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The Effect of Torsional Rigidity and Approximations in Analytical Solutions for the Critical Moment of Beams Considering Prebuckling Deflections

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Abstract

Solutions for the critical moment of lateral-torsional buckling of beams commonly used in design codes, are based on initially straight undeflected beams. However, studies indicate that when the weak-to-strong moment of inertia ratio is sufficiently high, prebuckling deformations can significantly impact critical loads. The proposed formulae in the literature often vary, and discussions on the effect of torsional rigidity are limited, with most studies focusing on open sections. This study provides a comprehensive review of the problem and relevant literature, exploring variations and simplifications used to derive closed-form solutions for the critical moment, while accounting for prebuckling deformations through the energy method. Several variations of the critical moment formula are presented and compared, with a detailed investigation into the influence of torsional rigidity. Prebuckling deformations were confirmed to have a significant effect on the critical moment for specific sections, and the conditions for appropriate simplifications were identified. Additionally, torsional rigidity was found to exert a non-negligible influence, with closed sections demonstrating a greater effect of prebuckling deformations.

Keywords

lateral-torsional buckling, prebuckling deflection, analytical solution, torsional rigidity

1 Introduction

Most solutions for the critical moment of lateral-torsional buckling (LTB) of beams are derived from linear buckling analysis (LBA), meaning that the analytical solutions are usually based on an undeflected straight beam. However, as the load increases, before buckling happens, the beam is already deflected in the major plane of the loading, which deflection is termed here "prebuckling deflection". Several studies have found that LBA leads to accurate results only in the case where the major-axis moment of inertia is much larger than the weak-axis moment of inertia, while the effect of prebuckling deformations can have a significant influence on the critical loads in the case of more compact members.

This paper investigates the effect of prebuckling deformations, as discussed in the literature. Upon reviewing the literature, it was found that most studies which included the prebuckling deformations focus mainly on the most basic case, doubly symmetric I-beams with pinned (forked) supports subjected to uniform moment (Fig. 1), with a handful of studies discussing other cases (mono-symmetric beams and some other loading conditions). These papers, although not too numerous, give several different formulas for the same case. The differences between these formulas vary in their significance, with some being minor and some being more significant. Furthermore, the effect of torsional rigidity is hardly discussed, with most studies focusing on open sections with low torsional rigidities,

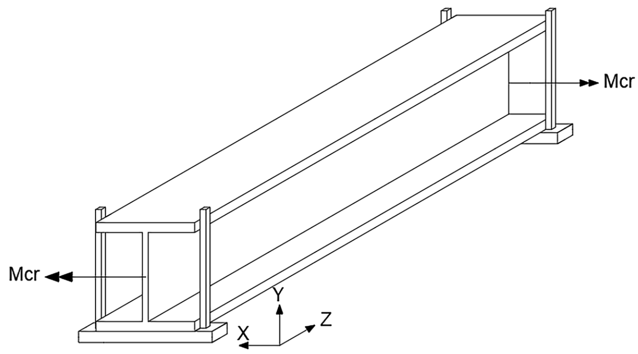

Fig. 1 Doubly symmetric beam with forked support

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and a clear distinction between open and closed sections not made in any of the reviewed papers.

This study, therefore, addresses these differences and makes a distinction between sections with low and high torsional rigidities. In Section 2, a detailed review of the literature is given. Section 3 discusses the analytical derivations, highlighting the different variations of used functions, as well as the decision points during the derivations, which can lead to different solutions. In Section 4, several possible closed-form solutions are given based on these variations, for both open and closed doubly symmetric cross-sections, and a numerical study is presented to assess these different formulae.

2 Detailed review of earlier studies

In Section 2, a literature review is provided, focusing on literature where the effect of prebuckling deflections on the LTB of doubly symmetric beams is discussed. The review is unusually detailed. While acknowledging the researchers who have contributed to this topic would be commendable, the primary reason of the detailed review is that the available papers are not too numerous but still include slightly different formulae for the same case, as well as include certain contradictory statements or suggestions.

