A COMPARATIVE STUDY OF THE NUMERICAL SOLUTION OF PLATES

A. LOVAS

Faculty of Civil Engineering, Department of Mechanics
Technical University, H-1521, Budapest

Received July 20, 1989

Presented by Prof. dr. Kaliszky

Abstract

In a comparative study by the finite element technique has been applied to the analysis of elasto-plastic thin plates under static and dynamic loading. The aim of the research was to determine the factors affecting the results and to follow the behavior of the elasto-plastic thin plates and the formation of plastic zones under gradually increasing transversal static load.

The error surfaces show the results which were obtained in order to characterize the factors influencing the numerical analysis: mesh size, aspect ratio of the element, effect of partial loading, effect of diagonal and consistent mass matrix, effect of the Rayleigh damping etc. Some figures show the relation between maximal deflection of the plate and the shape of the impact loading function. Sample solutions are given to demonstrate the applicability of the layered finite element model proposed to the description of the elasto-plastic behaviour of the plate.

1. Errors of the numerical solution

The solution of plate problems via the classical route is limited to relatively simple plate geometry, load and boundary conditions. If these conditions are more complex numerical and approximate methods are the only approaches that can be employed. In the engineering application it is important to be aware of the magnitude of the error of the solution and of its components. It must be mentioned that already differential equations comprise approximations and assumptions, thus, even the so called "exact solution" gives only an approximation of the actual behavior of the plate.

The first group is the *error of input data*. The external loads are known only with a certain degree of accuracy and in addition the material properties such as the Young moduli, the Poisson ratio can contain considerable inaccuracies. Furthermore the actual boundary conditions are merely approximations of the theoretical ones.

Employing approximate methods an additional inaccuracy is introduced which is called the *error of calculation*. Naturally this must be smaller than the error of data.

The *economy* of the solution is also an important factor to be considered in selecting the method to be employed in the analysis.

Problem solving by computers may introduce another type of error called *machine error*.

Finally a clear and systematic presentation of the computation not only permits an easier check by other persons but also mitigates the chances for *human error*.

2. Static and dynamic matrix equation of thin plates

In the application of the finite element method the compatible and complete rectangular finite element recommended by Bogner, Fox and Schmidt [2] was used (Fig. 1), the degree of freedom of which is 16, with w, w_x, w_y, w_{xy} unknown per node (BFS element).

It should be noted that by the omission of the degree of freedom of the nodal displacement w_{xy}, the compatible but incomplete rectangular element of 12 degrees of freedom is obtained. It was first used by Papenfuss (*P* element).

The basic equation of the displacement method can be expressed with the first order, third degree, one variable Hermitian interpolation polynomials.

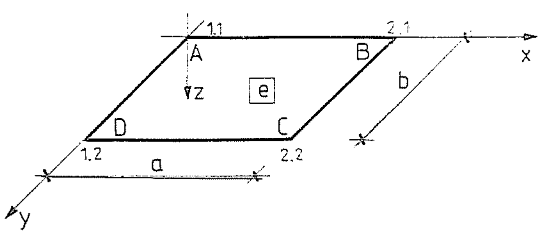

Fig. 1. The plate element marked with "e"

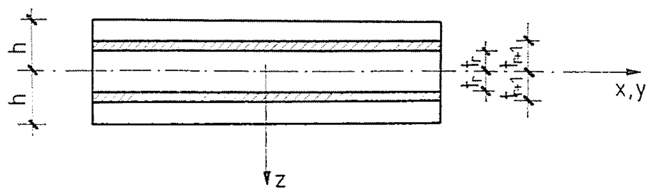

Fig. 2. The *r*-th elasto-plastic layer pair

Using layered plate finite elements for elasto-plastic analysis of plates (Fig. 2) we can write the incremental relationship between the moments m_{ij}^r of the r-th pair of layers and the curvature $\varkappa_{ij}$:

$$dm_{ij}^r = \frac{4G(t_{r+1}^3 - t_r^3)}{3}\left[d\varkappa_{ij} + \delta_{ij}\frac{\nu}{1-2\nu}\,d\varkappa_{kk} - m_{Dij}^r\frac{m_{Dki}^r\,d\varkappa_{kl}}{S}\right]$$

$$(i, j, k = 1, 2)$$

where $S = \frac{2}{3}\,\sigma_{\text{yield}}^2(t_{r+1}^2 - t_r^2)^2$, m_{Dij}^r denotes the deviatoric moment of the r-th pair of layers, G the modulus of elasticity in shear and ν the Poisson ratio.

The stiffness matrix of the elasto-plastic plate element can be obtained by the summation of the element stiffness matrices of the pairs of the layers.

