EXPECTED VALUE AND STANDARD DEVIATION OF PRODUCT-SUMS OF PERMUTATION

L. BALCZA

Department of Civil Engineering Mathematics,
Technical University, H-1521 Budapest

Received June 20, 1987
Presented by Prof. Dr. J. Reimann

Abstract

In this paper we introduce the expected value and the standard deviation of product-sums of permutations. We know that the product-sums of permutations are:

$$S(\pi) = \sum_{i=1}^{n} i\pi(i)$$

where $S(\pi)$ denotes the value of product-sums in π. It is shown that every element occurs at least once from R_n to Q_n where Q_n stands for the sum of the squares of the first natural numbers and $R_n = \frac{n(n+1)(n+2)}{6}$ for the product-sums corresponding to the permutation $n(n-1)\dots 21$, and hence product-sum Q_n corresponding to $12\dots(n-1)n$ is the maximal, and R_n corresponding to n . . . 21 is the minimal one. A mode for the production of the product-sums, is indicated.

It is further that $M(S(\pi)) = \frac{n(n+1)^2}{4}$ and $D^2(S(\pi)) = \frac{n^2(n+1)^2(n-1)}{144}$ where $M(S(\pi))$ is the expected value and $D(S(\pi))$ is the standard deviation of product-sums. We also introduced the complement permutations so that

$$\pi(i) + \pi'(i) = n + 1$$

where π' is a complement of π since $\pi = (\pi(1), \dots, \pi(n))$ and $\pi' = (\pi'(1), \dots, \pi'(n))$ denotes the images of $(1, \dots, n)$.

Of course we will do work in the probability field where each permutation has the same probability.

Introduction

The problem mentioned above is important both for theorical and applied mathematics.

The aim of this paper is to determine also product-sums of permutations and their expected value and standard deviations. The suggested theorem is introduced by some notations and definitions and then they will be proved.

P_N denotes the set of permutations of the finite set N, where $N = \{1, 2, \dots, n\}$. For any $\pi \in P_N$, $S(\pi)$ denotes the value of product-sums in π, where

$$S(\pi) = \sum_{i=1}^{n} i\pi(i) \tag{1}$$

and $\pi = (\pi(1), \dots, \pi(n))$.

Further let the expected value and standard deviation of product-sums of permutations of N be denoted by $M(S(\pi))$ and $D(S(\pi))$. First of all the complement permutation is introduced because then we can calculate with $\frac{n!}{2}$ only.

Definition: Let π' be the complement of π where

$$\pi(i) + \pi'(i) = n + 1 \qquad (1 \leq i \leq n). \tag{2}$$

We can then prove the following

Lemma:

$$S(\pi) + S(\pi') = (n + 1) \binom{n+1}{2} \tag{3}$$

Proof: Let $\pi = (\pi(1), \ldots, \pi(n)$ and $\pi' = (\pi'(1), \ldots, \pi'(n))$ denote the images of $\{1, 2, \ldots, n\}$ so that

$$\pi(i) + \pi'(i) = n + 1 \qquad (1 \leq i \leq n).$$

Then:

$$S(\pi) + S(\pi') = \sum_{i=1}^{n} i\pi(i) + \sum_{i=1}^{n} i\, \pi'(i) = \sum_{i=1}^{n} i \cdot [\pi(i) + \pi'(i)_j] =$$

$$= \frac{n(n+1)}{2}(n+1) = (n+1)\binom{n+1}{2}$$

Q.E.D.

In [1] we have determined the possible values of $S(\pi)$ for each n and π. We omit the trivial cases $n < 4$. Let Q_n stand for the sum of the squares of the first n natural numbers and $R_n = \frac{n(n+1)(n+2)}{6}$ for the product-sum corresponding to permutation $n \ldots 21$.

Theorem: Let n and k be natural numbers, $n \geq 4$. The following statements are equivalent:

(1) $R_n \leq k \leq Q_n$

(2) There exists a permutation π of the set $\{1, 2, \ldots, n\}$ so that $S(\pi) = k$.

This theorem is proved in [1].

