u:

$$\mathbf{p}^{\mathrm{T}} = [p_x\ p_y\ p_z], \qquad \mathbf{r}^{\mathrm{T}} = [r_x\ r_y\ r_z].$$

All the mentioned vectors relate to the global system, e.g.:

$$u_x = u_x(x, y, z), \qquad v_x = v_x(x, y, z), \qquad g_x = g_x(x, y, z),$$

$$p_x = p_x(x, y, z), \qquad r_x = r_x(x, y, z).$$

In the case of a state change analysis, the quasi-static velocities of these state variables will be used, marked by a dot like $\dot{\mathbf{u}}$, $\dot{\boldsymbol{\sigma}}$, etc.

The inequality conditions of the ideal unilateral connections of a single element are

$$\mathbf{F} = \mathbf{N}\boldsymbol{\sigma} - \boldsymbol{\alpha} \leq \mathbf{0}, \tag{1}$$

relating to plastificating points and

$$\mathbf{f} = \mathbf{M}\boldsymbol{\epsilon} - \boldsymbol{\beta} \leq \mathbf{0} \tag{2}$$

in the case of locking points, where vectors α and β contain stress and strain type scalars, resp.

Further we introduce the plastic and locking potential velocities in the form of

$$\dot{\mathbf{F}} = \mathbf{N}\dot{\boldsymbol{\sigma}} \quad \text{and} \quad \dot{\mathbf{f}} = \mathbf{M}\dot{\boldsymbol{\epsilon}}, \tag{3}$$

which are positive by a loading process, zero by a plastic or a locking state and negative when unloading takes place.

Hypermatrices $\mathbf{M}$ and $\mathbf{N}$ contain all the gradients of the planes belonging to the plastic and locking polyhedrons illustrating the conditions

$$\mathbf{F} = \mathbf{0} \quad \text{and} \quad \mathbf{f} = \mathbf{0}$$

relating to the node points of the element. For example the k-th element of hypermatrix $\mathbf{N}$ is

$$\mathbf{N}_k = \begin{bmatrix} n_{11}^k & n_{12}^k & \dots & n_{16}^k \\ n_{21}^k & n_{22}^k & \dots & n_{26}^k \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ n_{i1}^k & n_{i2}^k & \dots & n_{i6}^k \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ n_{r1}^k & n_{r2}^k & \dots & n_{r6}^k \end{bmatrix} \tag{4}$$

where k represents the serial number of the nodal point and n_{ij}^k is the j-th component of the unit normal vector $\mathbf{n}_i^k$ relating to the i-th plane of the six-dimensional convex polyhedron of r pieces of plane. Since the number of the nodal points of the element is p, the gradient hypermatrix $\mathbf{N}$ reads:

$$\mathbf{N}^T = [\mathbf{N}_1 \ \mathbf{N}_2 \ \dots \ \mathbf{N}_k \ \dots \ \mathbf{N}_p]. \tag{5}$$

The modified variational principles relating to the inequality subsidiary conditions (1), (2), (3) and the concerning sign-dependent state variables due to the simultaneous elastic-plastic-locking-contact behaviour has been published elsewhere [17, 18, 19, 20]. On the basis of the modified variational principles the relating mathematical programming formulaations have also been worked out.

2. Finite element approximations with respect to the strains or stresses

During the incremental process of a state change analysis, the rate of the state variable functions $\dot{\boldsymbol{\sigma}}$, $\dot{\boldsymbol{\epsilon}}$, $\dot{\mathbf{u}}$, $\dot{\mathbf{r}}$ and $\dot{\boldsymbol{\sigma}}_l$, $\dot{\boldsymbol{\epsilon}}_p$ have to be determined, where $\dot{\boldsymbol{\sigma}}_l$ contains the functions of the locking stresses and $\dot{\boldsymbol{\epsilon}}_p$ gives the plastic strains.

Let us first approximate the displacement rate function, for example $\dot{u}_x$ by the linear combination of

$$\dot{u}_x = \sum_{i=1}^{n} \sum_{j=1}^{m} \dot{e}^i_{jx} \varphi^i_j, \tag{6}$$

related to an element of n nodal points if each of the elements has a kinematical degree of freedom m. Here φ^i_j are the basis functions of the interpolation and $\dot{e}^i_{jx}$ are the scalar values and the derivatives of the displacement function $\dot{u}_x$ at the nodal points. The expression reads in a matrix form

$$\dot{\mathbf{u}} = \mathbf{C}^u \dot{\mathbf{e}}, \tag{7}$$

where $\mathbf{C}^u$ contains the basis functions. So the compatibility condition reads

$$\dot{\boldsymbol{\epsilon}} + \mathbf{N}^T \dot{\boldsymbol{\Lambda}} - \mathbf{D}\mathbf{C}^u \dot{\mathbf{e}} = \mathbf{0}, \tag{8}$$

from which

$$\dot{\boldsymbol{\epsilon}} = \mathbf{D}\mathbf{C}^u \dot{\mathbf{e}} - \mathbf{N}^T \dot{\boldsymbol{\Lambda}} = \mathbf{B}\dot{\mathbf{e}} - \mathbf{N}^T \dot{\boldsymbol{\Lambda}}, \tag{9}$$

where $\mathbf{D}$ is a differential operator as follows

$$\mathbf{D}^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

and $\mathbf{B} = \mathbf{D}\mathbf{C}^u$ is the so-called geometry matrix.

Approximating the plastic strain rates $\dot{\boldsymbol{\epsilon}}_p$ in the same way, we have

$$\dot{\boldsymbol{\epsilon}}_p = \mathbf{N}^T \dot{\boldsymbol{\Lambda}}. \tag{10}$$

Let us remember to the definitions (4) and (5) of $\mathbf{N}$, we obtain equivalently

$$\dot{\boldsymbol{\epsilon}}^p = \sum_{k=1}^{p} \sum_{i=1}^{r} \sum_{j=1}^{m} \dot{\Lambda} n^k_{ij}, \quad m \leq 6, \tag{11}$$

consequently, we can regard the unit vectors n^k_{ij} as the basis functions of the plastic strain rates and the values $\dot{\Lambda}$ as the relating nodal point scalars.

For the formulation of the approximation of the stress functions $\dot{\boldsymbol{\sigma}}$, we introduce an elastic and a locking part of the stresses by the previously detailed way (7) and (10)

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}^{\sigma T} \dot{\mathbf{s}} \quad \text{and} \quad \dot{\boldsymbol{\sigma}}_l = \mathbf{M}^* \dot{\boldsymbol{\lambda}} \tag{12}$$

where $\mathbf{C}^\sigma$ and $\mathbf{M}$ contain the basis functions and $\dot{\mathbf{s}}$ and $\dot{\boldsymbol{\lambda}}$ represent the nodal point scalar values of the stress functions. So the equilibrium equation reads:

$$\mathbf{T}^T \mathbf{D}^T (\mathbf{C}^{\sigma T} \dot{\mathbf{s}} + \mathbf{M}^* \dot{\boldsymbol{\lambda}}) + \dot{\mathbf{g}} = \mathbf{0}. \tag{12a}$$

