CHEBYSHEV-TYPE INEQUALITIES FOR SPECIAL RANDOM VARIABLES

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Abstract

An elementary proof is given for the generalized Chebysev-type inequalities.

Let ξ be a random variable which satisfies the conditions:
The density function $f(x)$ of ξ exists and

$$f(-x) = f(x) \qquad f(x_1) \geqq f(x_2) \quad \text{if} \quad 0 < x_1 < x_2 < +\infty \tag{1}$$

moreover the central moments of order $2n$ exist and if it is denoted by M_{2n}, then for every $\varepsilon > 0$

$$\frac{M_{2n}}{\left(1+\frac{1}{2n}\right)^{2n}\varepsilon^{2n}} \geqq P(|\xi| > \varepsilon). \tag{2}$$

As a consequence of this statement it will also be proved that the inequality (2) holds if M_{2n} exists, $M(\xi) = 0$ and the density function increasing in the ray $(-\infty; 0)$ and decreasing in $(0; +\infty)$.

Let us consider first the function $\lambda_{2n}(\varepsilon)$ defined by

$$\lambda_{2n}(\varepsilon) = \frac{\int\limits_0^\varepsilon x^{2n} f(x)\,dx}{\int\limits_\varepsilon^\infty x^{2n} f(x)\,dx} = \frac{\int\limits_0^\varepsilon x^{2n} f(x)\,dx}{\frac{1}{2}M_{2n} - \int\limits_0^\varepsilon x^{2n} f(x)\,dx} \tag{3}$$

where n is a positive integer and $f(x)$ is a density function satisfying the conditions (1). From the expression (3) we can get

$$\int\limits_0^\varepsilon x^{2n} f(x)\,dx = \lambda_{2n}(\varepsilon)\int\limits_\varepsilon^\infty x^{2n} f(x)\,dx,$$

$$\int\limits_0^\varepsilon x^{2n} f(x)\,dx + \int\limits_\varepsilon^\infty x^{2n} f(x)\,dx = \lambda_{2n}(\varepsilon)\int\limits_\varepsilon^\infty x^{2n} f(x)\,dx + \int\limits_\varepsilon^\infty x^{2n} f(x)\,dx,$$

$$\frac{1}{2}M_{2n} = \left(\lambda_{2n}(\varepsilon)+1\right)\int\limits_\varepsilon^\infty x^{2n} f(x)\,dx, \qquad \frac{1}{2}M_{2n} \geqq \left(\lambda_{2n}(\varepsilon)+1\right)\varepsilon^{2n}\left(1 - F(\varepsilon)\right),$$

$$M_{2n} \geqq \left(\lambda_{2n}(\varepsilon)+1\right)\varepsilon^{2n} P(|\xi| \geqq \varepsilon).$$

This implies

$$\frac{M_{2n}}{\left(\lambda_{2n}(\varepsilon)+1\right)\varepsilon^{2n}} \geqq P(|\xi| \geqq \varepsilon) \tag{4}$$

and the well-known Chebyshev inequality

$$\frac{M_{2n}}{\varepsilon^{2n}} \geqq P(|\xi| \geqq \varepsilon). \tag{4'}$$

We can see from inequality (4) that $\frac{M_{2n}}{\varepsilon^{2n}}$ is a rough estimation for the probability of (4'), because the function $\lambda_{2n}(\varepsilon)$ is increasing and

$$\lambda_{2n}(0)=0, \lim_{n\to\infty} \lambda_{2n}(\varepsilon)=+\infty. \tag{5}$$

We could not determine the function $\lambda_{2n}(\varepsilon)$ in general but we can approximate it in every practical case by the following form and the empirical distribution function

$$\lambda_{2n}(\varepsilon)=\frac{\varepsilon^{2n} F(\varepsilon)-2n\int\limits_0^{\varepsilon} x^{2n-1} F(x)\,dx}{\frac{1}{2} M_{2n}-\varepsilon^{2n}+2n\int\limits_0^{\varepsilon} x^{2n-1} F(x)\,dx}. \tag{6}$$