When the loads are time dependent the equilibrium equations are:

$$\mathbf{M\ddot{u}} + \mathbf{C\dot{u}} + \mathbf{Ku} = \mathbf{F}$$

where $\mathbf{u}$ is the deflection of the plate, $\mathbf{M}$ is the consistent or diagonal mass matrix, $\mathbf{C}$ is the damping matrix and $\mathbf{K}$ is the stiffness matrix of the plate.

3. Effect of mesh size and aspect ratio

A plate subjected to uniform load as shown in Fig. 3 was investigated. The boundary condition was simple supported and clamped.

The net density (m and n) of the plate quarter varied from 1 to 8. The errors ($\Delta\%$) in the central deflection and the errors in the m_x bending moment arising in the middle of the plate are plotted in terms of m, n, $\Delta\%$ (see Figs 4, 5, 6, 7).

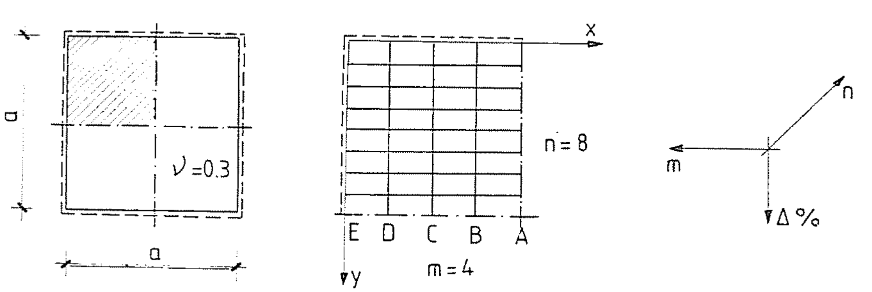

Fig. 3. Mesh of the square plate

4. Effect of partial loading

Let us investigate the effect on the solving when a load of given magnitude is uniformly distributed at the surface of the plate, or only partially, or along the edge or in the extreme case it is exerted only as a central concentrated force. The simple supported square plate shown in Fig. 8 was investigated at uniform $m = n = 8$ density.

The plate is divided into rectangles, determined by points $[-u, v]$, $[-u, -v]$ and $[u, -v]$, which are obtained by the projection of co-ordinates $[u, v]$,