3. Mathematical programming formulation of plastic connections on the basis of strain approximation

Approximating the compatible strains by (9), the quasistatic velocity of the potential energy completed by the sign-dependent variable $\dot{\Lambda} \geq 0$ reads:

$$\dot{\pi}_p(\dot{e}, \dot{\Lambda}) = \frac{1}{2}[\dot{e}^T \ \dot{\Lambda}^T] \cdot [\mathbb{K}] \cdot \begin{bmatrix} \dot{e} \\ \dot{\Lambda} \end{bmatrix} - [\dot{e}^T \ \dot{\Lambda}^T] \cdot \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \tag{13}$$

relating to a single finite element. Here $\mathbb{K}$ is the modified stiffness matrix:

$$\mathbb{K} = \int_V \begin{bmatrix} \mathbb{B}^T \mathbb{H} \mathbb{B} & -\mathbb{B}^T \mathbb{H} \mathbb{N}^T \\ -\mathbb{N} \ \mathbb{H} \ \mathbb{B} & \mathbb{N} \mathbb{H} \mathbb{N}^T \end{bmatrix} dV = \begin{bmatrix} \mathbb{K}_{11} & \mathbb{K}_{12} \\ \mathbb{K}_{21} & \mathbb{K}_{22} \end{bmatrix} \tag{14}$$

in which $\mathbb{H}$ is the matrix of the Hooke's law:

$$\sigma = \mathbb{H} \epsilon \tag{15}$$

relating to the elastic behaviour. Note that the matrix

$$\mathbb{K}_{11} = \int_V \mathbb{B}^T \mathbb{H} \mathbb{B} \, dV \tag{16}$$

is the classical stiffness matrix of the displacement method, which has been extended to (14) due to the plastic conditions. Furthermore, $\dot{q}$ contains the load conditions:

$$\dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \int_V \mathbb{C}^T \mathbb{T} \dot{g} \, dV + \int_V \mathbb{B}^T \mathbb{H} \dot{\epsilon}_0 \, dV + \int_{S_p} \mathbb{C}^T \mathbb{T} \dot{p} \, dS + \mathbb{T} \dot{q}_0 \\ - \int_V \mathbb{N} \mathbb{H} \dot{\epsilon}_0 \, dV \end{bmatrix} \tag{17}$$

in which $\dot{\epsilon}_0$ is the initial strain increasing by the same load parameter similarly to the concentrated force load $\dot{q}_0$ acting directly to the nodal points. $\mathbb{T}$ matrix transforms the global coordinate system to the local one.

For formulating the mathematical programming problems, the variational problem of the potential energy (13) has to be reduced to the sign-dependent state variable $\dot{\Lambda}$ only. From the first variation of functional (13) we obtain the equilibrium condition of the element:

$$[\delta\dot{e}^T \ \delta\dot{\Lambda}^T] \cdot \left(\begin{bmatrix} \mathbb{K}_{11} & \mathbb{K}_{12} \\ \mathbb{K}_{21} & \mathbb{K}_{22} \end{bmatrix} \cdot \begin{bmatrix} \dot{e} \\ \dot{\Lambda} \end{bmatrix} - \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \right) \geq 0. \tag{18}$$

Since $\delta\dot{e}$ is arbitrary, from the first inequality of (18) we obtain an equation from which we have

$$\dot{e} = -\mathbb{K}_{11}^{-1} \mathbb{K}_{12} \dot{\Lambda} + \mathbb{K}_{11}^{-1} \dot{q}_1 \tag{19}$$

in term of $\dot{\Lambda}$. Substituting (19) to functional (13), we obtain the expression of the potential energy with respect to the sign-dependent state variable $\dot{\Lambda}$ only:

$$\dot{\pi}_p(\dot{\Lambda}) = \frac{1}{2}\dot{\Lambda}^T \cdot \int_V \mathbf{N}(\mathbf{H} - \mathbf{H}\mathbf{B}\mathbf{K}_{11}^{-1}\mathbf{B}^T\mathbf{H})\mathbf{N}^T dV \cdot \dot{\Lambda} - \\ - \dot{\Lambda}^T \cdot \left[\dot{\mathbf{q}}_2 + \int_V \mathbf{N}\mathbf{H}\mathbf{B}\mathbf{K}_{11}^{-1} dV \cdot \mathbf{q}_1\right], \tag{20a}$$

or in a simplified form

$$\dot{\pi}_p(\dot{\Lambda}) = \frac{1}{2}\dot{\Lambda}^T \mathbf{A}\dot{\Lambda} - \dot{\Lambda}^T \dot{\mathbf{a}}, \tag{20}$$

where

$$\mathbf{A} = \mathbf{K}_{22} - \mathbf{K}_{12}\mathbf{K}_{11}^{-1}\mathbf{K}_{12}^T, \\ \dot{\mathbf{a}} = \dot{\mathbf{q}}_2 - \mathbf{K}_{12}^T\mathbf{K}_{11}^{-1}\dot{\mathbf{q}}_1. \tag{21}$$

The minimum conditions of the functional (20) lead to the variational inequalities

$$\delta\dot{\pi}_p(\dot{\Lambda}) = \delta\dot{\Lambda}^T \cdot (\mathbf{A}\dot{\Lambda} - \dot{\mathbf{a}}) \geq 0 \text{ and } \delta^2\dot{\pi}_p(\dot{\Lambda}) = \delta\dot{\Lambda}^T \mathbf{A}\delta\dot{\Lambda} \geq 0, \tag{22}$$

since matrix $\mathbf{A}$ is positive semidefinite. The first variation in (22) is equivalent to the flow law directly. Namely, taking into account expressions (3), (15), (9) and (19), (21), the plastic potential velocity is as follows:

$$\dot{\mathbf{F}} = \mathbf{N}\dot{\sigma} = \mathbf{N}\mathbf{H}\dot{\epsilon} = \mathbf{N}\mathbf{H}(\mathbf{B}\dot{\mathbf{e}} - \mathbf{N}^T\dot{\Lambda}) = \\ = \mathbf{N}\mathbf{H}\mathbf{B}(-\mathbf{K}_{11}^{-1}\mathbf{K}_{12}\dot{\Lambda} + \mathbf{K}_{11}^{-1}\dot{\mathbf{q}}_1) - \mathbf{N}\mathbf{H}\mathbf{N}^T\dot{\Lambda} = \\ = -(\mathbf{K}_{22} - \mathbf{K}_{12}^T\mathbf{K}_{11}^{-1}\mathbf{K}_{12})\dot{\Lambda} - \mathbf{K}_{12}^T\mathbf{K}_{11}^{-1}\dot{\mathbf{q}}_1 = -(\mathbf{A}\dot{\Lambda} - \dot{\mathbf{a}}). \tag{23}$$

Thus, changing the sign of the variational inequality (22), the stationarity principle of the potential energy in term of the sign-dependent plastic strain rates relates to the plastic potential velocities $\dot{\mathbf{F}}$ directly:

$$\delta\dot{\Lambda}^T\dot{\mathbf{F}} \leq 0. \tag{24}$$

Since $\dot{\Lambda} \geq \mathbf{0}$ is a sign-dependent variable, the variation $\delta\dot{\Lambda}$ of it has to be also sign-dependent. Namely if $\dot{\Lambda} = \mathbf{0}$ then $\delta\dot{\Lambda} \geq \mathbf{0}$ because of the validity of $\dot{\Lambda} + \delta\dot{\Lambda} \geq \mathbf{0}$. But if $\dot{\Lambda} > \mathbf{0}$ then $\delta\dot{\Lambda}$ may be arbitrary. The variational inequality (24) says that $\delta\dot{\Lambda}$ and $\dot{\mathbf{F}}$ have opposite sign unless they are zero. For example, if $\dot{\mathbf{F}} = \mathbf{0}$, then $\delta\dot{\Lambda}$ is arbitrary, indicating the always changing border selecting the plastic and elastic domains V_p and $V - V_p$, respectively. And if $\dot{\mathbf{F}} \leq \mathbf{0}$, then $\delta\dot{\Lambda} \geq \mathbf{0}$, according to the plastic unloading, when $\dot{\Lambda} = \mathbf{0}$. But in elastic state, if $\dot{\mathbf{F}} \geq \mathbf{0}$, then $\delta\dot{\Lambda}$ can hardly be nonpositive because here $\dot{\Lambda} = \mathbf{0}$.