As far as is known, the effect of pre-buckling deformations on the lateral-torsional buckling of beams was first reported by Michell [1]. In his paper, differential equations (D.E.-s) are formulated and solved for various beam-column cases, and two simple experiments are reported to validate the theoretical results. A particular case considered in the derivations is the basic case. A solution is derived for the critical moment M_{cr}, written (using the notations normally used nowadays) as:

$$M_{cr} = M_{cr0} \Big/ \sqrt{\left(1 - \frac{EI_y}{EI_x}\right)} \quad (1)$$

with $M_{cr0} = \frac{\pi}{L}\sqrt{EI_y GJ}$,

where M_{cr0} is the critical moment without the effect of prebuckling deformations, I_x and I_y are the second moments of area for the x (major) and y (minor) axes, respectively, J is the torsional inertia, L is the length of the beam, and E and G are the Young's modulus and shear modulus, respectively. It is to observe that the warping effect is totally disregarded. The results obtained from this formula are in line with later results if the cross-section has a small warping constant, e.g., in the case of a narrow rectangular section. In fact, while the paper does not explicitly state this limitation to narrow rectangular shapes, such members are considered in the conducted experiments. Another remark is that the paper does not discuss the "effect of prebuckling deformations" separately, i.e., there is no distinct solution provided with and without prebuckling deformations; instead, it explicitly assumes the presence of prebuckling deformations and solves the equations accordingly. Another early work was done by Prandtl [2], who also established the D.E.-s for the LTB problem (without considering warping), and solved them for a few cases, but without considering the prebuckling deflections.

An important contribution to LTB was made by Timoshenko [3], where the effect of warping for thin-walled members in twist was first introduced in the context of I-section members. Timoshenko [4] published the first critical moment formula with considering warping, particularly for I-shaped members. The formula is identical to the one known nowadays, though in [4] it is formally different and expressed specifically for I-shaped beams only. It is:

$$M_{cr0} = \frac{\pi}{L}\sqrt{EI_y GJ\left(1 + \frac{\pi^2 EI_w}{GJL^2}\right)}, \quad (2)$$

where I_w is the warping modulus. Later, in [5] a solution is presented for clamped-clamped beams (without the prebuckling effect).

Chwalla [6] formulated the D.E.-s of the beam-column buckling problem considering the warping effect and accounting for the prebuckling deflections and provided closed-form solutions for several cases. Regarding the effect of prebuckling deformations, the derivation is presented for the basic case with narrow rectangular sections, leading to the following formula:

$$M_{cr} = M_{cr0} \Big/ \sqrt{\left(1 - \frac{EI_y}{EI_x}\right)\left(1 - \frac{2GJ}{EI_x}\right)} \quad (3)$$

with M_{cr0} as given by Eq. (1). In [6], it is also proposed to introduce an equivalent lateral bending stiffness to consider the effect of prebuckling deflections, suggesting that the proposed equivalent bending stiffness can be employed to any LTB case. It is also commented that GJ/EI_x is typically small, therefore, can be neglected.

LTB is discussed by Davidson [7], also for I-sections (and as particular cases: narrow rectangular sections). Analytical solutions for both without and with considering the prebuckling deflections are given, assuming elastic end supports with separate stiffnesses for the global rotation of the beam ends (about the minor axis) and rotation of the flanges (which latter one could be "translated" to today's terminology as elastic warping support). Analytical solutions are given, but typically not in a closed format, due to the problem's complexity. Explicit formulae can be derived only for some simple cases, e.g., if the supports stiffnesses are zero (i.e., forked supports) and $I_w = 0$ (e.g., rectangular narrow section), the derivation leads to the following formula:

$$M_{cr} = M_{cr0} \Big/ \sqrt{\left(1-\frac{EI_y}{EI_x}\right)\left(1-\frac{GJ}{EI_x}\right)} \tag{4}$$

with M_{cr0} as given by Eq. (1). This is nearly (but not exactly) identical to the solution in [6].

The next appearance of the same problem is in Pettersson's [8] work, focusing on mono-symmetric cross-sections. The displacement functions of beams subjected to combined loading (biaxial bending and torsion) are derived. Mostly simply supported beams are considered, but 3-span beams are also discussed. Critical moment expressions with and without the prebuckling deflection are provided for a few cases. For rectangular section beams under uniform major-axis moment, the derived formula is identical to the one in [7], see Eq. (4).

Kerensky et al. [9] summarized the background of the then-current British Standard, and for the calculation of critical moment, a formula with considering the effect of prebuckling deflection was proposed, using the $1\Big/\sqrt{1-I_y/I_x}$ factor as in Eq. (1).

A few years later, Clark and Knoll [10] extended the critical moment formula for a few cases. For clamped beams with narrow rectangular cross-sections, they derived the formula as follows:

$$M_{cr} = M_{cr0}\left(1-\frac{EI_y}{EI_x}\right) \Big/ \sqrt{\left(1-\frac{EI_y}{EI_x}\right)\left(1-\frac{GJ}{EI_x}\right)} \tag{5}$$

with $M_{cr0} = \frac{2\pi}{L}\sqrt{EI_y GJ}$.

