CERTAIN COVERINGS OF FINITE HOMOCYCLIC ABELIAN GROUPS

S. Szabó

Department of Civil Engineering Mathematics,
Technical University, H-1521 Budapest

Received June 20, 1987
Presented by prof. Dr. J. Reimann

Abstract

Let G be a finite Abelian group written additively which is the n-fold direct sum of cyclic groups of order k. Denote A the union of these direct summands of G. Let $r(k, n)$ be the minimum of the cardinality of subsets $B \subseteq G$ for which $G = A + B$ holds. The inequality $r(tk, n) \leq t^{n-1}r(k, n)$ is proved in this note.

Introduction

Let G be a finite Abelian group written additively which is the direct sum of n copies of cyclic groups of order k, and let $g_1, \ldots, g_n$ be the basis elements for these cyclic groups respectively. Subset A will denote the union of these subgroups, that is,

$$A = \{0, g_1, 2g_1, \ldots, (k-1)g_1\} \cup \ldots \cup \{0, g_n, 2g_n, \ldots, (k-1)g_n\}$$

Obviously, A has $(k-1)n + 1$ distinct elements.

We will speak of packing covering and tiling of group G by translates of A as follows.

Let B be a subset of G. If each element g of G can be expressible in the form $g = a + b$, where $a \in A$ and $b \in B$, then we say that translates of A cover G with covering set B.

If each element g of G can be expressible in only one way in the form $g = a + b$, where $a \in A$ and $b \in B$, then we will say that translates of A pack G with packing set B.

When B is a packing and covering set as well we call it the tiling set of G for A.

Packing covering and tiling of G by translates of A are in an intimate connection with such problems as error correcting codes, latin squares and the so-called gambling problem. For the background and the history of the problem see references [1] and [5].

The problem of finding $r(k, n)$, the smallest cardinality of subsets B for which translates of A cover G, is posed by O. Taussky and J. Todd in [6]. It also occurs as Problem 10.2 in [1].

S. K. Zaremba [8] proved the next result on tiling of G which provides an answer for the previous question in some special cases.

If k is a prime power e is an integer greater than 1 and $n = (k^e - 1)/(k - 1)$, then the group G which is the direct sum of n cyclic groups of order k has a tiling by translates of A. The following Table, which was first given by R. G. Stanton [4], lists $r(k, n)$ for some small values.

n \ k	2	3	4	5	6	7	8
2	2	3	4	5	6	7	8
3	2	5	8	13	18	25	32
4	4	9	24	—	72	—	—
5	7	27	4^3	—	—	—	—
6	12	—	—	5^4	—	—	—
7	16	—	—	—	—	—	—
8	32	—	—	—	—	7^6	—

The next new upper bounds of $r(3, n)$ are listed in [9].

n	6	7	8	9
$r(3, n) \leq$	74	216	567	1458

Result

The problem of packing covering and tiling of G by translates of A can be expressed in a more intuitive way by means of chess boards and so-called rook domains. We intend to use them so we describe their constructions. Let vectors $e_1, \ldots, e_n$ be the standard orthonormal basis for the n-dimensional Euclidean space.

An n-dimensional chess board with side length k consists of k^n translates of a unit cube by vectors $c_1e_1 + \ldots + c_ne_n$, where $c_1, \ldots, c_n$ are integers between 0 and $k - 1$. A rook on this chess board attacks $(k - 1)n + 1$ positions. Namely, the position of the rook and those positions whose coordinates differing from it in exactly one coordinate. Figure 1 shows rook domains in the cases $(k, n) = (4, 2)$ and $(k, n) = (2, 3)$.

We will use a special type of starbodies, the semicross. A (k, n) semicross is a union of translates of unit n-dimensional cube by vectors je_i where

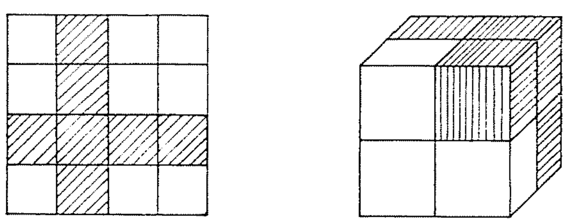

Fig. 1

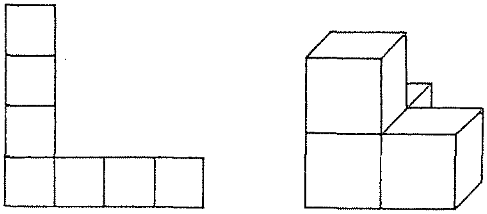

Fig. 2

$1 \leq i \leq n$ and $0 \leq j \leq k-1$. Thus it consists of a corner cube together with n arms of length $k-1$ attacked at non-opposite facets.