We know that $S(\pi) = k$ is a random variable, so we can prove the next theorem:

Theorem: The expected value of product-sums of permutations is given:

$$M(S(\pi)) = \frac{n(n+1)^2}{4} \tag{4}$$

Proof: a) According to (3) the product-sums of permutations are symmetrical with respect to the centre of $[R, Q]$. It is thus true that

$$M(S(\pi)) = \frac{1}{2}[S(\pi) + S(\pi')] = \frac{n(n+1)^2}{4}.$$

b) We can perform this proof on an elementary method too, where the constructs of permutations can be seen:

$$1 \cdot 2 \sum_{i=2}^{n} i(n-1)! = 1(2+n)(n-1)(n-1)! = 1(n^2+n-1\cdot 2)(n-1)!$$

$$2 \cdot 2 \sum_{i=3}^{n} i(n-1)! = 2(3+n)(n-2)(n-1)! = 2(n^2+n-2\cdot 3)(n-1)!$$

$$3 \cdot 2 \sum_{i=4}^{n} i(n-1)! = 3(4+n)(n-3)(n-1)! = 3(n^2+n-3\cdot 4)(n-1)!$$

$$\vdots \qquad\qquad \vdots$$

$$(n-1)\cdot 2 \sum_{i=(n-1)}^{n} i(n-1)! = (n-1)[(n+n)(n-(n-1)](n-1)! =$$
$$= (n-1)[n^2+n-(n-1)\cdot n]$$

and

$$(1^2+2^2+\ldots+n^2)(n-1)! = \frac{n(n+1)(2n+1)}{6}(n-1)!\cdot$$

Then:

$$(n-1)!\Big\{(n^2+n-1\cdot 2) + 2(n^2+n-2\cdot 3) + \ldots +$$
$$+ (n-1)[n^2+n-(n-1)\cdot n] + \frac{n(n+1)(2n+1)}{6}\Big\} =$$
$$= (n-1)!\left[\sum_{i=1}^{n-1} i(n^2+n) - \sum_{i=1}^{n-1} i^2(i+1) + \frac{n(n+1)(2n+1)}{6}\right] =$$
$$= (n-1)!\left[\frac{n(n-1)}{2}(n^2+n) - \frac{n(n+1)(n-1)(3n-2)}{12} +\right.$$
$$\left. + \frac{n(n+1)(2n+1)}{6}\right] = (n-1)!\frac{n(n+1)}{12}(3n^2+3n) = n!\frac{n(n+1)^2}{4}.$$

Where

$$-\sum_{i=1}^{n-1} i^2(i+1) = \frac{n(n^2-1)(3n-2)}{12}$$

Finally:

$$M(S(\pi)) = \frac{1}{n!} n! \frac{n(n+1)^2}{4} = \frac{n(n+1)^2}{4}$$

Q.E.D.

c) In general the transformations are performed on the permutations and for the permutations. We know that

$$M(S(\pi))=\frac{1}{n\,!}\sum_{\pi\in P_n}\sum_{i=1}^{n} i\,\pi(i) \tag{5}$$

Then

$$M(S(\pi))=\frac{1}{n\,!}\sum_{\pi\in P_N}\sum_{i=1}^{n} i\,\pi(i)=\frac{1}{n\,!}\sum_{i=1}^{n} i\sum_{\pi\in P_N}\pi(i)=$$

$$=\frac{1}{n\,!}\sum_{i=1}^{n} i\sum_{j=1}^{n}\sum_{\substack{\pi\in P_N\\ \pi(i)=j}}\pi(i)=\frac{1}{n\,!}\sum_{i=1}^{n} i\sum_{j=1}^{n} j\sum_{\pi\in P_N}1=$$

$$=\frac{1}{n\,!}\left(\sum_{i=1}^{n} i\right)^2\cdot(n-1)\,!=\frac{n(n+1)^2}{4},$$

since there are $(n-1)\,!$ permutations with $\pi(i)=j$.

Q.E.D.

Further we can prove that the standard deviation of product-sums is:

$$D^2(S(\pi))=M[(S(\pi))^2]-[M(S(\pi))]^2$$

Theorem:

$$D^2(S(\pi))=\frac{n^2(n+1)^2(n-1)}{144} \tag{6}$$

Proof: The expected value is known, so we must determine only the second moment. Namely according to (5):

$$M\left[(S(\pi))^2\right]=\frac{1}{n\,!}\sum_{\pi\in P_N}\sum_{i=1}^{n} i\,\pi(i)\sum_{k-1}^{n} k\,\pi(k)$$

Then there are the second moments of product-sums of permutations which depend on two variables. So now it will be partitioned depending on either $i=k$ and $i\neq k$. Then:

$$\frac{1}{n\,!}\sum_{i=1}^{n} i\sum_{k=1}^{n} k\sum_{\substack{\pi\in P_N\\ i=k\\ i\neq k}}\pi(i)\,\pi(k)=\frac{1}{n\,!}\sum_{i=1}^{n} i^2\sum_{\substack{\pi\in P_N\\ i=k}}[\pi(i)]^2+$$

$$+\frac{1}{n\,!}\sum_{i=1}^{n} i\sum_{k=1}^{n} k\sum_{\substack{\pi\in P_N\\ i\neq k}}\pi(i)\,\pi(k)=\frac{1}{n}\left(\sum_{i=1}^{n} i^2\right)^2+$$

$$+\frac{1}{n(n-1)}\left(\frac{n^2(n+1)^2}{4}-\frac{n(n+1)(2n+1)}{6}\right)^2,$$

according to (5).