Proof of the inequality (2)

If the density function of a random variable satisfies the conditions (1), then the central moment of order $2n$

$$M_{2n}=2\int\limits_0^{\infty} x^{2n} f(x)\,dx. \tag{7}$$

The right hand side of this equation can be reordered as follows

$$\begin{aligned}
&M_{2n}=2\int\limits_0^{(2n+1)\varepsilon} x^{2n} f(x)\,dx+2\int\limits_{(2n+1)\varepsilon}^{\infty} x^{2n} f(x)\,dx, \quad \varepsilon>0,\\
&M_{2n}\geqq 2\int\limits_0^{(2n+1)\varepsilon} x^{2n} f(x)\,dx+2\big((2n+1)\varepsilon\big)^{2n}\int\limits_{(2n+1)\varepsilon}^{\infty} f(x)\,dx, \qquad (8)\\
&M_{2n}\geqq 2\left\{[x^{2n} F(x)]\cdot\int\limits_0^{(2n+1)\varepsilon} -2n\int\limits_0^{(2+1)\varepsilon} x^{2n-1}F(x)dx+\big((2n+1)\varepsilon\big)^{2n}\int\limits_{(2n+1)\varepsilon}^{\infty} f(x)dx\right\},\\
&M_{2n}\geqq 2\big((2n+1)\varepsilon\big)^{2n} F\big((2n+1)\varepsilon\big)+2\big((2n+1)\varepsilon\big)^{2n}\big(1-F((2n+1)\varepsilon)\big)-\\
&\qquad -2\cdot 2n\int\limits_0^{(2n+1)\varepsilon} x^{2n-1} F(x)\,dx,\\
&M_{2n}\geqq 2\left[\big((2n+1)\varepsilon\big)^{2n}-2n\int\limits_0^{(2n+1)\varepsilon} x^{2n-1} F(x)\,dx.\right]
\end{aligned}$$

Now we are going to construct an upper bound for the integral in the form (8) using the condition (1) that is the graph of the distribution function $F(x)$ is concave down in the ray $(0; +\infty)$.

Consider the region under the graph of the function $F(x)$ in the interval $(0; (2n+1)\varepsilon)$.

This region has a moment of order $2n-1$ concerning the y-axis not longer than the same moment of the region under the straight line which passes through points $P_1(\varepsilon; F(\varepsilon))$ and $P_2(2n\varepsilon; F(2n\varepsilon))$.

The case $n = 2$ is shown in Fig. 1.

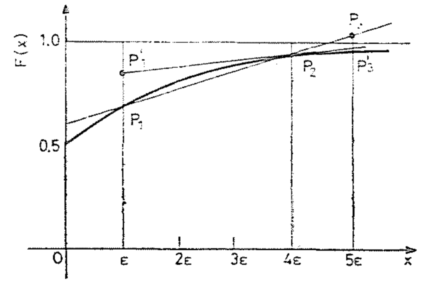

Fig. 1

The concavity of $F(x)$ implies that the moment of the region under the line P_1P_2 is not less than the moment of the region under the graph of $F(x)$ in the interval $(0; \varepsilon)$ with respect to the y-axis (see Fig 1).

Then we prove that the moment of order $2n-1$ of the region under the section P_1P_3 is not less than the same moment of the region under the section $P_1'P_3'$ with respect to the y-axis, where the straight line $P_1'P_3'$ is the tangent line of the graph of $F(x)$ at point P_2. The last mentioned moment of the region under the section P_1P_3 is obviously not less than the corresponding moment of the region under the graph of $F(x)$ in the interval $(\varepsilon, (2n+1)\varepsilon)$.