So the variational inequality (24) relates only to the plastic domain V_p expressing that

$$\dot{\mathbf{F}} \leq \mathbf{0} \text{ and } \dot{\Lambda} \geq \mathbf{0} \text{ on } V_p. \tag{25}$$

Besides, the variational inequality (24) includes the complementary or orthogonality condition, as well:

$$\dot{\Lambda}^{\mathrm{T}} \cdot \dot{\mathbf{F}} = 0. \tag{26}$$

Now we give the primal-dual formulations of the *quadratic programming problem* of the above detailed extremum problem:

$$\begin{aligned} &\mathrm{Q1:}\ \min \left\{ \frac{1}{2} \dot{\Lambda}^{\mathrm{T}} \mathbf{A} \dot{\Lambda} - \dot{\Lambda}^{\mathrm{T}} \dot{\mathbf{a}} \mid \dot{\Lambda} \geq \mathbf{0} \right\}, \\ &\mathrm{Q2:}\ \max \left\{ -\frac{1}{2} \dot{\Lambda}^{\mathrm{T}} \mathbf{A} \dot{\Lambda} \mid -\mathbf{A}\dot{\Lambda} + \dot{\mathbf{a}} \leq \mathbf{0},\ \dot{\Lambda} \geq \mathbf{0} \right\}, \end{aligned} \tag{27}$$

with the relating *linear complementary problem*

$$\mathrm{LC:}\ \{\mathbf{A}\dot{\Lambda} - \dot{\mathbf{a}} + \dot{\mathbf{F}} = \mathbf{0} \mid \dot{\Lambda} \geq \mathbf{0},\ \dot{\mathbf{F}} \leq \mathbf{0},\ \dot{\Lambda}^{\mathrm{T}}\mathbf{F} = 0\}. \tag{28}$$

The primal problem Q1 is the most general principle, because it relates to the entire domain V, problems Q2 and LC are limited to the plastic domain V_p.

*Problem Q*1 says that the velocity of the potential energy is minimized by the actual plastic strain rates $\dot{\epsilon}_p$, since

$$\begin{aligned} \frac{1}{2} \dot{\Lambda}^{\mathrm{T}} \mathbf{A} \dot{\Lambda} - \dot{\Lambda}^{\mathrm{T}} \dot{\mathbf{a}} &= \frac{1}{2} \int_V \dot{\Lambda}^{\mathrm{T}} \mathbf{N} (\mathbf{H} - \mathbf{H}\mathbf{B}\mathbf{K}_{11}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{H}) \mathbf{N}^{\mathrm{T}} \dot{\Lambda}\, dV - \\ &- \int_V \dot{\Lambda}^{\mathrm{T}} \mathbf{N} \mathbf{H} \dot{\epsilon}_0\, dV + \int_V \dot{\Lambda}^{\mathrm{T}} \mathbf{N} \mathbf{H} \mathbf{B}\, dV \cdot \mathbf{K}_{11}^{-1} \dot{\mathbf{q}}_1 = \\ &= \frac{1}{2} \int_V \dot{\epsilon}_p^{\mathrm{T}} (\mathbf{H} - \mathbf{H}\mathbf{B}\mathbf{K}_{11}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{H}) \dot{\epsilon}_p\, dV - \\ &- \int_V \dot{\epsilon}_p^{\mathrm{T}} (\mathbf{H}\dot{\epsilon}_0 - \mathbf{H}\mathbf{B}\mathbf{K}_{11}^{-1} \dot{\mathbf{q}}_1)\, dV = \dot{\pi}_p(\dot{\epsilon}_p) = \min! \end{aligned} \tag{29}$$

*Problem Q*2 represents a complementary energy principle as follows:

$$\begin{aligned} -\frac{1}{2} \dot{\Lambda}^{\mathrm{T}} \mathbf{A} \dot{\Lambda} &= -\frac{1}{2} \int_V \dot{\Lambda}^{\mathrm{T}} \mathbf{N} (\mathbf{H} - \mathbf{H}\mathbf{B}\mathbf{K}_{11}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{H}) \mathbf{N}^{\mathrm{T}} \dot{\Lambda}\, dV = \\ &= -\frac{1}{2} \int_V \dot{\Lambda}^{\mathrm{T}} \mathbf{N} \mathbf{H} (\mathbf{H}^{-1} - \mathbf{B}\mathbf{K}_{11}^{-1}\mathbf{B}^{\mathrm{T}}) \mathbf{H} \mathbf{N}^{\mathrm{T}} \dot{\Lambda}\, dV = \\ &= -\frac{1}{2} \int_V \dot{\sigma}_p^{\mathrm{T}} (\mathbf{H}^{-1} - \mathbf{B}\mathbf{K}_{11}^{-1}\mathbf{B}^{\mathrm{T}}) \dot{\sigma}_p\, dV = \dot{\pi}_c(\dot{\sigma}_p) = \max! \end{aligned} \tag{30}$$

which says that the actual stress rates $\dot{\sigma}_p$ due to the plastic strains, calculated on the basis of the elastic behaviour of the structure, make the velocity of the complementary energy maximal.

Problem LC expresses that the actual non-negative plastic strain rates are always orthogonal to the non-positive plastic potential velocities.

4. Mathematical programming formulation of locking connections on the basis of stress approximation

Approximating the stresses in equilibrium by (12), the quasi-static velocity of the complementary energy completed by the sign-dependent variable $\dot{\lambda} \geq 0$ reads:

$$\dot{\pi}_c(\dot{\mathbf{s}}, \dot{\lambda}) = -\frac{1}{2}[\dot{\mathbf{s}}^T \ \dot{\lambda}^T] \cdot [\mathbb{L}] \cdot \begin{bmatrix} \dot{\mathbf{s}} \\ \dot{\lambda} \end{bmatrix} + [\dot{\mathbf{s}}^T \ \dot{\lambda}^T] \cdot \begin{bmatrix} \dot{\mathbf{t}}_1 \\ \dot{\mathbf{t}}_2 \end{bmatrix} \tag{31}$$

relating to a single finite element. Here $\mathbb{L}$ is the modified flexibility matrix:

$$\mathbb{L} = \int_V \begin{bmatrix} \mathbf{C}\mathbf{H}^{-1}\mathbf{C}^T & \mathbf{C}\mathbf{H}^{-1}\mathbf{M}^T \\ \mathbf{M}\mathbf{H}^{-1}\mathbf{C} & \mathbf{M}\mathbf{H}^{-1}\mathbf{M}^T \end{bmatrix} dV = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix}, \tag{32}$$

in which $\mathbf{C} = \mathbf{C}^{\sigma}$ contains the basis functions of the stresses. Note that the matrix

$$\mathbf{L}_{11} = \int_V \mathbf{C}\mathbf{H}^{-1}\mathbf{C}^T dV \tag{33}$$

is the classical flexibility matrix of the force method, which has been extended to (32) due to the contact or locking conditions. Furthermore, we introduce the load vector

$$\dot{\mathbf{t}} = \begin{bmatrix} \dot{\mathbf{t}}_1 \\ \dot{\mathbf{t}}_2 \end{bmatrix} = \begin{bmatrix} -\int_V \mathbf{C}\dot{\boldsymbol{\epsilon}}_0 dV + \int_{S_u} \mathbf{C}\mathbf{n}\mathbf{T}\dot{\mathbf{v}} dS + \mathbf{T}\dot{\mathbf{t}}_0 \\ -\int_V \mathbf{M}\dot{\boldsymbol{\epsilon}}_0 dV + \int_{S_u} \mathbf{M}\mathbf{n}\mathbf{T}\dot{\mathbf{v}} dS \end{bmatrix} \tag{34}$$

which contains only kinematical-type loads, where $\dot{\mathbf{t}}_0$ acts directly to the nodal points, concentratedly.