Finally:

$$D^2[S(\pi)] = \frac{n(n+1)^2(2n+1)^2}{36} + \frac{1}{n(n-1)}\left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right]^2 -$$

$$- \frac{n^2(n+1)^4}{16} = \frac{n^2(n+1)^2(n-1)}{144}.$$

Q.E.D.

A method can also be given for the construction of subsequences of $\{\pi\}$ where $\{\pi\}$ is the construction of $\{S(\pi)\}$. In [1] we describe how a permutation can be constructed $\pi = (\pi(1), \ldots, \pi(n))$ with given degree n and product-sum k. We know that:

(*) $S(1234) = 30$, $S(2134) = 29$, $S(2143) = 28$, $S(2314) = 27$,
$S(3214) = 26$, $S(3142) = 25$, $S(4123) = 24$, $S(4132) = 23$,
$S(3412) = 22$, $S(4312) = 21$, $S(4321) = 20$.

Here it is given a $\{\pi\}$ where $\{S(\pi)\} = S_4$ and $S_n = \{S(\pi) : \pi \in P_N\}$ the set of possible values of $S(\pi)$ for $\pi \in P_N$.

$$|S_n| = \sum_{i=0}^{n-5} S_{n-4,\,i} \quad \text{and} \quad S_j \cap S_k = 0. \quad \text{where } j \neq k. \tag{7}$$

If we know the S_{n-1} than we can construct the S_n. Namely when we achieve the $(n-1)\ldots 21\ldots$ than in the next we will start from $S_{n-4,0,n-1}$ such that for the place of $\pi(1)$ put n and for the place of $\pi(n)$ put $(n-1)$, where we must go back with $n-1$ spaces in S_{n-1}. So for every n we know that from R_n to Q_n sums of product-sums of permutations are $\binom{n+1}{3} + 1$. It is true because

$$Q_n - R_n = \binom{n+1}{3} \tag{8}$$

From (8) it is followed that

$$11 + \sum_{i=5}^{n} \binom{i}{2} = \binom{n+1}{3} + 1 \tag{9}$$

The (9) is clear because

$$|S_n| = \binom{n}{2} \tag{10}$$

since $R_{n-1} + n^2 - R_n = \frac{n(n-1)}{2} = \binom{n}{2}$ and we can prove the (9) with induction but $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$. So we can choose n now some subsequences of $\{S(\pi)\}$:

(9) with induction, but $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$. So we can choose some subsequences of $\{S(\{\pi\})\}$.

Inset: some subsequences of $\{S(\{\pi\})\}$:

(**)

		1	2	3	4	5	6	7	8	9	.	.	.
		2	1	3	4	5	6	7	8	9	.	.	.
		2	1	4	3	5	6	7	8	9	.	.	.
		2	3	1	4	5	6	7	8	9	.	.	.
		3	2	1	4	5	6	7	8	9	.	.	.
S_4	$S_{0,0}$	3	1	4	2	5	6	7	8	9	.	.	.
		4	1	2	3	5	6	7	8	9	.	.	.
		4	1	3	2	5	6	7	8	9	.	.	.
		3	4	1	2	5	6	7	8	9	.	.	.
		4	3	1	2	5	6	7	8	9	.	.	.
		4	3	2	1	5	6	7	8	9	.	.	.
		5	2	1	3	4	6	7	8	9	.	.	.
		5	2	1	4	3	6	7	8	9	.	.	.
		5	2	3	1	4	6	7	8	9	.	.	.
		5	3	2	1	4	6	7	8	9	.	.	.
		5	3	1	4	2	6	7	8	9	.	.	.
S_5	$S_{1,0}$	5	4	1	2	3	6	7	8	9	.	.	.
		5	4	1	3	2	6	7	8	9	.	.	.
		5	3	4	1	2	6	7	8	9	.	.	.
		5	4	3	1	2	6	7	8	9	.	.	.
		5	4	3	2	1	6	7	8	9	.	.	.
		6	4	1	2	3	5	7	8	9	.	.	.
		6	4	1	3	2	5	7	8	9	.	.	.
	$S_{2,0}$	6	3	4	1	2	5	7	8	9	.	.	.
		6	4	3	1	2	5	7	8	9	.	.	.
		6	4	3	2	1	5	7	8	9	.	.	.
		—	—	—	—	—	—	—	—	—	—	—	—
		6	5	2	1	3	4	7	8	9	.	.	.
S_6		6	5	2	1	4	3	7	8	9	.	.	.
		6	5	2	3	1	4	7	8	9	.	.	.
		6	5	3	2	1	4	7	8	9	.	.	.
		6	5	3	1	4	2	7	8	9	.	.	.
	$S_{2,1}$	6	5	4	1	2	3	7	8	9	.	.	.
		6	5	4	1	3	2	7	8	9	.	.	.
		6	5	3	4	1	2	7	8	9	.	.	.
		6	5	4	3	1	2	7	8	9	.	.	.
		6	5	4	3	2	1	7	8	9	.	.	.