If the last statement is true, it follows from the next integral, (Fig. 2), that

$$\int_{\varepsilon}^{(2n+1)\varepsilon} x^{2n-1}\left(\frac{b}{2n\,\varepsilon}\,x - b\right)dx = b\int_{\varepsilon}^{(2n+1)\varepsilon}\left(\frac{1}{2n\,\varepsilon}\,x^{2n} - x^{2n-1}\right)dx =$$

$$= b\left[\frac{1}{2n\,\varepsilon}\,\frac{x^{2n+1}}{2n+1} - \frac{x^{2n}}{2n}\right]_{\varepsilon}^{(2n+1)\varepsilon} =$$

$$= b\left[\frac{1}{2n\,\varepsilon}\,\frac{((2n+1)\,\varepsilon)^{2n+1}}{2n+1} - \frac{((2n+1)\,\varepsilon)^{2n}}{2n} - \frac{1}{2n\,\varepsilon}\,\frac{\varepsilon^{2n+1}}{2n+1} + \frac{\varepsilon^{2n}}{2n}\right] =$$

$$= \frac{b}{2n}\,\varepsilon^{2n}\left[(2n+1)^{2n} - (2n+1)^{2n} - \frac{1}{2n+1} + 1\right] > 0.$$

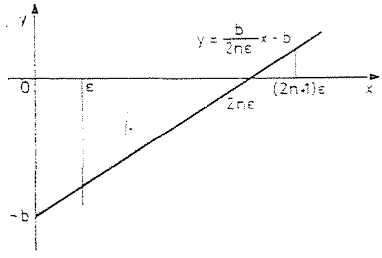

Fig. 2

Hence, we get the inequality

$$\int_0^{(2n+1)\varepsilon} x^{2n-1} F(x)\,dx \leq \int_0^{(2n+1)\varepsilon} x^{2n-1}\left(F(\varepsilon) + \frac{F(2n\varepsilon) - F(\varepsilon)}{(2n-1)\,\varepsilon}(x-\varepsilon)\right)dx. \tag{9}$$

Let us determine the integral of the right hand side, then we have

$$\int_0^{(2n+1)\varepsilon} x^{2n-1}\left(F(\varepsilon) + \frac{F(\varepsilon)}{2n-1} - \frac{F(2n\varepsilon)}{2n-1} + \frac{F(2n\,\varepsilon) - F(\varepsilon)}{(2n-1)\,\varepsilon}\,x\right)dx =$$

$$= \int_0^{(2n+1)\varepsilon}\left(\left(\frac{2n}{2n-1}\,F(\varepsilon) - \frac{F(2n\,\varepsilon)}{2n-1}\right)x^{2n-1} + \frac{F(2n\varepsilon) - F(\varepsilon)}{(2n-1)\,\varepsilon}\,x^{2n}\right)dx =$$

$$= \left[\left(\frac{2n\,F(\varepsilon)}{2n-1} - \frac{F(2n\,\varepsilon)}{2n-1}\right)\frac{x^{2n}}{2n} + \frac{F(2n\varepsilon) - F(\varepsilon)}{(2n-1)\,\varepsilon}\,\frac{x^{2n+1}}{2n+1}\right]_0^{(2n+1)\varepsilon} =$$

$$= \frac{F(\varepsilon)}{2n-1}\left((2n+1)\,\varepsilon\right)^{2n} - \frac{F(\varepsilon)}{2n-1}\left((2n+1)\,\varepsilon\right)^{2n} +$$

$$+ \frac{F(2n\,\varepsilon)}{(2n-1)\,\varepsilon}\,\frac{\left((2n+1)\,\varepsilon\right)^{2n+1}}{2n+1} - \frac{F(2n\,\varepsilon)}{2n-1}\,\frac{\left((2n+1)\,\varepsilon\right)^{2n}}{2n} =$$

$$= \left((2n+1)\,\varepsilon\right)^{2n} F(2n\,\varepsilon)\left(\frac{1}{2n-1} - \frac{1}{(2n-1)\,2n}\right) =$$

$$= \left((2n+1)\,\varepsilon\right)^{2n} F(2n\,\varepsilon)\,\frac{2n-1}{(2n-1)\,2n} = \left((2n+1)\,\varepsilon\right)^{2n}\frac{F(2n\,\varepsilon)}{2n} =$$

$$= \left(1 + \frac{1}{2n}\right)^{2n}(2n\,\varepsilon)^{2n}\,\frac{F(2n\,\varepsilon)}{2n}.$$