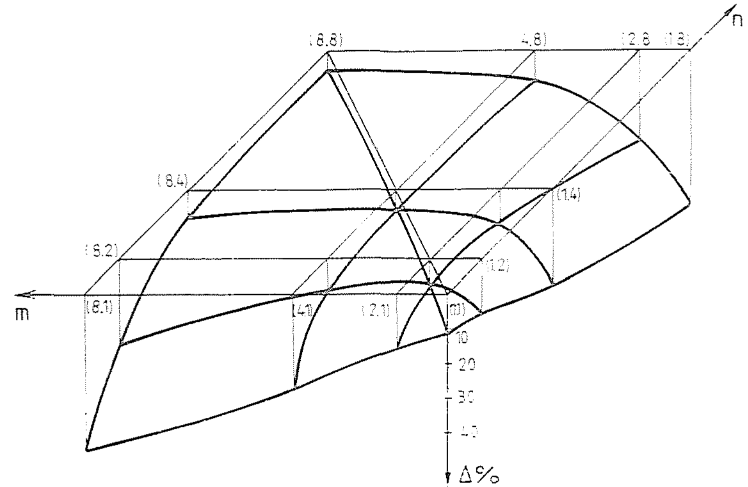

Fig. 4. Simple supported square plate, error of the central deflection

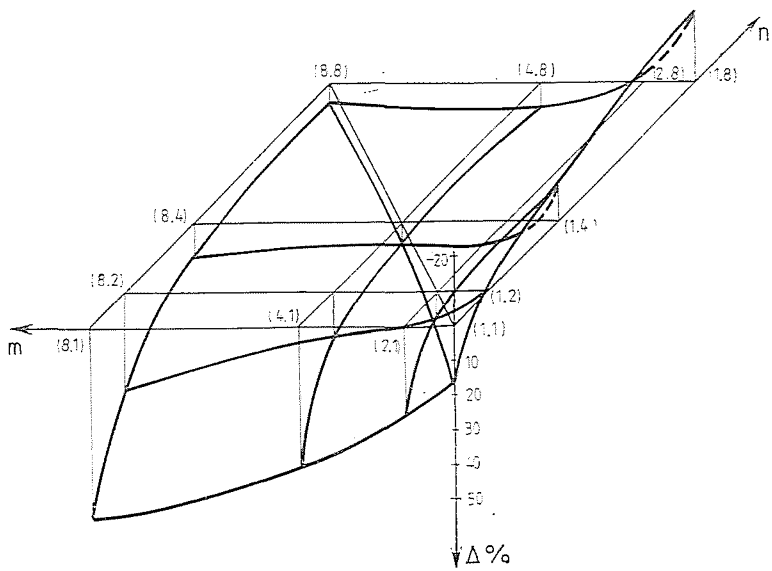

Fig. 5. Simple supported square plate, error of the central m_x bending moment

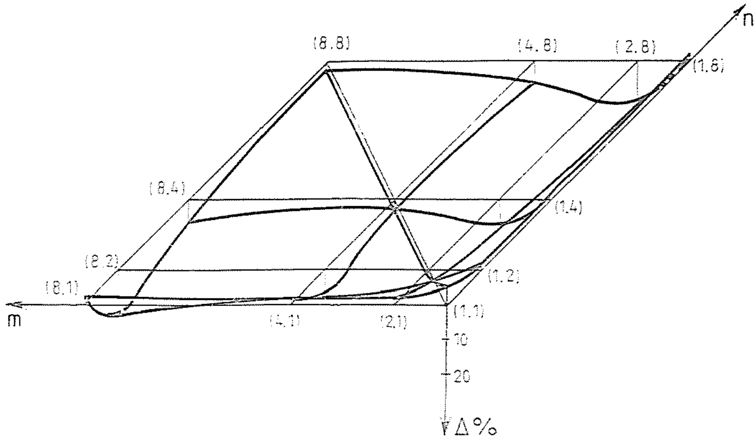

Fig. 6. Clamped square plate, error of the central deflection

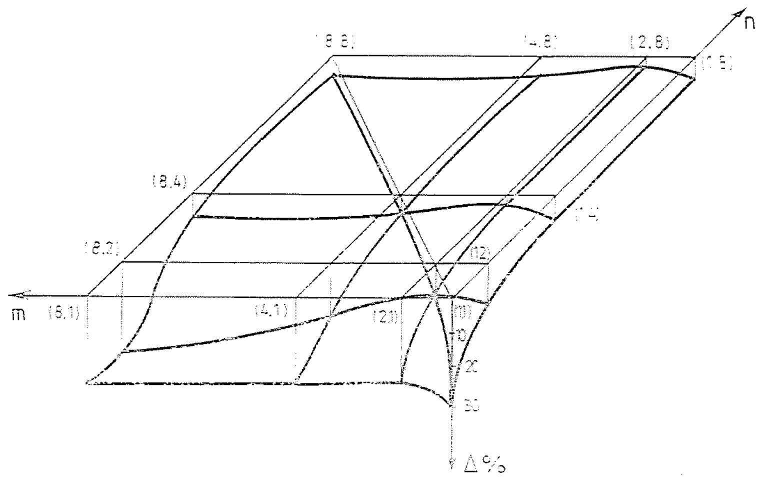

Fig. 7. Clamped square plate, error of the central m_x bending moment

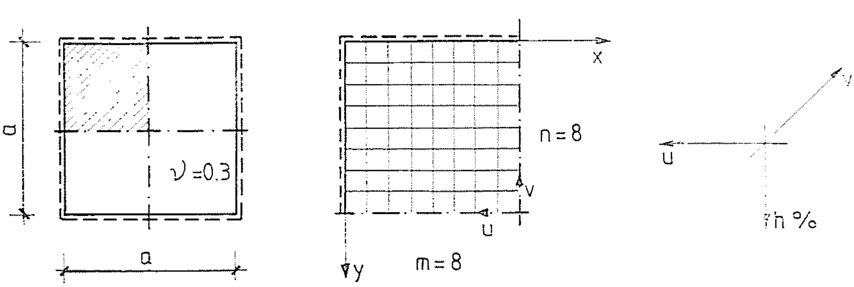

Fig. 8. Square plate and the co-ordinate systems

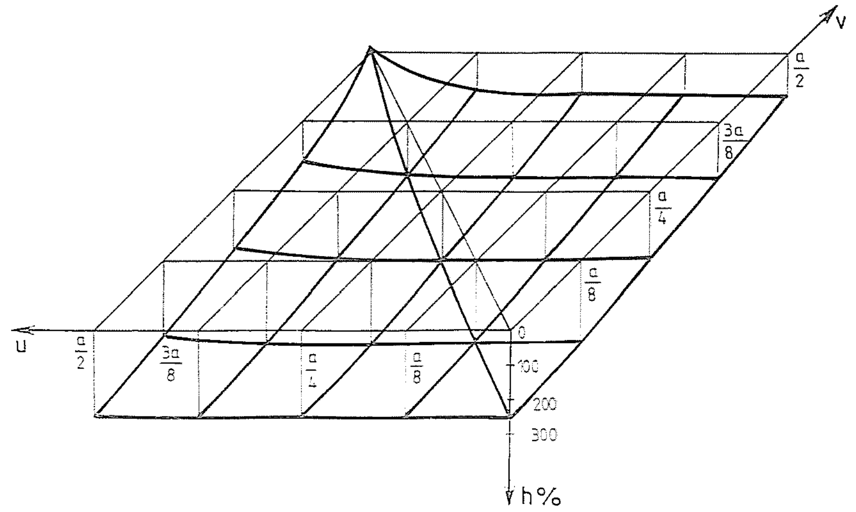

Fig. 9. Ratio of the central deflection

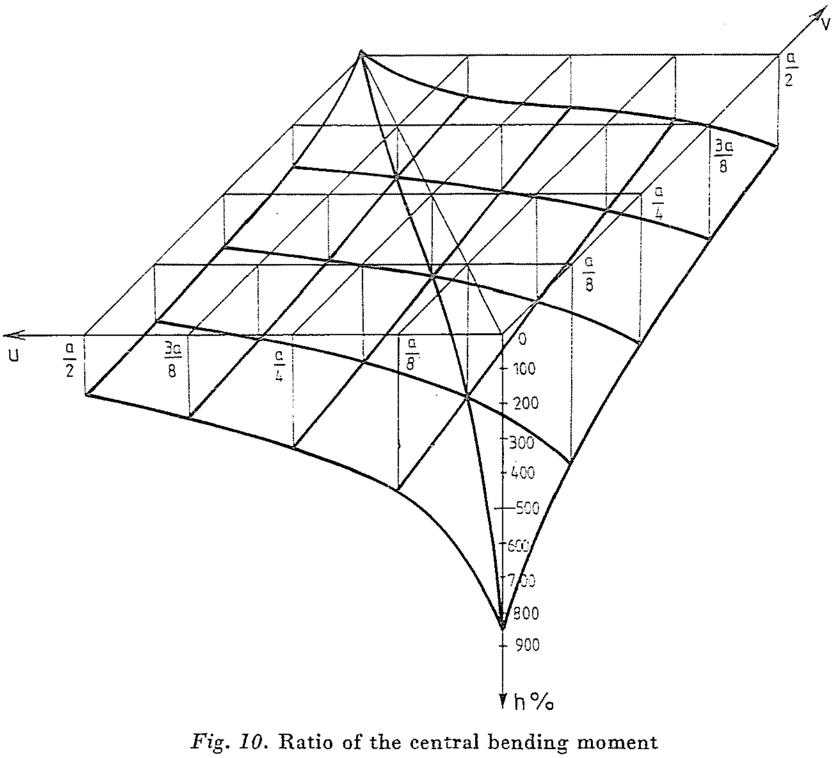

Fig. 10. Ratio of the central bending moment

given by the coordinate system whose origin is the centre of the plate. The total load of the plate is in each case identical.

The ratios of central deflection and bending moment values to values exactly calculated from uniformly distributed load over the whole surface of the plate ($h\%$) are plotted in the co-ordinate system u, v, $h\%$ (see in Figs 9, 10).

5. Effect of the development of the plastic zone

A solution is given to demonstrate the applicability of the proposed layered finite element model to the description of the elasto-plastic behaviour of the plate. The simple supported or clamped square plate shown in Fig. 3, with 4×4 mesh size and 3 pairs of layers was investigated. The dimensionless load-deflection relationship of the center point of the plate subjected to uniform load and the progression of the yielded regions are shown in Figs 11, 12.

6. Effect of diagonal and consistent mass matrix

The simple supported square plate (see Fig. 3) was investigated at $m = n$ uniform net divison. The $\Delta\%$ error of the smallest eigenfrequency is shown in Fig. 13 as a function of the NB^2 parameter where N is the number of the equations and B is the half bandwith of the stiffness matrix.