S_7	$S_{3,0}$	7	5	3	1	4	2	6	8	9	.	.	.
		7	5	4	1	2	3	6	8	9	.	.	.
		7	5	4	1	3	2	6	8	9	.	.	.
		7	5	3	4	1	2	6	8	9	.	.	.
		7	5	4	3	1	2	6	8	9	.	.	.
		7	5	4	3	2	1	6	8	9	.	.	.
		—	—	—	—	—	—	—	—	—	—	—	—
	$S_{3,1}$	7	6	4	1	2	3	5	8	9	.	.	.
		7	6	4	1	3	2	5	8	9	.	.	.
		7	6	3	4	1	2	5	8	9	.	.	.
		7	6	4	3	1	2	5	8	9	.	.	.
		7	6	4	3	2	1	5	8	9	.	.	.
		—	—	—	—	—	—	—	—	—	—	—	—
	$S_{3,2}$	7	6	5	2	1	3	4	8	9	.	.	.
		7	6	5	2	1	4	3	8	9	.	.	.
		7	6	5	2	3	1	4	8	9	.	.	.
		7	6	5	3	2	1	4	8	9	.	.	.
		7	6	5	3	1	4	2	8	9	.	.	.
		7	6	5	4	1	2	3	8	9	.	.	.
		7	6	5	4	1	3	2	8	9	.	.	.
		7	6	5	3	4	1	2	8	9	.	.	.
		7	6	5	4	3	1	2	8	9	.	.	.
		7	6	5	4	3	2	1	8	9	.	.	.

S_8	$S_{4,0}$	8	6	5	3	2	1	4	7	9	.	.	.
		8	6	5	3	1	4	2	7	9	.	.	.
		8	6	5	4	1	2	3	7	9	.	.	.
		8	6	5	4	1	3	2	7	9	.	.	.
		8	6	5	3	4	1	2	7	9	.	.	.
		8	6	5	4	3	1	2	7	9	.	.	.
		8	6	5	4	3	2	1	7	9	.	.	.
		—	—	—	—	—	—	—	—	—	—	—	—
	$S_{4,1}$	8	7	5	3	1	4	2	6	9	.	.	.
		8	7	5	4	1	2	3	6	9	.	.	.
		8	7	5	4	1	3	2	6	9		.	.
		8	7	5	3	4	1	2	6	9	.	.	.
		8	7	5	4	3	1	2	6	9	.	.	.
		8	7	5	4	3	2	1	6	9	.	.	.

References

1. BALCZA, L.: Product-sums of Permutations. Department of Mathematics Karl Marx University of Economics, Budapest, 1985-3.
2. CSÁKÁNY, B.: Algebra Tankönyvkiadó, Budapest, 1973.
3. ERDŐS, P.—SPENCER, J.: Probabilistic Methods in Combinatorics. Publishing House of the Hungarian Academy of Sciences, 1974.
4. HARDY, G. H.—LITTLEWOOD, J. E.—PÓLYA, G.: Inequalities Cambridge University Press, 1934.
5. В. Н. Сачков: Вероятностные Методы в Комбинаторном Аналзе. Москва, «Наука» Главная Редакции Физико-математической, Литературы, 1978.
6. TOMESCU, I.: Introducere in Combinatorica Editura Technica, Bucuresti, 1973.

Dr. Lajos BALCZA H-1521 Budapest