Figure 2 shows the (4,2) and (2,3) semicrosses. Note that the semicross (k, n) is a special rook domain for an n-dimensional chess board with side length k.

We need these concepts to prove the next result.

Theorem. For positive integers k, n, t the inequality $r(tk, n) \leq t^{n-1}\, r(k, n)$ holds.

Proof. Let G and G' be the n-fold direct sum of cyclic group of order k and tk respectively, and let A and A' be the union of the direct summands respectively.

We will prove that if translates of A cover G with covering set B, then G' can be covered by translates of B' with covering set B' so that $|B'| \leq t^{n-1}|B|$.

Group G may be represented as a set of n-tuples of integers between 0 and $k-1$ with operation of component wise addition modulo k. The coordinates of positions of an n-dimensional chess board with side length k may be viewed as elements of G. In short we identify G with a chess board.

The elements of subset B give the positions of the rooks and subset $A+b$, $b \in B$ gives a rook domain.

Translate the chess board by vectors of lattice L spanned by vectors $ke_1, \dots, ke_n$. These translates tile the n-space. Take a rook domain and all of its translates by elements of L. These translates form a structure which reminds us of a timber framework.

Now colour the nodes of our timber framework. Place the corner cube of a (tk, n) semicross at a node cube. Then translate it by elements of lattice L' spanned by vectors $tke_1, \dots, tke_n$. If these semicrosses cover the nodes of the timber framework our work is finished otherwise we must choose an uncovered node and repeat the previous process. Since elements of L' translate the semicross system and the timber framework into themself it is enough to consider the uncovered nodes within a bounded domain. It means that the previous process terminates at finitely many steps. Let the nodes of the timber framework covered by corner cubes of semicrosses be coloured black and the others white.

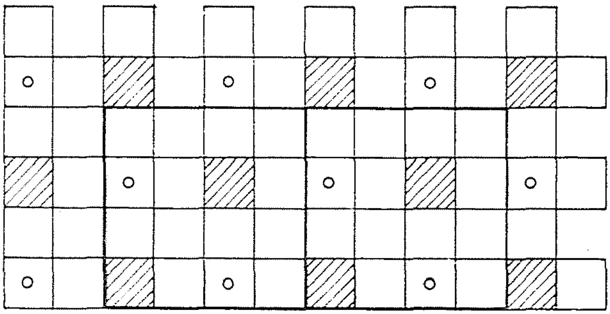

Fig. 3

The semicrosses cover the timber framework. The white node cubes are covered n times but the other cubes of the timber framework are covered ones. Figure 3 illustrates the proof in the case of $k = 2$, $n = 3$, $t = 2$.

Since the rook domains cover the chess board the n-space is covered by translates of this timber framework. Summing up our information we have a system of (tk, n) semicrosses cover the n-space and elements of L' are periods of this system. Consider a new n-dimensional chess board with side length tk and all of its translates by vectors of lattice L'.

Since $tke_1, \ldots, tke_n$ are periods of the semicross and the big chess board systems as well the new big chess boards divide the semicross system into rook domains. The semicross system covers the n-space so the rook domains cover each big chess board.

If $|B|$ rook domains cover the original chess board then the timber framework has $t^n|B|$ nodes in a new chess board. So there are $t^{n-1}|B|$ black nodes in it. The number of the black nodes is not other than the number of the rook domains which cover the new chess board. This completes the proof.

References

1. DÉNES, J.—KEEDWELL, A. D.: Latin squares and their application. Akadémiai Kiadó, Budapest, 1974.
2. LOSEY, G.: Note on a theorem of Zaremba, J. Comb. Theory, *6*, 208—209 (1969).
3. MAULDON, J. G.: Covering theorem for groups, Quart. J. Math. Oxford. *2*, 284—287 (1950).
4. STANTON, R. G.: Covering theorem in groups, Recent Progress, in Combinatorics (Proc. Third Waterloo Conf. on Combinatorics, 1968), 21—26. Academic Press, New York, 1969.
5. STEIN, S. K.: Algebraic tiling, Amer. Math. Monthly *81*, 445—462 (1974).
6. TAUSSKY, O.—TODD, J.: Covering theorems for groups, Ann. Soc. Polonaise de Math. *21*, 303—305 (1948).
7. ZAREMBA, S. K.: A covering theorem for abelian groups, J. Math. Soc., *26*, 71—72 (1951).
8. ZAREMBA, S. K.: Covering problems concerning abelian groups, J. London Math. Soc., *27*, 242—246 (1952).
9. WILLE, L. T.: The football pool problem for 6 matches: A new upper bound obtained by simulated annealing, Journal of Comb. Theory, Serie A *45*, 171—177 (1987).

Dr. Sándor SZABÓ H-1521 Budapest