A FAMILY OF ITERATIVE METHODS IN THE THEORY OF MATHEMATICAL MODELLING

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Received March 18, 1987

Abstract

Two types of recent iterative modelling methods are discussed. The first of them is based upon the theory of generalized integro-differential operators and applies to systems satisfying some kind of linear (ordinary or partial) differential equation. It seems to be useful, however, rather from theoretical than from practical point of view. The other procedure has a completely different character: it can be considered as a strong extension of the well-known logarithmization method combined with least squares adjustment, and leads to the analytical modelling of a lot of physical, engineering and biomedical systems or phenomena which could be described only empirically up to the present. Some ideas of the results have been published by the author e.g. in: Lecture Notes in Mathematics, *457* (Springer, 1975) as well as in the volume "Mathematical Modelling in Science and Technology" (Pergamon Press, 1984).

1. Preliminaries on Generalized Differential and Integral Operators

Since the publication of Heaviside's "Electromagnetic Theory" (1893—1912), it has been common knowledge that the formal calculation by the operator $D = d/dt$ yields non-integral powers or transcendental functions of D for numerous linear partial differential equations of practice, this being a natural motivation to define "fractional" integrals and derivatives. Introducing the Laplace transformation and other rigorous operational methods of the recent past, we can lay the exact foundation of the "linear operators of arbitrary order" or (roughly speaking) that of the "fractional calculus" in a wider framework.

During recent decades, many important further equations have been treated in literature by fractional operators for which the utilization of this technique is not immediate at all. Among numerous examples from potential theory, electrodynamics, hydro- and aerodynamics, chemical kinetics, etc. (cf. [1], [3], [9], [10], [22], [23], [25]), let us stress Marcel Riesz's fundamental results on the m-dimensional wave equation (see [24]), further the comprehensive investigations of Erdélyi and Sneddon on axially symmetric potential problems and dual integral equations ([2], [4], [5], [6], [26]). In these works the "fractional calculus" appears not only as a "short-hand" method for a more concise and more lucid presentation of certain analytical processes or mathematical deducations, but it suggests also the validity of some essential interconnections, thus becoming a useful "catalyst" of development.

We add that often, according to the problem in consdieration, appropriate generalizations have been needed. M. Riesz gave e.g. in [24] the solution of the Cauchy problem for the Lorentz—Minkowski space of m-dimensions (especially for the relativistic space-time) by means of a remarkable version of the so-called *Riemann—Liouville fractional integral*

$$ {}_{x_0}I_x^\nu f = \Gamma(\nu)^{-1} \int_{x_0}^{x} f(t)\,(x-t)^{\nu-1}\,dt \quad (Re\,\nu > 0), \tag{1} $$

which exists always when the function f is bounded and Riemann integrable, as well as with fixed ν for "almost all x" if f is Lebesgue integrable (cf. e.g. Mikolás [14]), furthermore satisfies the following "index law" or "semigroup property":

$$ {}_{x_0}I_x^{\nu_1}({}_{x_0}I_u^{\nu_2}) = {}_{x_0}I_x^{\nu_2}({}_{x_0}I_u^{\nu_1}) = {}_{x_0}I_x^{\nu_1+\nu_2} \quad (\nu_1 > 0,\ \nu_2 > 0;\ x_0 < u \leq x). \tag{2} $$

2. A New Iterative — Interpolatoric Principle for Solving Differential Equations

For our present purposes, the idea of "*derivative of order* μ" is of the greatest importance where μ means an arbitrary (real or complex) number. Instead of its usual definition (due to Riemann) which yields the fractional derivative via ordinary differentiations of (1) or (as it was proposed by M. Riesz) by analytic continuation of the Riemann—Liouville integral as a holomorphic function of the order ν, we consider the limit (cf. Mikolás [11]—[12] and [21], [27], [28]):

$$ {}_{x_0}D_x^\mu f = \lim_{n\to\infty} \left(\frac{x-x_0}{n}\right)^{-\mu} \sum_{k=0}^{n-1} (-1)^k \binom{\mu}{k} f\left(x - k\,\frac{x-x_0}{n}\right). \tag{3} $$

This is clearly a direct common extension of the notions of ordinary derivatives and Riemann integral by

$$ {}_{x_0}D_x^p f = f^{(p)}(x), \quad {}_{x_0}D_x^{-\nu} f = \Gamma(\nu)^{-1} \int_{x_0}^{x} f(t)(x-t)^{\nu-1}\,dt $$

for $\mu = p$ $(p = 0, 1, 2, \ldots)$, f p-times differentiable, and for $\mu = -\nu < 0$, f Riemann integrable, respectively; hence it is also motivated the recent designation "*integro-derivative*" (introduced for (3) by the author).

