ANALYSIS OF ELASTIC STRUCTURES BY MATHEMATICAL PROGRAMMING

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Abstract

The physical nonlinearity of the structures is examined in the elastic state. The problem is solved by finite element method and nonlinear mathematical programming. For the second case both the primal and the dual problems are discussed. The different methods of the solutions are compared.

1. Introduction

The basic idea of the finite element method is to solve an unconstrained minimization problem.

Nonlinearities occur in two different forms. The first one is the physical or material nonlinearity from which nonlinear constitutive laws are derived. In this case the theory of small displacements is considered. The geometric nonlinearity is the second one, which is derived from the finite changes in the geometry of a deforming body. This category encompasses large strains and large displacements. Most generally, nonlinear problems are the combination of the above cases.

The solution of nonlinear problems by the finite element method is usually done by one of the three basic techniques: incremental or stepwise procedures, iterative or Newton methods and step-iterative or mixed procedures.

The basic idea of the incremental or stepwise procedure is the subdivision of load into many small partial loads or increments. The solution for each step of loading is obtained as an increment of the displacements. Usually, the tangent modul is used to compute the tangent stiffness matrix in the incremental procedure. The equilibrium equations are satisfied in each step [1], [7].

The iterative procedure is a sequence of calculations in which the structure is fully loaded in each iteration. After each iteration the portion of the total loading that is not balanced is calculated and used in the next step to compute an additional increment of the displacements. This process is repeated until the equilibrium is approximated [2], [7].

The step-iterative procedure is the combination of the incremental and iterative methods [7].

In this paper the physically nonlinear problems are discussed as a constrained minimization one and solved by nonlinear mathematical programming.

2. The primal and dual problems of nonlinear elastic analysis

The structure is divided into finite elements in the usual way. The shape functions are supposed in polynomial form on an unique domain in a curvilinear coordinate system. The relationships between the displacements and the strains ($\mathbb{B}$) are determined according to the type of the element in the curvilinear coordinate system.

$$\begin{gathered}\boldsymbol{\epsilon} = \mathbb{B}\mathbf{u} + \boldsymbol{\epsilon}_0 \\ [z_j]\,[z_j s_i]\,[s_i]\,[z_j]\end{gathered} \tag{1}$$

where: $\boldsymbol{\epsilon}$: vector of strains
$\mathbf{u}$: vector of displacements
$\boldsymbol{\epsilon}_0$: initial value of strains
z_j: stress freedom in the j-th point of the element
s_i: displacement's freedom at the i-th node.

Using Jacobian and — if neccessary — linear transformations, equations (1) are in the global Cartesian system of the structure. The correlations between the loads and inner forces are given by transposing matrix $\mathbf{B}$ in a similar way. Integrating the stresses on the domain of element we get the inner forces. The equilibrium equations of an element are the following in the global Cartesian system:

$$\int_{\Omega} \mathbf{T}^*(\mathbf{J}^{-1})^* \mathbf{B}^*(\boldsymbol{\sigma} + \boldsymbol{\sigma}_0)\, d\Omega + \mathbf{f} = 0 \tag{2}$$

where Ω: domain of an element
$\boldsymbol{\sigma}$: vector of stresses
$\boldsymbol{\sigma}_0$: initial value of stresses
$\mathbf{f}$: vector of loads
$\mathbf{J}$: matrix of Jacobi transformation
$\mathbf{T}$: matrix of linear transformation.

The equilibrium equations of the structure are presented by the compilation of (2). It is realized according to the joined nodes in the hyper rows and to the elements in the hyper columns. The boundary conditions are taken into consideration in the usual way.

It means that the displacements are joined on the finite element nodes but the unknowns are the stresses of the elements.

According to the principle of stationary complementary strain energy, among the stress distributions of a structure which are in equilibrium with forces those will be valid where the complementary strain energy has stationarity. In this way the principle of small displacements are supposed, therefore the stationary point is the global minimum and the compatibility equations are satisfied as well.

The complementary strain energy of an element is:

$$\mathbf{U}_i^* = \int_{\sigma_0}^{\sigma} g(\tau)\, d\tau\, d\Omega_i \tag{3}$$

where $\varepsilon = g(\sigma)$ is a function of the nonlinear elastic material's model according to Fig. 1.