For the mathematical programming formulation we have to eliminate the non sign-dependent variables from the functional (31). From the first variation of (31), we obtain the compatibility condition of the element:

$$[\delta\dot{\mathbf{s}}^T \ \delta\dot{\Lambda}^T] \cdot \left(-\begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{s}} \\ \dot{\lambda} \end{bmatrix} + \begin{bmatrix} \dot{\mathbf{t}}_1 \\ \dot{\mathbf{t}}_2 \end{bmatrix} \right) \leq 0 \tag{35}$$

Since $\delta\dot{\mathbf{s}}$ is arbitrary, from the first inequality of (35) we obtain an equation from which we have

$$\dot{\mathbf{s}} = -\mathbf{L}_{11}^{-1}\mathbf{L}_{12}\dot{\lambda} + \mathbf{L}_{11}^{-1}\dot{\mathbf{t}}_1 \tag{36}$$

in term of $\dot{\boldsymbol{\lambda}}$. Substituting (36) to functional (31), we obtain the complementary energy vith respect to the sign-dependent state variable $\dot{\boldsymbol{\lambda}}$ only:

$$\dot{\pi}_c(\dot{\boldsymbol{\lambda}}) = -\frac{1}{2}\dot{\boldsymbol{\lambda}}^{\mathrm{T}} \cdot \int_V \mathbf{M}(\mathbf{H}^{-1} - \mathbf{H}^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{L}_{11}^{-1}\mathbf{C}\mathbf{H}^{-1})\mathbf{M}^{\mathrm{T}}dV \cdot \dot{\boldsymbol{\lambda}} + \\ + \dot{\boldsymbol{\lambda}}^{\mathrm{T}} \cdot \left[\dot{\mathbf{t}}_2 - \int_V \mathbf{C}\mathbf{H}^{-1}\mathbf{M}^{\mathrm{T}}\mathbf{L}_{11}^{-1}dV \cdot \dot{\mathbf{t}}_1\right], \tag{37a}$$

or in a simplified form

$$\dot{\pi}_c(\dot{\boldsymbol{\lambda}}) = -\frac{1}{2}\dot{\boldsymbol{\lambda}}^{\mathrm{T}}\mathbf{B}\dot{\boldsymbol{\lambda}} + \dot{\boldsymbol{\lambda}}^{\mathrm{T}}\dot{\mathbf{b}}\,, \tag{37}$$

where

$$\mathbf{B} = \mathbf{L}_{22} - \mathbf{L}_{12}^{\mathrm{T}}\mathbf{L}_{11}^{-1}\mathbf{L}_{12}, \\ \dot{\mathbf{b}} = \dot{\mathbf{t}}_2 - \mathbf{L}_{12}\mathbf{L}_{11}^{-1}\dot{\mathbf{t}}_1. \tag{38}$$

The maximum conditions of the functional (37) lead to the variational inequalities

$$\delta\dot{\pi}_c(\dot{\boldsymbol{\lambda}}) = \delta\dot{\boldsymbol{\lambda}}^{\mathrm{T}}(-\mathbf{B}\dot{\boldsymbol{\lambda}} + \dot{\mathbf{b}}) \leq 0 \text{ and } \delta^2\dot{\pi}_c(\dot{\boldsymbol{\lambda}}) = -\delta\dot{\boldsymbol{\lambda}}^{\mathrm{T}}\mathbf{B}\,\delta\dot{\boldsymbol{\lambda}} \leq 0\,, \tag{39}$$

since matrix $\mathbf{B}$ is positive semidefinite. The first variation in (39) is equivalent to the locking law directly. Namely, considering the expressions (3), (15), (12a) and (36), (38) the locking potential velocity reads:

$$\dot{\mathbf{f}} = \mathbf{M}\dot{\boldsymbol{\epsilon}} = \mathbf{M}\mathbf{H}^{-1}\dot{\boldsymbol{\sigma}} = -\mathbf{M}\mathbf{H}^{-1}(\mathbf{C}^{\mathrm{T}}\dot{\mathbf{s}} + \mathbf{M}^{\mathrm{T}}\dot{\boldsymbol{\lambda}}) = \mathbf{M}\mathbf{H}^{-1}\mathbf{C}^{\mathrm{T}}(\mathbf{L}_{11}^{-1}\mathbf{L}_{12}\dot{\boldsymbol{\lambda}} - \mathbf{L}_{11}^{-1}\dot{\mathbf{t}}_1) - \\ - \mathbf{M}\mathbf{H}^{-1}\mathbf{M}^{\mathrm{T}}\dot{\boldsymbol{\lambda}} = -(\mathbf{L}_{22} - \mathbf{L}_{12}^{\mathrm{T}}\mathbf{L}_{11}^{-1}\mathbf{L}_{12}^{\mathrm{T}})\dot{\boldsymbol{\lambda}} - \mathbf{L}_{12}^{\mathrm{T}}\mathbf{L}_{11}^{-1}\dot{\mathbf{t}}_1 = -\mathbf{B}\dot{\boldsymbol{\lambda}} + \dot{\mathbf{b}}. \tag{40}$$

So the stationarity principle of the complementary energy in term of the sign-dependent locking stress rates relates to the locking potential velocities $\dot{\mathbf{f}}$ directly:

$$\delta\dot{\boldsymbol{\lambda}}^{\mathrm{T}}\dot{\mathbf{f}} \leq 0. \tag{41}$$

We know that the sign-dependent $\dot{\boldsymbol{\lambda}} \geq \mathbf{0}$ has sign-dependent variation $\delta\dot{\boldsymbol{\lambda}}$ which has to be positive, if $\dot{\boldsymbol{\lambda}} = \mathbf{0}$, because $\dot{\boldsymbol{\lambda}} + \delta\dot{\boldsymbol{\lambda}} \geq \mathbf{0}$. Since the variational inequality (41) says that $\delta\dot{\boldsymbol{\lambda}}$ and $\dot{\mathbf{f}}$ have opposite sign unless they are zero, it relates only to the locking domain $V_l \subset V$. Namely, if $\dot{\mathbf{f}} \geq \mathbf{0}$ on $V - V_l$, then $\delta\dot{\boldsymbol{\lambda}}$ can hardly be nonpositive because here $\boldsymbol{\Lambda} = \mathbf{0}$. The condition $\dot{\mathbf{f}} = \mathbf{0}$, if $\delta\dot{\boldsymbol{\lambda}}$ can be arbitrary, separates the domains V_l and $V - V_l$. The condition (41) states that

$$\dot{\mathbf{f}} \leq \mathbf{0} \text{ and } \dot{\boldsymbol{\lambda}} \geq \mathbf{0} \text{ on } V_l. \tag{42}$$