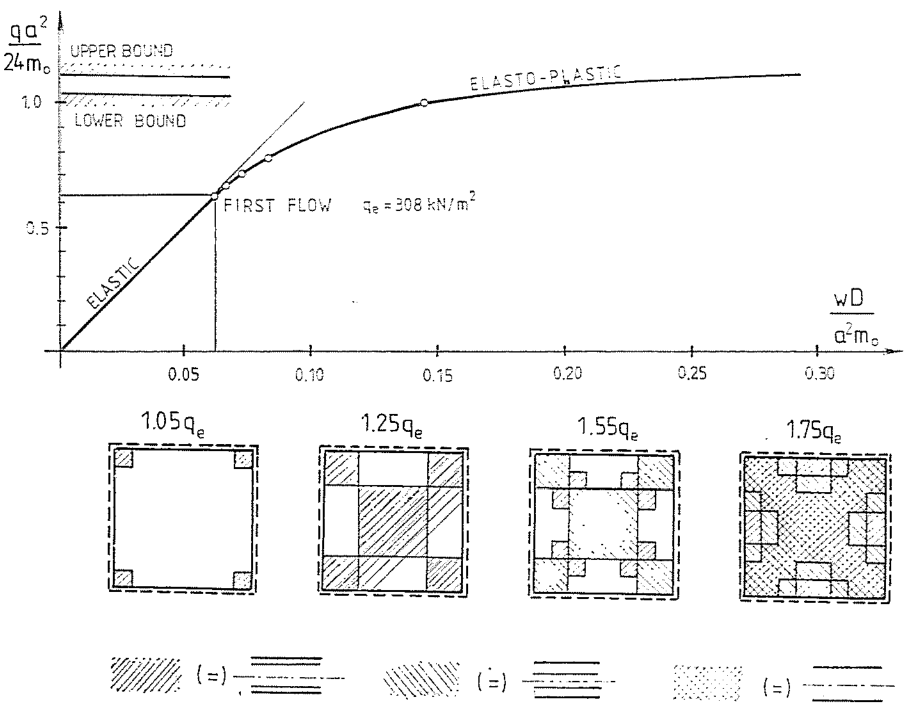

Fig. 11. Simple supported square plate

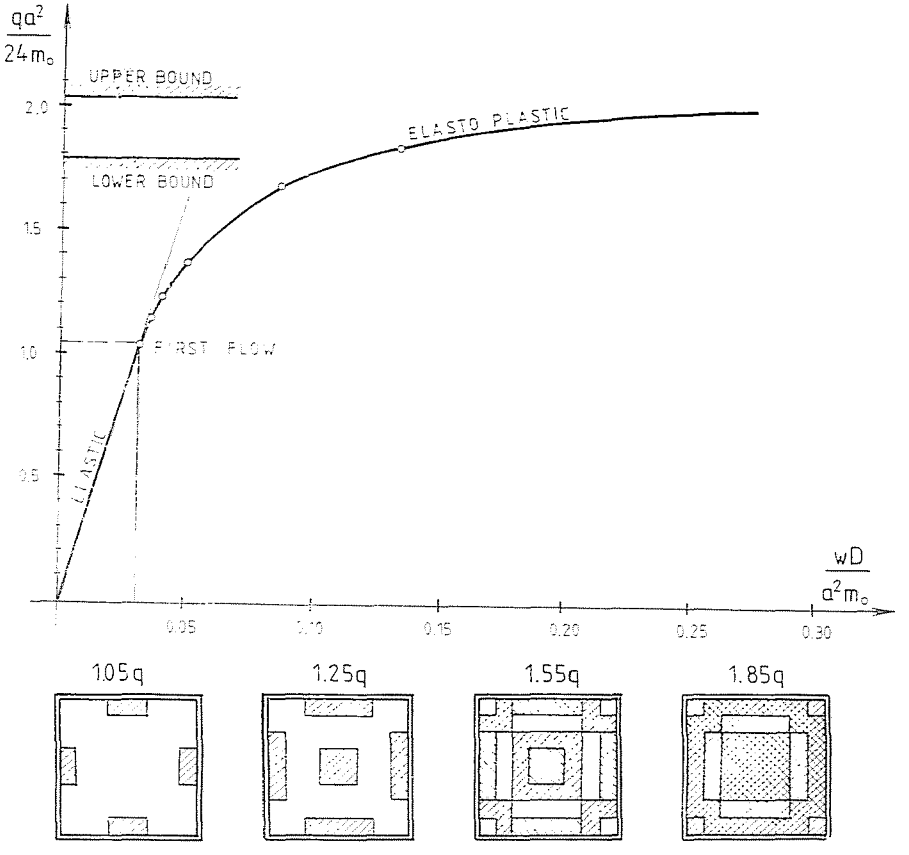

Fig. 12. Clamped square plate

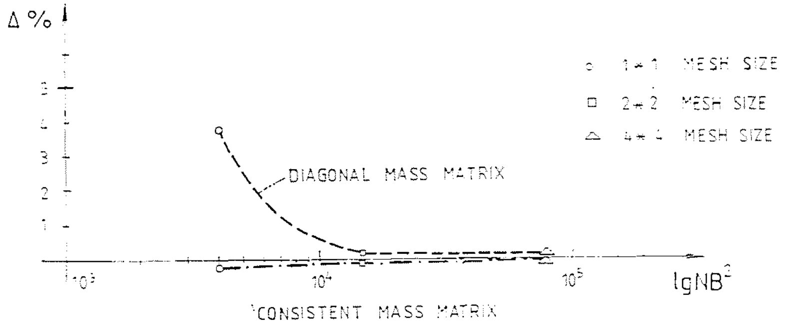

Fig. 13. Effect of diagonal and consistent mass matrix

λ_{0ij}	Exact (i^2+j^2)	Diagonal mass matrix	Consistent mass matrix
λ_{11}	2.000	2.000	2.000
$\lambda_{12} = \lambda_{21}$	5.000	4.995	5.011
λ_{22}	8.000	7.995	8.014
$\lambda_{13} = \lambda_{31}$	10.000	9.977	10.147
$\lambda_{32} = \lambda_{23}$	13.000	12.935	13.123
$\lambda_{14} = \lambda_{41}$	17.000	16.881	17.171
λ_{33}	18.000	17.841	18.184

The λ_{0ij} reduced eigenvalues are compared in the Table at a net division of $m = n = 4$

$$\lambda_{0ij} = \frac{a^2}{\pi^2} \frac{3G(1-\nu)}{2Gh^2}$$

where ϱ is the density, G is the modulus of elasticity in shear, ν the Poisson ratio, a the size of the plates and $2h$ is the thickness of the plates.

7. Effect of the Rayleigh damping

The determination of the damping matrix $\mathbf{C}$ is in practice difficult as the knowledge of the viscous matrix $\boldsymbol{\mu}$ is lacking. It is often assumed [1] therefore that the damping matrix is a linear combination of the stiffness and the mass matrices

$$\mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K}.$$

Here α and β have been determined from two given damping ratios that correspond to two unequal frequencies of the vibration.