Summing up, the following mathematical programming problem has to be solved in the case of the whole structure.

$$\varkappa \left[\int_{\Omega} \mathbf{T}^* (\mathbf{J}^{-1})^* \mathbf{B}^* (\boldsymbol{\sigma} + \boldsymbol{\sigma}_0)\, d\Omega + \mathbf{f} \right] = 0 \tag{4.a}$$

$$\sum^{m} \int_{\Omega} \int_{\sigma_0}^{\sigma} g(\tau)\, d\tau\, d\Omega \Rightarrow \min \tag{4.b}$$

where: m is the number of members and the compilation — denoted by $\varkappa$ — is performed according to the nodes and members.

Using Wolfe's procedure the dual problem is:

$$-\varkappa \int_{\Omega} \big(g(\boldsymbol{\sigma}) - g(\boldsymbol{\sigma}_0)\big)\, d\Omega = \varkappa \int_{\Omega} \boldsymbol{\epsilon} \mathbf{T}^* (\mathbf{J}^{-1})^* \mathbf{B}^*\, d\Omega \tag{5.a}$$

$$\sum^{m} \int_{\Omega} \int_{\sigma_0}^{\sigma} g(\tau)\, d\tau\, d\Omega + \sum^{m} \int_{\Omega} \boldsymbol{\epsilon} \mathbf{T} (\mathbf{J}^{-1}) \mathbf{B} (\boldsymbol{\sigma} + \boldsymbol{\sigma}_0)\, d\Omega + \int_{\Omega} \boldsymbol{\epsilon}\, d\Omega\, \mathbf{f} \Rightarrow \min. \tag{5.b}$$

Equation (5.a) can be written in the following form:

$$\varkappa \left[\int_{\Omega} \big(g(\boldsymbol{\sigma}) - g(\boldsymbol{\sigma}_0) + \varepsilon \mathbf{T}^* (\mathbf{J}^{-1})^* \mathbf{B}^*\big)\, d\Omega \right] = 0. \tag{5.c}$$

The value of the integral is equal to zero if the integrandus is zero. Multiplying by $(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0)$ the next form is valid:

$$\sum^{m} \big(g(\boldsymbol{\sigma}) - g(\boldsymbol{\sigma}_0)\big)(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) + \sum^{m} \boldsymbol{\epsilon} \mathbf{T} (\mathbf{J}^{-1}) \mathbf{B} (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) = 0. \tag{5.d}$$

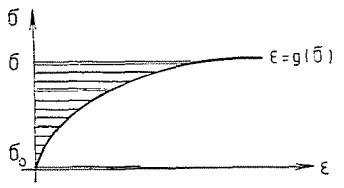

Fig. 1

Inserting into the objective function, using the displacements instead of the integral of the strains and transforming into the minimization problem the final form of the dual is:

$$\varkappa\left(\mathbb{B}(\mathbb{J}^{-1})\mathbb{T}u + \int_{\Omega}\left(g(\boldsymbol{\sigma}) - g(\boldsymbol{\sigma}_0)\right)d\Omega\right) = 0 \tag{6.a}$$

$$\sum^{m}\int_{\Omega}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0)\left(g(\boldsymbol{\sigma}) - g(\boldsymbol{\sigma}_0)\right) - \int_{\sigma_0}^{\sigma} g(\tau)\,d\tau\,d\Omega - \int_{\Omega}\epsilon\,d\Omega\,\mathbf{f} \Rightarrow \min. \tag{6.b}$$

It corresponds to the principle of stationary strain energy: among the displacements of a structure which satisfy the compatibility equations those will be valid where the strain energy has stationarity and similarly to the former case this is the global minimum. The structure has an equilibrium state with these displacements.

3. Numerical approximations

Approximating the integrals by Gauss-Legendre's numerical method, problems (4) and (6) are in the following form:
Primal:

$$\varkappa\left[\sum^{N}\varrho\mathbb{T}^{*}(\mathbb{J}^{-1})^{*}\mathbb{B}^{*}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) + \mathbf{f}\right] = 0 \tag{7.a}$$

$$\sum^{m}\sum^{N}\varrho\sum^{L}\varrho g(\hat{\sigma}) \Rightarrow \min \tag{7.b}$$

where: $\hat{\sigma}$ the Gaussian point on the axis σ,
N number of the Gaussian points on the domain Ω,
L number of the Gaussian points on the axis σ.