Besides, the variational inequality (41) includes the complementary or orthogonality condition, too:

$$\dot{\boldsymbol{\lambda}}^T \cdot \dot{\mathbf{f}} = 0. \tag{43}$$

We now give the primal-dual formulations of the *quadratic programming problem* relating to the extremum problem:

$$\begin{aligned} &\text{Q1: } \max \left\{ -\frac{1}{2}\dot{\boldsymbol{\lambda}}^T \mathbf{B}\dot{\boldsymbol{\lambda}} + \dot{\boldsymbol{\lambda}}^T \dot{\mathbf{b}} \,\middle|\, \dot{\boldsymbol{\lambda}} \geq \mathbf{0} \right\}, \\ &\text{Q2: } \min \left\{ \frac{1}{2}\dot{\boldsymbol{\lambda}}^T \mathbf{B}\dot{\boldsymbol{\lambda}} \,\middle|\, \mathbf{B}\dot{\boldsymbol{\lambda}} - \dot{\mathbf{b}} \leq 0,\ \dot{\boldsymbol{\lambda}} \geq \mathbf{0} \right\}, \end{aligned} \tag{44}$$

with the relating *linear complementary problem :*

$$\text{LC: } \{ -\mathbf{B}\dot{\boldsymbol{\lambda}} + \dot{\mathbf{b}} + \dot{\mathbf{f}} = \mathbf{0} \mid \dot{\boldsymbol{\lambda}} \geq \mathbf{0},\ \dot{\mathbf{f}} \leq \mathbf{0},\ \dot{\boldsymbol{\lambda}}^T \dot{\mathbf{f}} = 0 \}, \tag{45}$$

Note that these problems are equivalent to the following ones, changing the signs and the type of the extremum simultaneously

$$\begin{aligned} &\text{Q1: } \min \left\{ \frac{1}{2}\dot{\boldsymbol{\lambda}}^T \mathbf{B}\dot{\boldsymbol{\lambda}} - \dot{\boldsymbol{\lambda}}^T \dot{\mathbf{b}} \,\middle|\, \dot{\boldsymbol{\lambda}} \geq \mathbf{0} \right\} \\ &\text{Q2: } \max \left\{ -\frac{1}{2}\dot{\boldsymbol{\lambda}}^T \mathbf{B}\dot{\boldsymbol{\lambda}} \,\middle|\, -\mathbf{B}\dot{\boldsymbol{\lambda}} + \dot{\mathbf{b}} \leq \mathbf{0},\ \dot{\boldsymbol{\lambda}} \geq \mathbf{0} \right\} \\ &\text{LC: } \{ \mathbf{B}\dot{\boldsymbol{\lambda}} - \dot{\mathbf{b}} - \dot{\mathbf{f}} = \mathbf{0} \mid \dot{\boldsymbol{\lambda}} \geq \mathbf{0},\ \dot{\mathbf{f}} \leq \mathbf{0},\ \dot{\boldsymbol{\lambda}}^T \dot{\mathbf{f}} = 0 \} \end{aligned}$$

which are formally equal to the mathematical programming problems (27) and (28) relating to the potential energy formulation of the inequality problem of the plastic behaviour. This fact is also a proof of the high duality existing between the plastic and locking behaviour of the materials or structures.

*Problem Q*1 in (44) says that the velocity of the complementary energy is maximized by the actual locking stress rates $\dot{\boldsymbol{\sigma}}_l$, since:

$$\begin{aligned} -\frac{1}{2}\dot{\boldsymbol{\lambda}}^T \mathbf{B}\dot{\boldsymbol{\lambda}} + \dot{\boldsymbol{\lambda}}^T \dot{\mathbf{b}} &= -\frac{1}{2}\int_V \dot{\boldsymbol{\lambda}}^T \mathbf{M}(\mathbf{H}^{-1} - \mathbf{H}^{-1}\mathbf{C}^T \mathbf{L}_{11}^{-1} \mathbf{C}\mathbf{H}^{-1})\mathbf{M}^T \dot{\boldsymbol{\lambda}}\, dV + \\ &+ \int_V \dot{\boldsymbol{\lambda}}^T \mathbf{M}\dot{\boldsymbol{\varepsilon}}_0\, dV + \int_{S_u} \dot{\boldsymbol{\lambda}}^T \mathbf{M}\mathbf{n}\mathbf{T}\dot{\mathbf{v}}\, dS - \int_V \dot{\boldsymbol{\lambda}}^T \mathbf{M}\mathbf{H}^{-1}\mathbf{C}^T dV \cdot \mathbf{L}_{11}^{-1}\dot{\mathbf{t}}_1 = \\ &= -\frac{1}{2}\int_V \dot{\boldsymbol{\sigma}}_l^T(\mathbf{H}^{-1} - \mathbf{H}^{-1}\mathbf{C}^T \mathbf{L}_{11}^{-1}\mathbf{C}\mathbf{H}^{-1})\dot{\boldsymbol{\sigma}}_l\, dV + \int_V \dot{\boldsymbol{\sigma}}_l^T(\dot{\boldsymbol{\varepsilon}}_0 - \mathbf{H}^{-1}\mathbf{C}^T \mathbf{L}_{11}^{-1}\dot{\mathbf{t}}_1)\, dV + \\ &+ \int_{S_u} \dot{\boldsymbol{\sigma}}_l^T \mathbf{n}\mathbf{T}\dot{\mathbf{v}}\, dS = \dot{\pi}_c(\dot{\boldsymbol{\sigma}}_l) = \max! \end{aligned} \tag{46}$$

Problem Q2 in (44) represents a potential energy principle as follows:

$$\frac{1}{2}\dot{\lambda}^T B\dot{\lambda} = \frac{1}{2}\int_V \dot{\lambda}^T M(H^{-1} - H^{-1}C^T L_{11}^{-1} C H^{-1}) M^T \dot{\lambda} dV =$$

$$= \frac{1}{2}\int_V \dot{\lambda}^T M H^{-1}(H - C^T L_{11}^{-1} C) H^{-1} M^T \dot{\lambda} dV = \qquad (47)$$

$$= \frac{1}{2}\int_V \dot{\epsilon}_l^T (H - C^T L_{11}^{-1} C)\dot{\epsilon}_l dV = \dot{\pi}_p(\dot{\epsilon}_l) = \min!$$

which says that the actual strain rates $\dot{\epsilon}_l$ due to the locking stresses, calculated on the basis of the elastic behaviour of the structure, make the velocity of the potential energy minimal.

Problem LC expresses that the actual non-negative locking stress rates are always orthogonal to the non-positive locking potential velocities.

5. Mathematical programming formulation of plastic-locking connections on the basis of strain approximations

On the basis of the duality existing between the plastic and locking phenomena, a hybrid variational principle has been worked out. This principle relates only to the sign-dependent state variables, namely to the multipliers $\dot{\Lambda}$ and $\dot{\lambda}$ of the plastic strain rates $\dot{\epsilon}_p$ and the locking stress rates $\dot{\sigma}_l$, respectively. For eliminating any variables from a functional of a variational principle, relations among the variables have to be introduced previously.

Let us start from the generalized Hu-Washizu variational principle relating to the state change analysis of a single finite element with plastic-locking points:

$$\dot{\pi}_{HW}(\dot{\sigma}, \dot{r}, \dot{\lambda}, \dot{\epsilon}, \dot{u}, \dot{\Lambda}) = \int_V \left(\frac{1}{2}\dot{\epsilon}^T H\dot{\epsilon} + \dot{\lambda}^T M\dot{\epsilon}\right) dV -$$

$$-\int_V \dot{u}^T \dot{g} dV - \int_V [(\dot{\epsilon} + \Lambda^T \dot{N} - DT\dot{u})^T \cdot (\dot{\sigma} + M^T\lambda)] dV - \qquad (48)$$

$$-\int_{S_u} \dot{u}^T \dot{p} dS - \int_{S_p} [(\dot{u} - \dot{v})^T \cdot \dot{r}] dS = \text{stationary}.$$

For eliminating the non sign-dependent variables $\dot{\sigma}$, $\dot{r}$, $\dot{\epsilon}$, u from the functional, we express them in term of the sign-dependent ones $\dot{\Lambda}$ and $\dot{\lambda}$ by using the compatibility and equilibrium equations with all the boundary conditions and the constitutive transformation, as well.