Thus the following inequality holds

$$\int_0^{(2n+1)\varepsilon} x^{2n-1} F(x)\,dx \leq \left(1 + \frac{1}{2n}\right)^{2n}(2n\,\varepsilon)^{2n}\,\frac{1}{2n}\,F(2n\,\varepsilon). \tag{10}$$

If we substitute the expression of the right hand side of inequality (10) into inequality (8), we get

$$M_{2n} \geqq 2\left(((2n+1)\,\varepsilon)^{2n} - 2n\left(1+\frac{1}{2n}\right)^{2n}(2n\,\varepsilon)^{2n}\,F(2n\,\varepsilon)\,\frac{1}{2n}\right),$$

$$M_{2n} \geqq 2\left(\left(1+\frac{1}{2n}\right)^{2n}(2n\,\varepsilon)^{2n} - \left(1+\frac{1}{2n}\right)^{2n}(2n\,\varepsilon)^{2n}\,F(2n\,\varepsilon)\right),$$

$$M_{2n} \geqq \left(1+\frac{1}{2n}\right)^{2n}(2n\,\varepsilon)^{2n}\cdot 2\big(1-F(2n\,\varepsilon)\big).$$

Using the notation $\tilde{\varepsilon} = 2n\varepsilon > 0$, the statement is proved

$$\frac{M_{2n}}{\left(1+\frac{1}{2n}\right)^{2n}\cdot\tilde{\varepsilon}^{2n}} \geqq 2\big(1-F(\tilde{\varepsilon})\big),$$

$$\frac{M_{2n}}{\left(1+\frac{1}{2n}\right)^{2n}\cdot\tilde{\varepsilon}^{2n}} \geqq P(|\,\xi\,| \geqq \tilde{\varepsilon}),\quad \tilde{\varepsilon} > 0.$$

4. Let us consider the case when ξ is a random variable with a non-even density function, its expected value $M(\xi) = 0$, M_{2n} exists and its distribution function $F(x)$ is convex in the ray $(-\infty; 0)$ and concave in $(0; +\infty)$.

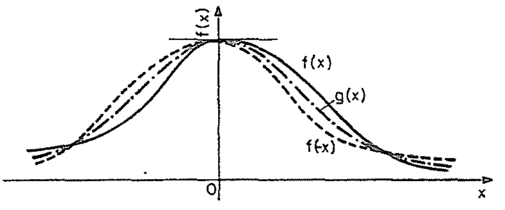

Fig. 3

In Fig. 3 drawn by a continuous line a density function is shown which satisfies the above mentioned conditions. The dotted line shows the reflected density function to the y-axis.

One can easily see that function

$$g(x) = \frac{1}{2}\big(f(x) + f(-x)\big)\quad (-\infty < x < +\infty)$$

satisfies every conditions used in the previous section and is obviously a density function and

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx = 2\int_0^{\infty} x^2 \frac{1}{2}\big(f(x)+f(-x)\big)\, dx,$$

$$P(|\xi| \geq \varepsilon) = \int_{-\infty}^{-\varepsilon} f(x)\, dx + \int_{\varepsilon}^{\infty} f(x)\, dx = 2\int_{\varepsilon}^{\infty} \frac{1}{2}\big(f(x)+f(-x)\big)\, dx.$$

Using these formula we get the corresponding inequalities the same way as earlier.

Notations

a) Inequality (2) is the best with respect to this class of distribution functions.

b) Inequality (1) also holds for discrete random variables if the probabilities are increasing until the expected value, then are decreasing (for example binomial distribution). The proofs of this statement are in papers [4] and [5].

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