$$\alpha = \bar{\xi}\,\overline{\omega}$$

$$\beta = \bar{\xi}/\overline{\omega}$$

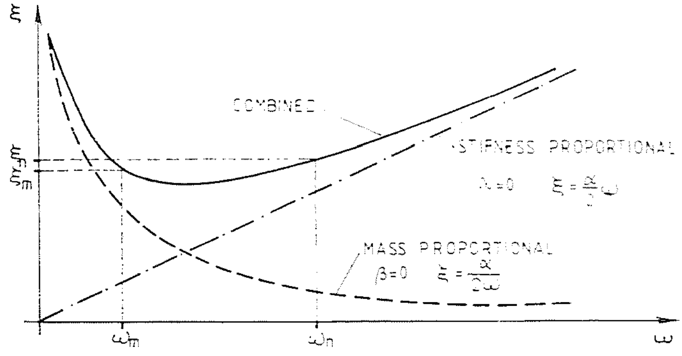

Fig. 14. Rayleigh damping

where $\bar{\xi}$ is the minimum damping factor, $\bar{\omega}$ the relevant circular frequency. Choosing $\bar{\omega} = 600/s$ for the given plate (see Fig. 3) α and β proportionality factors can be calculated. Using different damping factors the variation with time of the central dimensionless deflection is shown in Fig. 15. The case when the damping is zero and the static solution are also indicated in the figure. The load was a central concentrated force, acting dynamically.

8. Effect of the impact loading function

The simple supported square plate shown in Fig. 3 was investigated. Rayleigh's damping factors were chosen as $\alpha = 12.0$; $\beta = 3.3 \cdot 10^{-5}$ and six different shapes of the impact loading function were compared (see Fig. 16).

It can be seen (Fig. 17) that the shape of the impact loading function affects to a very high degree the response of the plate in the dimensionless central deflection.

9. Relationship between element type and economic aspects

Figures in chapters 3 and 4 were plotted using the P element with 12 degrees of freedom. When using the BFS element with 16 degrees of freedom errors calculated are about a fifth. Naturally when using the BFS element the