We can word now a general "*iterative-interpolatoric principle*" for solving differential equations of the form

$$ \Theta\,u = f \tag{4} $$

where Θ is a homogeneous linear (ordinary or partial) differential operator and $f \neq 0$ is a known function. (See also Mikolás [13].)

I. step: on the basis of the representation (3), we write the first term of (4) as a limit expression involving binomial coefficients.

II. step: iterate the operator Θ p-times; the resulting $\Theta^p u$ will again have a limit form of the just mentioned type.

III. step: try now to extend the definition of the latter to arbitrary real ν instead of p so that it should be a continuous function of ν.

IV. step: if it comes off, we merely have to put $\nu = -1$ and obtain as a particular solution of (4)

$$u = \Theta^{-1} f \tag{5}$$

provided that the operator relation $\Theta^{-1}\Theta = \Theta^0$ (where Θ^0 is the identity operator) can be verified.

For example, in case of the equation $u' + u = f(x)$ $(u = u(x);\ x_0 < x \leq x_1)$ the calculation will be:

$$\Theta^p u = \left(1 + \frac{d}{dx}\right)^p u =$$

$$= \lim_{n\to\infty} \left(\frac{x - x_0}{n}\right)^{-p} \sum_{k=0}^{n-1} (-1)^k \binom{p}{k} \left(1 + \frac{x - x_0}{n}\right)^{p-k} u\left(x - k\frac{x - x_0}{n}\right);$$

and hence we get:

$$u = \Theta^{-1} f = \lim_{n\to\infty} \frac{x - x_0}{n} \sum_{k=0}^{n-1} \left(1 + \frac{x - x_0}{n}\right)^{-(k+1)} f\left(x - k\frac{x - x_0}{n}\right) = \int_{x_0}^{x} f(t)\, e^{t-x}\, dt,$$

as it may be pointed out by another way, too.

It is obvious that this method enables us to model analytically several problems in physics and engineering, but the effective representation of the inverse operator Θ^{-1} is often too complicated. Nevertheless we see that, whenever applicable, the procedure permits simultaneously to give also numerical approximations and computations for the particular solution looked for.

3. The ILP — Method for Modelling Systems of Empirical Data

In the following we shall study in detail *another modelling process* which is much nearer to practice. As far as the preliminaries are concerned, let us mention first, that since 1977 a lot of observations have been investigated by logarithmization and least squares adjustment at our Mathematics Department as well as at the Department for Production Engineering (headed by Prof. I. Kalászi); partly on some *characteristic properties of several plastics*, partly on the main *factors governing the grinding operation* which is *funda-*

mental in mechanical technology. (Cf. KALÁSZI [7]—[8]; MIKOLÁS—BARDÓCZ [17]—[18].)

At about the same time, an independent work was begun in collaboration of the author's team with the neonatological research group in the Heim Pál Children's Clinic of Budapest (the head of which has been Prof. I. Sárkány). In that institute, a big experimental and statistical material was collected during a period of ten years, containing e.g. the data of more than 5000 infant deaths occurring in the first 24 hours after birth. So it was natural to ask: how the logarithmization method could be developed further in order to describe the dynamics of the phenomena in question — if a kind of iteration was utilized?

Based on this line of investigations, elaborating also over ten thousand recent perinatal events (in Hungary and abroad), a new „*iterative logarithmic procedure*" (ILP) could be given, applicable to arbitrary physical, engineering and biomedical systems, characterized by positive and strictly monotone increasing or decreasing — especially convex or concave — sequences of data. (Cf. MIKOLÁS [15], MIKOLÁS—SÁRKÁNY [19].)