First we introduce the approximation of (7) type:

$$\dot{\mathbf{u}} = \mathbf{T}^T \mathbf{C} \dot{\mathbf{e}}$$

in which $\dot{\mathbf{e}}$ consists of the values of the displacement function $\dot{\mathbf{u}}$, at the nodal points of the element.

In this way from the compatibility equation of the element

$$\dot{\boldsymbol{\epsilon}} + \mathbf{N}^T \dot{\boldsymbol{\Lambda}} - \mathbf{D}\mathbf{T}\dot{\mathbf{u}} = \mathbf{0} \text{ on } V \tag{49}$$

we can obtain the strain function $\dot{\boldsymbol{\epsilon}}$ in term of the nodal point displacement values

$$\dot{\boldsymbol{\epsilon}} = \mathbf{D}\mathbf{C}\dot{\mathbf{e}} - \mathbf{N}^T \dot{\boldsymbol{\Lambda}} \tag{50}$$

related to the local coordinate system.
Substituting the form of the Hooke's law

$$\dot{\boldsymbol{\epsilon}} = \mathbf{H}^{-1} \dot{\boldsymbol{\sigma}} \tag{51}$$

for (49), we have

$$\mathbf{H}^{-1} \dot{\boldsymbol{\sigma}} + \mathbf{N}^T \dot{\boldsymbol{\Lambda}} - \mathbf{D}\mathbf{C}\dot{\mathbf{e}} = \mathbf{0}, \tag{52}$$

from which $\dot{\boldsymbol{\sigma}}$ can be expressed in the similar form to (50):

$$\dot{\boldsymbol{\sigma}} = \mathbf{H}\mathbf{D}\mathbf{C}\dot{\mathbf{e}} - \mathbf{H}\mathbf{N}^T \dot{\boldsymbol{\Lambda}}. \tag{53}$$

The equilibrium equation

$$\mathbf{T}^T \mathbf{D}^T (\dot{\boldsymbol{\sigma}} + \mathbf{M}^T \dot{\boldsymbol{\lambda}}) + \dot{\mathbf{g}} = \mathbf{0} \text{ on } V \tag{54}$$

relates to the volume of the element but we have to reduce all the force type loadings to the nodal points because most of the loads are generally acting on the nodal points directly.

The equilibrium boundary conditions

$$\begin{aligned} \dot{\mathbf{p}} - \mathbf{T}^T \mathbf{n}^T (\dot{\boldsymbol{\sigma}} + \mathbf{M}^T \dot{\boldsymbol{\lambda}}) &= \mathbf{0} \text{ on } S_p \\ \dot{\mathbf{r}} - \mathbf{T}^T \mathbf{n}^T (\dot{\boldsymbol{\sigma}} + \mathbf{M}^T \dot{\boldsymbol{\lambda}}) &= \mathbf{0} \text{ on } S_u \end{aligned} \tag{55}$$

relates to the surface load functions and we have a load $\mathbf{q}_0$ too, acting directly on the nodal points. So the loads in the equilibrium equation (54) and in the boundary conditions (55) have to be concentrated to the nodal points by using the same reduction functions $\mathbf{C}$ of (48):

$$\dot{\mathbf{q}} = \mathbf{T}\dot{\mathbf{q}}_0 + \int_{S_p} \mathbf{C}^T \mathbf{T} \dot{\mathbf{p}} \, dS - \int_V \mathbf{C}^T \mathbf{T} \dot{\mathbf{g}} \tag{56}$$

related to the local coordinate system. Consequently the equilibrium conditions (54) and (55) have to be drawn together in a common expression relating to the local system:

$$\int_V \mathbf{D}^T \mathbf{C}^T (\dot{\boldsymbol{\sigma}} + \mathbf{M}^T \dot{\boldsymbol{\lambda}}) \, dV - \dot{\mathbf{q}} = \mathbf{0}. \tag{57}$$

Now we can use the expression (53) of $\dot{\sigma}$ by substituting it for (57)

$$\int_V \mathbf{D}^T\mathbf{C}^T(\mathbf{H}\mathbf{D}\mathbf{C}\dot{\mathbf{e}} - \mathbf{H}\mathbf{N}^T\dot{\mathbf{\Lambda}} + \mathbf{M}^T\dot{\mathbf{\lambda}})\,dV - \dot{\mathbf{q}} = \mathbf{0}. \tag{58}$$

Considering that $\dot{\mathbf{e}}$, $\dot{\mathbf{\Lambda}}$ and $\dot{\mathbf{\lambda}}$ contain equally scalars and so they can be taken out from the integrals, we have:

$$\int_V \mathbf{D}^T\mathbf{C}^T\mathbf{H}\mathbf{D}\mathbf{C}dV\cdot\dot{\mathbf{e}} - \int_V \mathbf{D}^T\mathbf{C}^T\mathbf{H}\mathbf{N}^T dV\cdot\dot{\mathbf{\Lambda}} + \int_V \mathbf{D}^T\mathbf{C}^T\mathbf{M}^T dV\cdot\dot{\mathbf{\lambda}} - \dot{\mathbf{q}} = \mathbf{0}. \tag{59}$$

Finally we can express $\dot{\mathbf{e}}$ in term of the sign-dependent $\dot{\mathbf{\Lambda}}$ and $\dot{\mathbf{\lambda}}$:

$$\dot{\mathbf{e}} = \mathbb{K}^{-1}\int_V \mathbf{B}^T\mathbf{H}\mathbf{N}^T dV\cdot\dot{\mathbf{\Lambda}} - \mathbb{K}^{-1}\int_V \mathbf{B}^T\mathbf{M}^T dV\cdot\dot{\mathbf{\lambda}} + \mathbb{K}^{-1}\dot{\mathbf{q}}. \tag{60}$$

Here

$$\mathbb{K} = \mathbb{K}_{11} = \int_V \mathbf{D}^T\mathbf{C}^T\mathbf{H}\mathbf{D}\mathbf{C}dV = \int_V \mathbf{B}^T\mathbf{H}\mathbf{B}dV \tag{61}$$

is the classical stiffness matrix of the element related to the local system and

$$\mathbf{B} = \mathbf{D}\mathbf{C} \tag{62}$$

is the so-called geometry matrix which contains the derivatives of the basic functions. If we have any kinematical type loadings, for example an initial strain $\dot{\epsilon}_0$ then the expression (60) of $\dot{\mathbf{e}}$ is completed by the term of

$$\mathbb{K}^{-1}\int_V \mathbf{B}^T\mathbf{H}\dot{\epsilon}_0 dV = \mathbb{K}^{-1}\dot{\mathbf{t}}. \tag{63}$$