Dual:

$$\varkappa\left[\mathbb{B}(\mathbb{J}^{-1})\mathbb{T}\mathbf{u} + \sum^{N}\varrho g(\hat{\sigma}) - g(\sigma_0)\right] = 0 \tag{8.a}$$

$$\sum^{m}\sum^{N}\varrho(\hat{\sigma} - \sigma_0)\left(g(\hat{\sigma}) - g(\sigma_0)\right) - \sum^{L}\left(\varrho g(\hat{l})\right) - \mathbf{u}\mathbf{f} \Rightarrow \min \tag{8.b}$$

where: $\hat{\sigma}$ is the Gaussian point.

At the optimal point the constraints of both primal and dual problems are satisfied.

In the linear case (when $g(\sigma) = \mathbb{F}\sigma$) it is easy to check that the wellknown expression of the stiffness matrix is obtained by inserting the stresses from (8.a) into (7.a).

Now we have three possibilities to determine the stresses and the displacements of the structure:

i: In the usual way we solve the system of equations with nonlinear stiffness matrix. This procedure is described in the literature. [7, 9] The number of the unknowns: $\sum_{i=1}^{N} s_i$.

ii: Solving the nonlinear system of equations which contains (7.a) and (8.a). The structure of the system of equations can be seen in Fig. 2.

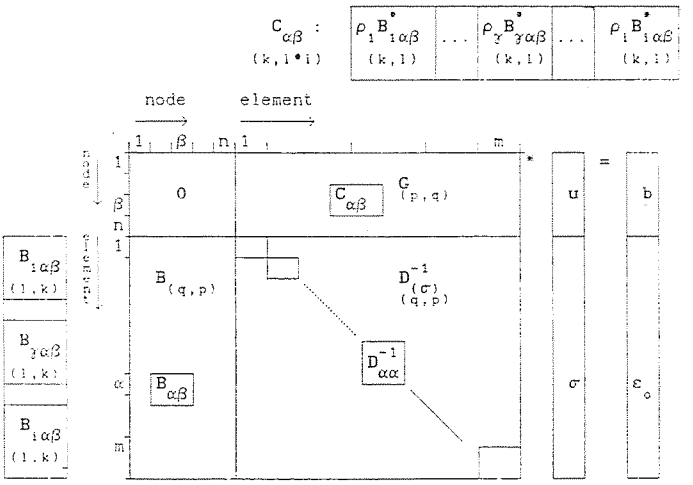

Fig. 2

The coefficient matrix has not a symmetric form. This is caused by the fact that the displacement and the stresses are determined at the nodes and at the Gaussian points, respectively.

The next problem is the choice of the number of the Gaussian points. It is neccessary that the degree of the shape polynoms should be in accordance with them [8].

In the case of one-dimensional approximation [8]

$$\alpha = 2r - 1$$

where: r is the degree of the shape polynoms.

In multi-dimensional cases the number of Gaussian points can be determined by multiplying the numbers which are neccessary on the coordinate axis. The number α is equal to the number of the Legendre polynom's roots and in this case the approximation has minimal error.

— If $\alpha > 2r - 1$ the equations will be linear,

— if $\alpha < 2r - 1$ some information contained in shape functions is lost.

The number of the unknowns: $\sum_{i=1}^{N} s_i + \sum_{i=1}^{m} \sum_{j=1}^{\alpha_i} z_j$

iii: The mathematical programming problems (7) or (8) are solved directly. In this case it does not cause any difficulty if $\alpha > 2r - 1$.

If $\alpha < 2r - 1$ the result will not be right.

In the primal problem the matrix of constraints has full row rank. Separating the free variables the number of the unknowns is: $\sum_{i=1}^{m} \sum_{j=1}^{z_i} z_j - \sum_{i=1}^{N} s_i$.

The dual problem contains nonlinear functions in the constraints, hence the separation of the free unknowns is difficult.

The number of unknowns remains: $\sum_{i=1}^{N} s_i + \sum_{i=1}^{m} \sum_{j=1}^{z_i} z_j$.

Summarizing the i;-th case generally has less unknowns than the primal but the solution needs more difficult procedure than using a nonlinear programming method. The ii;-th case and the dual resulted the same number of unknowns but they are too large for the practice.