The main points of the method are:

1^0 If $0 < x_1 < x_2 \ldots < x_n$ are the values of a variable x, $0 < y_1 < \ldots < y_n$ the corresponding values of another variable y, we "delineate" first the connection between

$$Y_i^{(1)} = \lg \frac{x_i}{x_1}, \quad Y_i^{(1)} = \lg \frac{y_i}{y_1} \quad (i = 1, 2, \ldots, n), \tag{6}$$

where "*lg*" means logarithm to the base 10.

2^0 We consider the connection of

$$X_i^{(2)} = \lg (1 + 10\, X_i^{(1)}) \text{ and } Y_i^{(2)} = \lg(1 + 10\, Y_i^{(1)}) \quad (i = 1, 2, \ldots, n). \tag{7}$$

3^0 Logarithmizing again, we connect

$$X_i^{(3)} = \lg (1 + 10\, X_i^{(2)}) \text{ with } Y_i^{(3)} = \lg (1 + 10\, Y_i^{(2)}) \quad (i = 1, 2, \ldots, n), \tag{8}$$

and so on.

The final step will yield a straight line or a graph composed of two straight lines, containing approximately all the points $(X_i^{(N)}, Y_i^{(N)})$ with some fixed N and $i = 1, 2, \ldots, n$. Then adjusting by least squares for the point system just mentioned, we can get a more precise analytic expression describing the stochastic dependence of y and x. If this has come to pass at the N-th step, we say that the connection $x \to y$ obeys an *N-fold logarithmic law*.

We have to lay particular stress on the facts that 1) the procedure converges rather rapidly, namely it finishes in most cases for $N \leq 3$ (we speak about a "*simple*", "*double*", "*triple*" *logarithmic law* when $N = 1$, $N = 2$

and $N = 3$, respectively); 2) the detailed discussion is based upon the inequalities

$$\lg(1 + 10x) > x \quad (0 < x < \tau); \ \lg(1 + 10x) < x \quad (x > \tau),$$

where $\tau = 1.067\ldots$ denotes the only real root of the equation

$$\lg(1 + 10x) = x, \tag{9}$$

furthermore upon the limit relation $\lim\limits_{x \to \tau} [\lg(1 + 10x) - x] = 0.$

4. Applications of Late Years, Repeated Logarithmic Laws

A few typical *illustrations* of the iterative logarithmic method:

In Figure 1, we present the dynamics of the mortality during the first year of life, stated for the about 10.7 million population of Hungary in 1981. As known, the so-called *infant mortality* is one of the most important parameters for the health conditions of a population; thus it has been registered by the WHO since many years for almost all countries of the world. Unfortunately, till now it could be investigated empirically only, because no fit mathematical model has been known. We tried to obtain such a model in the following manner: firstly we considered, how many percents of infants died after birth in a time interval $[0, T]$ less one year (e.g. during 1 month, 2 months, ...) and then we investigated this percent number Q as a function of T, represented by a curve concave from below.

Turning now to the logarithms $\lg(Q/Q_1)$ and $\lg(T/1)$, i.e. applying the first step of the above procedure, we find that there exists a *linear* connection between these variables. In other words: we have a *simple logarithmic law for*

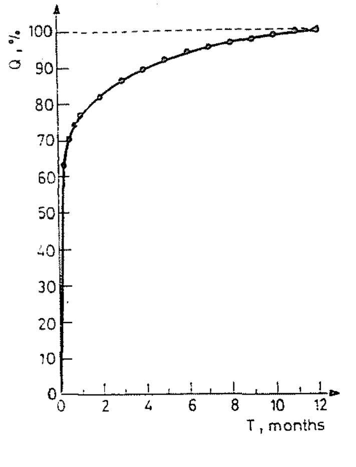

Fig. 1. The dynamics of mortality during the first year of life. Hungary, 1981

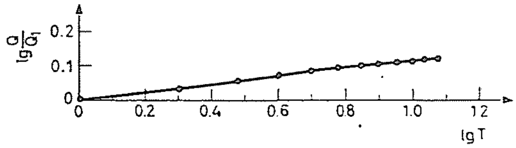

Fig. 2. The simple logarithmic law of infant mortality. Hungary. 1981

the infant mortality in Hungary. (Cf. Figure 2.) This statement seems to hold for other periods and countries, too.