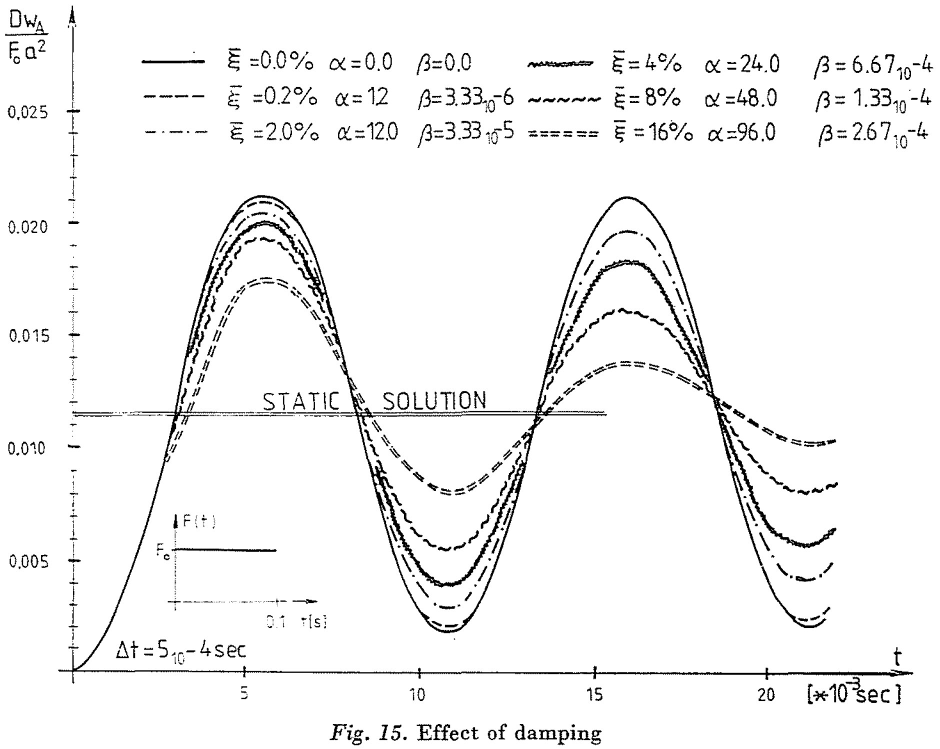

Fig. 15. Effect of damping

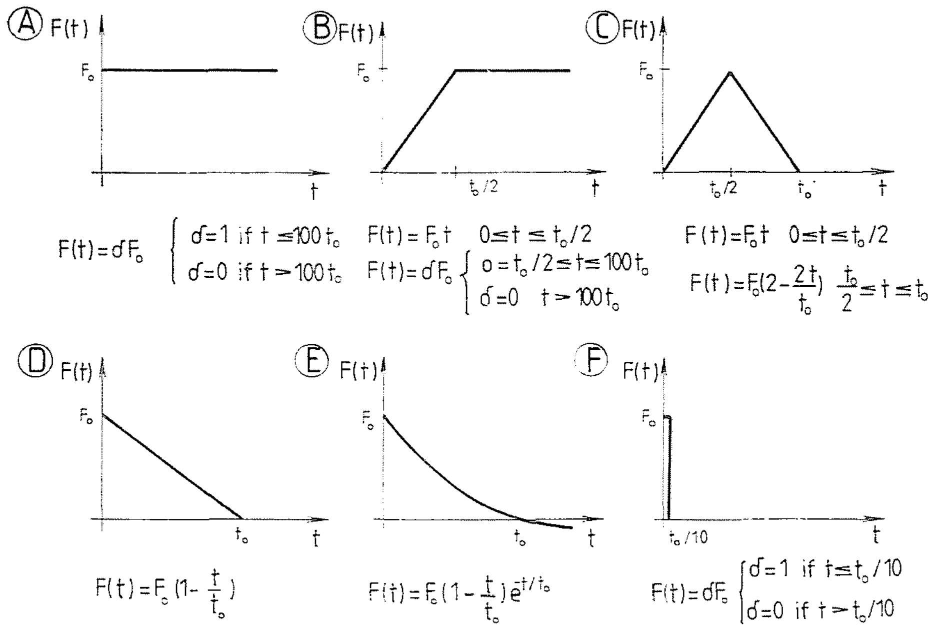

Fig. 16. Various load functions

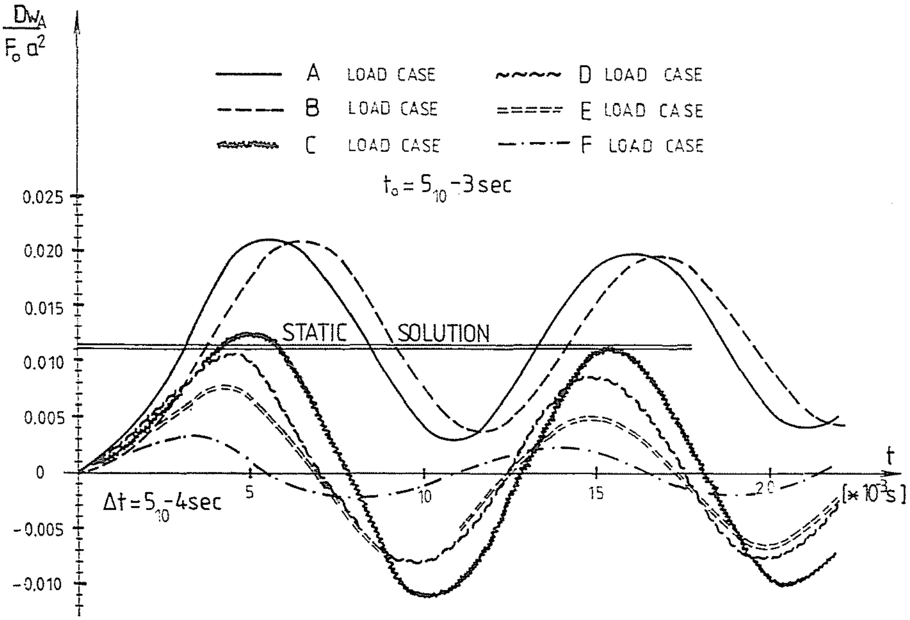

Fig. 17. **Effect of the load function**

number of unknowns is higher, and thus, calculation errors of the two elements must be composed as a function of solving time. The time of calculation depends mainly on