4. Numerical experiences

If Hooke's law is avaible (linear case) many examples give the same result in the above cases. The CPU time cannot be compared because the different cases were run on different computers.

If the phenomenon has a hyperelastic character the Green models can be used for the constitutive law. These suppose that the complementary energy function can be given in the polynomial form of the stress deviators. In this case the complementary energy function contains unknown stresses of more than second degree. This means it is nonconvex in the whole stress space, it has some convex part only [3], [6].

According to the Drucker theorem the phenomenon is stable if the Hessian matrix of the complementary energy function is positive definite. (The positive definity of the Hessian matrix is the necessary condition of convexity at a given point.) [4] Using the Green models the Hessian matrix depends on the stresses and its definity cannot be predicted. It is numerically impossible to handle this question in every iterative step.

In the following example we used the Green model in the next form: $\Omega = aJ_2 + bI_1J_2$

where a and b are material coefficients determined by experiments, I_1 and J_2 are the stress deviators, namely:

$$I_1 = \sigma_x + \sigma_y + \sigma_z \quad \sigma_0 = (\sigma_x + \sigma_y + \sigma_z)/3$$

$$J_2 = \begin{vmatrix} \sigma_x - \sigma_0 & \tau_{xy} \\ \tau_{yx} & \sigma_y - \sigma_0 \end{vmatrix} + \begin{vmatrix} \sigma_y - \sigma_0 & \tau_{yz} \\ \tau_{zy} & \sigma_z - \sigma_0 \end{vmatrix} + \begin{vmatrix} \sigma_x - \sigma_0 & \tau_{xz} \\ \tau_{zx} & \sigma_z - \sigma_0 \end{vmatrix}.$$

Analysing the function Ω (determining the coordinates of the extreme points, the sign of gradients etc.) we give limits for the stresses and by the help of them a convex domain is defined and the energy function is convex on it.

This means that the problem can be solved by nonlinear programming where beside the linear equations we have nonlinear inequalities and a nonlinear objective function.

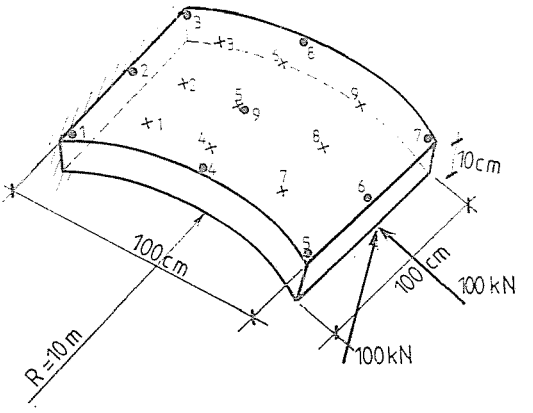

Fig. 3

If some inequalities are active at the optimal point (that is the optimal point lies on the bound), it cannot be known whether we have a limit state or the problem has no solution on the stable domain of the constitutive law. If the optimal point lies in the interior, the result satisfies the compatibility equations too.

Example:

A thick shell was computed by an isoparametrical element with 9 nodes. The stresses were definied at 9 Gaussian points. The normal stresses are neglected. Isotropic material is applied.

The constants are: a = 10.0

b = 0.3

The dimensions of the shell can be seen in Fig. 3.

The programpackage MINOS was used for solving the problem.

The number of unknown stresses : 45

equalities : 30

inequalities : 18

From the equalities 28 was eliminated so we had 17 free unknowns and 2 equalities in the MINOS.

The results:

Gaussian point	Stresses σ_x	σ_y	τ_{xy}	τ_{yz}	τ_{xz}
1	0.47208	0.25581	0.25638	0.53851	−0.46040
2	−0.70830	−0.16157	−0.64950	−0.045394	−0.0015999
3	−0.47022	−0.38811	−0.027903	0.025735	−0.088438
4	−0.19693	−0.52732	0.024904	0.12254	0.0036696
5	−0.77691	−1.0337	0.052662	−0.21104	0.21203
6	−0.065857	0.21370	−0.092120	−0.076517	0.14849
7	0.36573	0.31500	−0.34264	−0.26327	0.15212
8	0.81684	−0.54385	0.14057	0.12545	−0.20318
9	−0.16713	0.080269	−0.080043	−0.091544	0.12279

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