After determining $\dot{\mathbf{e}} = \dot{\mathbf{e}}(\dot{\mathbf{\Lambda}}, \dot{\mathbf{\lambda}})$, we can express $\dot{\mathbf{u}} = \dot{\mathbf{u}}(\dot{\mathbf{\Lambda}}, \dot{\mathbf{\lambda}})$, $\dot{\epsilon} = \dot{\epsilon}(\dot{\mathbf{\Lambda}}, \dot{\mathbf{\lambda}})$ and $\dot{\sigma} = \dot{\sigma}(\dot{\mathbf{\Lambda}}, \dot{\mathbf{\lambda}})$, by applying expression (60) in (48), (50) and (53), respectively. Finally, by substituting $\dot{\mathbf{u}}$, $\dot{\epsilon}$ and $\dot{\sigma}$ for the original expression of the Hu-Washizu functional (48), we obtain a new functional $\dot{\pi}_{pl}$ and a new variational principle, the so called activization energy principle related to the active state of the points, the plastic and locking behaviour of the finite element:

$$\dot{\pi}_{pl} = \dot{\pi}_{pl}(\dot{\mathbf{\Lambda}}, \dot{\mathbf{\lambda}}) =$$

$$= \frac{1}{2}[\dot{\mathbf{\Lambda}}^T\ \dot{\mathbf{\lambda}}^T]\int_V \begin{bmatrix} \mathbf{N} & \\ & \mathbf{MB} \end{bmatrix} \begin{bmatrix} \mathbf{H} - \mathbf{HBK}^{-1}\mathbf{B}^T\mathbf{H} & \mathbf{HBK}^{-1} \\ \mathbf{K}^{-1}\mathbf{B}^T\mathbf{H} & -\mathbf{K}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{N}^T & \\ & \mathbf{B}^T\mathbf{M}^T \end{bmatrix} dV \begin{bmatrix} \dot{\mathbf{\Lambda}} \\ \dot{\mathbf{\lambda}} \end{bmatrix} -$$

$$- [\dot{\mathbf{\Lambda}}^T\ \dot{\mathbf{\lambda}}^T]\int_V \begin{bmatrix} \mathbf{N} & \\ & \mathbf{HB} \end{bmatrix} \begin{bmatrix} \mathbf{H} - \mathbf{HBK}^{-1}\mathbf{B}^T\mathbf{H} & \mathbf{HBK}^{-1} \\ \mathbf{K}^{-1}\mathbf{B}^T\mathbf{H} & -\mathbf{K}^{-1} \end{bmatrix} dV \begin{bmatrix} \dot{\mathbf{t}} \\ \dot{\mathbf{q}} \end{bmatrix} +$$

$$+ \text{constants} = \text{stationary},$$

or in a more useful form

$$\dot{\pi}_{pl}(\dot{\Lambda}, \dot{\lambda}) = \frac{1}{2}[\dot{\Lambda}^T \ \dot{\lambda}^T]\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \dot{\Lambda} \\ \dot{\lambda} \end{bmatrix} - [\dot{\Lambda}^T \ \dot{\lambda}^T]\begin{bmatrix} \dot{a}_1 \\ \dot{a}_2 \end{bmatrix} = \text{stationary} \quad (65)$$

subject to $\dot{\Lambda} \geq 0$ and $\dot{\lambda} \geq 0$.

This functional is expressed only in term of the sign-dependent plastic strain rates and locking stress rates. But this extremum problem has a lot of other subsidiary conditions, namely all of the canonical equations of the elasticity, the equilibrium and the compatibility equations with all the boundary conditions and the elastic material law, as well. What can then this principle express? Of course, it describes the plastic and locking behaviour, namely, the plastic flow law and the locking law are characterized by it, just like the previously detailed principles $\dot{\pi}_p(\dot{\Lambda}) = \min!$ and $\dot{\pi}_c(\dot{\lambda}) = \max!$, but separately. Consequently, the functional $\dot{\pi}_{pl}(\dot{\Lambda}, \dot{\lambda})$ has a saddle point by the actual solution because of the contrasted characteristic of its variables, having $\dot{\Lambda}$ as a strain and $\dot{\lambda}$ as a stress type variable.

The first variations of the functional $\pi_{pl}(\dot{\Lambda}, \dot{\lambda})$ according to the variables $\dot{\Lambda}$ and $\dot{\lambda}$, respectively, yield the variational inequalities

$$\delta_{\Lambda}\dot{\pi}_{pl} = \delta\dot{\Lambda}^T(A_{11}\dot{\Lambda} + A_{12}\dot{\lambda} - \dot{a}_1) \geq 0, \quad (66)$$

$$\delta_{\lambda}\dot{\pi}_{pl} = \delta\dot{\lambda}^T(A_{22}\dot{\lambda} + A_{21}\dot{\Lambda} - \dot{a}_2) \leq 0, \quad (67)$$

representing the saddle point as a minimum point over the plastic strains and as a maximum point over the locking stresses.

Knowing that by (59) and (60) we obtain $\dot{F}$ as

$$\begin{aligned} \dot{F} &= N\dot{\sigma} = N(HB\dot{e} - HN^T\dot{\Lambda}) = \\ &= NHB\left(K^{-1}\int_V B^T HN^T dV \cdot \dot{\Lambda} - K^{-1}\int_V B^T M^T dV \cdot \dot{\lambda} + K^{-1}\dot{q}\right) - \\ &\quad - NHN^T\dot{\Lambda} = -(A_{11}\dot{\Lambda} + A_{12}\dot{\lambda} - \dot{a}_1), \end{aligned} \quad (68)$$

and from (50) and (60) $\dot{f}$ results:

$$\begin{aligned} \dot{f} &= M\dot{\epsilon} = M(B\dot{e} - N^T\dot{\Lambda}) = \\ &= MB\left(K^{-1}\int_V B^T HN^T dV \cdot \dot{\Lambda} - K^{-1}\int_V B^T M^T dV \cdot \dot{\lambda} + K^{-1}\dot{q}\right) - \\ &\quad - MN^T\dot{\Lambda} = A_{22}\dot{\lambda} + A_{21}\dot{\Lambda} - \dot{a}_2, \end{aligned} \quad (69)$$

since $MN^T = 0$ because $V_p \cup V_l = \emptyset$.

So the variational inequalities (66) and (67) can be written in the form, in harmony with the expressions (24) and (41)

$$\delta[\dot{\Lambda}^T \ \dot{\lambda}^T]\begin{bmatrix} \dot{F} \\ \dot{f} \end{bmatrix} \leq 0, \quad (70)$$

relating to the plastic and locking domains V_p and V_l. The variational inequality (60) is equivalent to the unified inequality and complementary conditions of (25), (26) and (42), (43):

$$\begin{bmatrix}\dot{\mathbf{\Lambda}}\\ \dot{\boldsymbol{\lambda}}\end{bmatrix} \geq \mathbf{0},\ \begin{bmatrix}\dot{\mathbf{F}}\\ \dot{\mathbf{f}}\end{bmatrix} \leq \mathbf{0} \quad \text{and} \quad [\dot{\mathbf{\Lambda}}^{\mathrm{T}}\dot{\boldsymbol{\lambda}}^{\mathrm{T}}]\cdot\begin{bmatrix}\dot{\mathbf{F}}\\ \dot{\mathbf{f}}\end{bmatrix} = 0, \tag{71}$$

or to the simplified form of

$$\dot{\mathbf{x}} \geq \mathbf{0},\quad \dot{\mathbf{y}} \leq \mathbf{0} \quad \text{and} \quad \dot{\mathbf{x}}^{\mathrm{T}}\dot{\mathbf{y}} = 0. \tag{72}$$

Then the expressions of the potential velocities (68) and (69) read

$$\dot{\mathbf{y}} = \begin{bmatrix}\dot{\mathbf{F}}\\ \dot{\mathbf{f}}\end{bmatrix} = \begin{bmatrix}-\mathbf{A}_{11} & -\mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{\Lambda}}\\ \dot{\boldsymbol{\lambda}}\end{bmatrix} - \begin{bmatrix}-\dot{\mathbf{a}}_1\\ \dot{\mathbf{a}}_2\end{bmatrix} = \hat{\mathbf{A}}\dot{\mathbf{x}} - \hat{\mathbf{a}} \tag{73}$$

and in this way, the mathematical programming problem relating to the saddle point problem (65) can be reduced to the linear complementary problem

$$\mathrm{LC}\colon \{\hat{\mathbf{A}}\dot{\mathbf{x}} - \dot{\mathbf{a}} - \dot{\mathbf{y}} = \mathbf{0} \mid \dot{\mathbf{x}} \geq \mathbf{0},\ \dot{\mathbf{y}} \leq \mathbf{0},\ \dot{\mathbf{x}}^{\mathrm{T}}\dot{\mathbf{y}} = 0\}. \tag{74}$$

This LC problem expresses that the actual non-negative plastic strain rates, simultaneously with the locking stress increments, are always orthogonal to the non-positive plastic and locking potential velocities, respectively.