the NB^2 parameter, where N is the number of equations, and B is the half bandwith. The central deflection errors of the plates are shown in Figs 18 and 19.

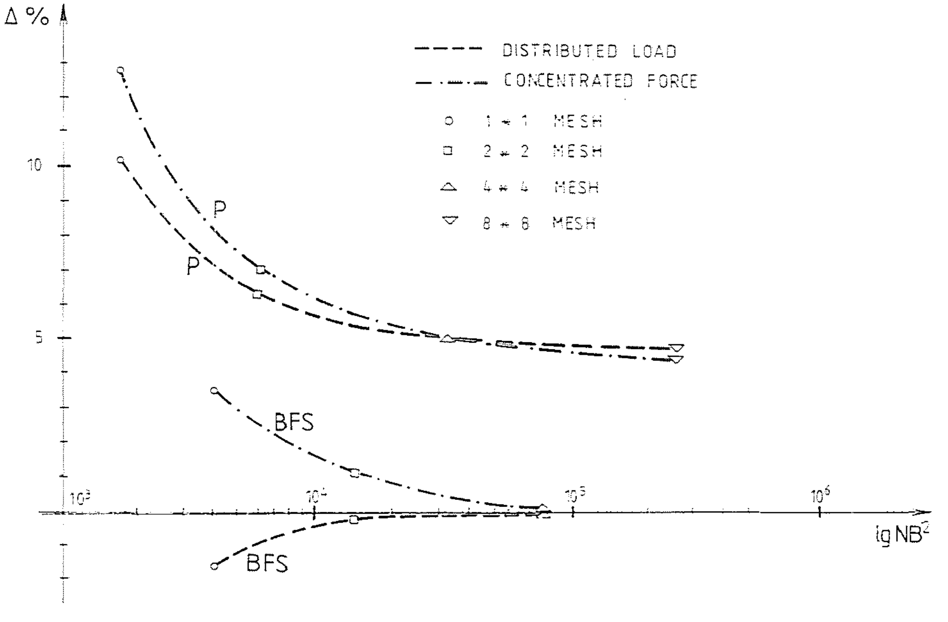

Fig. 18. Simple supported square plate

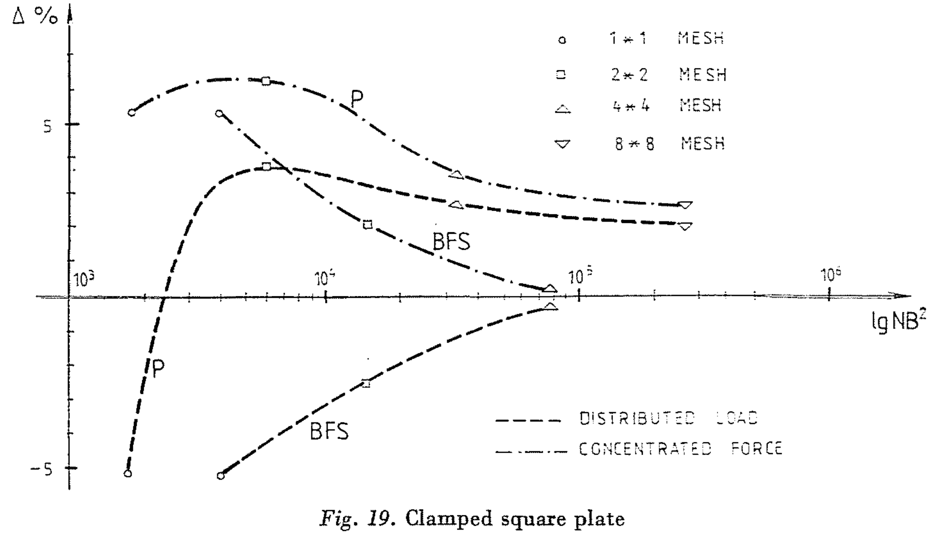

Fig. 19. Clamped square plate

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Dr. Antal LOVAS H-1521, Budapest