In Figures 3—4 similar interpretations are given for the *mortality during the first day of life* — again in case of Hungary for the year 1981. It is remarkable that now also the second step of the method is needed, i.e. we obtain a *double logarithmic law*. Going further: the same connection holds also for the dynamics of the mortalities within one month (which is called *"neonatal mortality")* or one week *("postnatal mortality")*, besides for the intrauterine mortality, too; so it is about a *general physiological law*.

Figures 5 and 6 show that the above procedure can be applied also to a much wider manifold of demographic and biomedical phenomena: if we investigate the *frequency of new born with birth order** $\leq$ *k among all born in the same year*, then we get a *"triple logarithmic law"* in the sense indicated before.

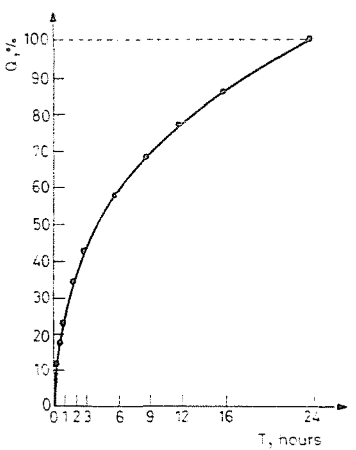

Fig. 3. The dynamics of mortality during the first day of life. Hungary, 1981

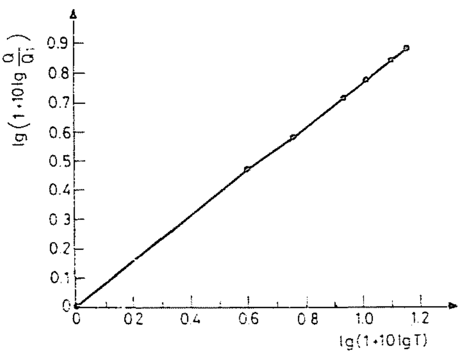

Fig. 4. The double logarithmic law of mortality within 24 hours. Hungary, 1981

* By the expression *"birth order"* we mean, as usual in demography, the ordinal numeral of the new born among all children of the same mother. (For example, the birth order of a second child is 2.)

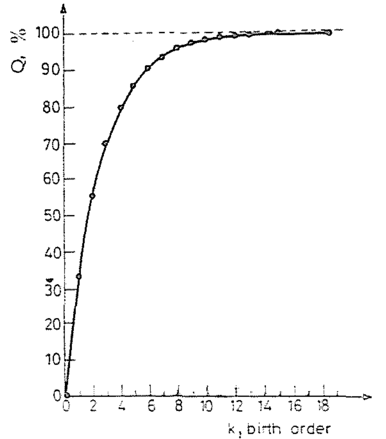

Fig. 5. Frequency of new born with birth order $\leq k$ among all born in the same year. Hungary 1938

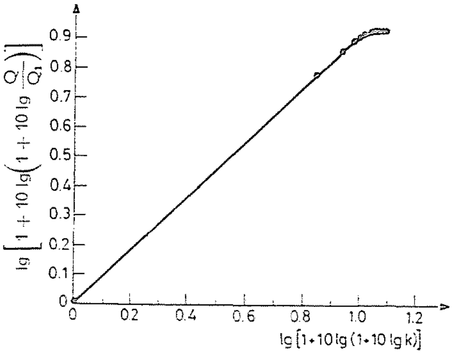

Fig. 6. The triple logarithmic law of the birth order before World War II. Hungary, 1938

Note that the biological aspects have been presented by invitation at the 29th World Congress of the International Union of Physiological Sciences (cf. [20]) and a brief outline on the general results appeared in a Proceedings volume at Pergamon Press in 1984. (Cf. Mikolás [16].)

5. Remark on Developing Possibilities

Finally, we remark that the role of "lg" in the above method ILP may be taken over by any simple analytic function $\lambda(x)$ for which $\lambda(1) = 0$, $\lambda(x)\uparrow\infty$ and $\lambda'(x)\downarrow 0$ $(1 < x < \infty)$. The use of logarithms, however, seems to be most convenient for the majority of applications.

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