Of course, we can create the relating quadratic programming problems, as well:

$$\mathrm{Q1}\colon \min\left\{-\frac{1}{2}\dot{\mathbf{x}}^{\mathrm{T}}\hat{\mathbf{A}}\dot{\mathbf{x}} + \dot{\mathbf{a}}\dot{\mathbf{x}} \mid \dot{\mathbf{x}} \geq \mathbf{0}\right\} \tag{75}$$

$$\mathrm{Q2}\colon \max\left\{\frac{1}{2}\dot{\mathbf{x}}^{\mathrm{T}}\hat{\mathbf{A}}\dot{\mathbf{x}} \mid \hat{\mathbf{A}}\dot{\mathbf{x}} - \dot{\mathbf{a}} \leq \mathbf{0},\ \dot{\mathbf{x}} \geq \mathbf{0}\right\}. \tag{76}$$

Taking the expressions of the potential velocities (68) and (69) into consideration, the form of the energy $\dot{\pi}(\dot{\mathbf{\Lambda}}, \dot{\boldsymbol{\lambda}})$ can be written in a form of

$$\dot{\pi}_{pl}(\dot{\mathbf{\Lambda}}, \dot{\boldsymbol{\lambda}}) = -\dot{\mathbf{\Lambda}}^{\mathrm{T}}\dot{\mathbf{F}} + \dot{\boldsymbol{\lambda}}^{\mathrm{T}}\dot{\mathbf{f}} \tag{77}$$

expressing that this functional contains the indicator functionals of the convex sets

$$K = \{\boldsymbol{\epsilon} \mid \mathbf{f}(\boldsymbol{\epsilon}) \leq \mathbf{0}\} \tag{78}$$

relating to the locking phenomena and similarly

$$K^c = \{\boldsymbol{\sigma} \mid \mathbf{F}(\boldsymbol{\sigma}) \leq \mathbf{0}\} \tag{79}$$

referring to the plastification [7], since thein dicator functionals, by definition, are

$$\begin{aligned} \dot{J}_K(\dot{\boldsymbol{\epsilon}}) &= \begin{cases}\dot{\boldsymbol{\lambda}}^{\mathrm{T}}\dot{\mathbf{f}} = \dot{\boldsymbol{\lambda}}^{\mathrm{T}}\mathbf{M}\dot{\boldsymbol{\epsilon}} = 0 \ \text{if}\ \boldsymbol{\epsilon} \in \mathbf{K}\\ \infty,\ \text{if}\ \boldsymbol{\epsilon} \notin K\end{cases}\\ J_K^c(\dot{\boldsymbol{\sigma}}) &= \begin{cases}\dot{\mathbf{\Lambda}}^{\mathrm{T}}\dot{\mathbf{F}} = \dot{\mathbf{\Lambda}}^{\mathrm{T}}\mathbf{N}\dot{\boldsymbol{\sigma}} = 0 \ \text{if}\ \boldsymbol{\sigma} \in K^c\\ \infty,\ \text{if}\ \boldsymbol{\sigma} \notin K^c\end{cases} \end{aligned} \tag{80}$$

Thus, we can state that the so called activization energy velocity $\dot{\pi}_{pl}$ of the finite element consists of the indicator functionals. Since the indicator functionals are really the Lagrange functions, extending the Lagrange multiplier method to the case of inequality problems, evidently, the functional of the activization energy is a generalized Lagrange function.

If the expression (77) has no subsidiary conditions relating to the variables $\dot{\Lambda}$, $\dot{\lambda}$, $\dot{F}$ and $\dot{f}$, then it can be taken into account as a functional of four variables:

$$\dot{\pi}_{pl} = \dot{\pi}_{pl}(\dot{\Lambda}, \dot{\lambda}, \dot{F}, \dot{f}) = -\dot{\Lambda}^T\dot{F} + \dot{\lambda}^T\dot{f}. \tag{81}$$

Thus, among the first variations of (81), beside the variations by $\dot{\Lambda}$ and $\dot{\lambda}$ given in (70), also the variations with respect to $\dot{F}$ and $\dot{f}$ have to be done, too:

$$\delta[\dot{F}^T \ \dot{f}^T]\begin{bmatrix}\dot{\Lambda}\\ \dot{\lambda}\end{bmatrix} \geq 0 \tag{82}$$

which are equivalent to the conditions (71), of course.

We can obtain the functional (81) by the Lagrange multiplier method, too. Let us start from the functional $\dot{\pi}_{pl}(\dot{\Lambda}, \dot{\lambda})$ of two variables, given in the expression (65). The subsidiary conditions relating to the variables are the inequalities of

$$\dot{\Lambda} \geq 0 \text{ and } \dot{\lambda} \geq 0.$$

Introducing $\dot{F} \leq 0$ and $\dot{f} \leq 0$ as Lagrange multipliers of inequality type, we obtain the Lagrange functions

$$\dot{\Lambda}^T\dot{F} = 0 \text{ and } \dot{\lambda}^T\dot{f} = 0. \tag{83}$$

Adding the Lagrange functions (83) to the functional (65) we obtain a functional of four independent variables:

$$\dot{\pi}_{pl}(\dot{\Lambda}, \dot{\lambda}, \dot{F}, \dot{f}) = \frac{1}{2}[\dot{\Lambda}^T \ \dot{\lambda}^T]\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}\dot{\Lambda}\\ \dot{\lambda}\end{bmatrix} - [\dot{\Lambda}^T \ \dot{\lambda}^T]\begin{bmatrix}\dot{a}_1\\ \dot{a}_2\end{bmatrix} -$$
$$- [\dot{\Lambda}^T \ \dot{\lambda}^T]\begin{bmatrix}-\dot{F}\\ \dot{f}\end{bmatrix} = \text{stationary}. \tag{84}$$

The first variations with respect to $\dot{\Lambda}$ and $\dot{\lambda}$ are

$$\begin{aligned}\delta\dot{\Lambda}^T(A_{11}\dot{\Lambda} + A_{12}\dot{\lambda} - \dot{a}_1 + \dot{F}) = 0,\\ \delta\dot{\lambda}^T(A_{21}\dot{\Lambda} + A_{22}\dot{\lambda} - \dot{a}_2 - \dot{f}) = 0,\end{aligned} \tag{85}$$

from which we have for $\dot{F}$ and $\dot{f}$ that

$$\begin{aligned}\dot{F} = -(A_{11}\dot{\Lambda} + A_{12}\dot{\lambda} - \dot{a}_1)\\ \dot{f} = A_{21}\dot{\Lambda} + A_{22}\dot{\lambda} - \dot{a}_2.\end{aligned} \tag{86}$$

The variational principle of the activization energy $\dot{\pi}_{pl}$ has been applied to frame structures. Numerical examples relating to the simultaneous elastic-plastic-locking behaviour of structures have been worked out and published in [3, 6, 17, 19, 20].

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