In Eq. (5), the M_{cr0} expression is the one normally used nowadays (if the warping effect is negligible), but the modification factor due to the prebuckling deflections is significantly different from those previously derived for the pinned-pinned case. This is, therefore, the first publication where the influence of the supports on the prebuckling effect is explicitly reported. Moreover, a formula is derived for a doubly-symmetric I-section beam in [10], where M_{cr0} is identical to that derived by Timoshenko [4], and the modification factor accounting for the effect of prebuckling deflections is the same as that derived by Davidson [7].

The problem was revisited by Trahair and Woolcock [11]. In [11] a set of D.E.-s with considering the effect of prebuckling deformations, assuming doubly symmetrical cross-sections and pinned end supports, is derived. The solutions for a few cases are discussed (mostly numerically). A closed-form solution is given for the basic case, which can be written as:

$$M_{cr} = M_{cr0} \Big/ \sqrt{\left(1-\frac{EI_y}{EI_x}\right)\left(1-\frac{GJ}{EI_x}\left(1+\frac{\pi^2 EI_w}{GJL^2}\right)\right)} \tag{6}$$

where M_{cr0} is the same as proposed by Timoshenko [44], see Eq. (2).

The next important contribution is made by Vacharajittiphan et al. [12], where a general approach is introduced for describing the three-dimensional behavior of thin-walled members in bending, assuming doubly symmetrical cross-sections. From the general description, a simplified set of D.E.-s is derived. Since the aim was to calculate the critical load, i.e., to capture the point of bifurcation of the equilibrium, it was assumed that the lateral and torsional displacements are infinitesimally small, while the primary (i.e., in the plane of the bending) displacements are moderately large. The simplifications are introduced accordingly, in a consistent way, as follows: the lateral and torsional displacements are approximated by linear terms, while the primary displacement is approximated by up to quadratic terms. The derived formula is identical to the one in [11], see Eq. (6).

Roberts and Azizian [13] developed a beam finite element model for the analysis of thin-walled members with open cross-sections. The used variational form of the problem is aimed to get weak (approximate) solutions numerically, as usual in any finite element implementation. Arbitrary open cross-sections, including asymmetrical ones, are considered. The developed beam finite element is based on Vlasov's thin-walled beam theory and is employed to solve simple column and beam problems using an incremental-iterative solution scheme. Though the effect of prebuckling deformations is not specifically discussed, it is mentioned that "…when these nonlinear strains are incorporated in a general instability analysis …, the influence of pre-buckling displacements is automatically taken into account" ([13]:p.565). Subsequent research [14] discusses the effect of prebuckling deformations, based on the same principles as in [13]. However, analytical solutions are also provided. For the basic case, a critical moment formula is derived, which is identical with the one in [11, 12]. An analytical solution for beams with monosymmetric T-shaped cross-sections, assuming that I_w is zero, is also provided.

The next noteworthy contribution is a pair of papers by Pi and Trahair [15, 16]. In [15], Roberts and Azizian's [14] earlier work is criticized. The criticism is on the basis that the finite element solution provided in [13, 14] leads to a quadratic eigenvalue problem due to the presence of second-order terms, and instead of using an iterative approach, the problem in [13] is solved as a linear eigenvalue problem. Other issues in [13] were addressed, too, such as the consideration of "constant prebuckling in-plane rotations and curvatures" ([15]:p.2949) along each element, which is not an accurate representation, as well as the negligence of the additional moments the axial loads cause due to the presence of prebuckling deformations.

The geometrical description of the problem in [15] is like the one in [12], but there are some significant differences and/or advancements. One is that energy equations are provided and used. Another one is that the equations are developed for mono-symmetrical cross-sections too. Moreover, the geometric description is more accurate and/or general. Finally, more terms are included in the approximations (compared to [12]), though when it comes to the derivation of actual M_{cr} formulae, the kept nonlinear terms are more-or-less the same. Two new versions of M_{cr} formulae are derived and presented in [16], but only for the basic case. One formula, termed as "linearized", is obtained by neglecting "the terms containing the second-order prebuckling deformations … in the energy equation" ([16]:p.2968), as follows:

$$M_{cr} = M_{cr0} \Big/ \left[\left(1 - \frac{EI_y}{EI_x}\right)\left(1 - \frac{GJ}{2EI_x}\left(1 + \frac{\pi^2 EI_w}{GJL^2}\right)\right)\right]. \quad (7)$$

This formula is immediately criticized, stating that this expression "overestimates the critical moment and shows that second-order terms in the energy equation should not be neglected" ([16]:p.2968). The other, believed to be more accurate, formula is as follows:

$$M_{cr} = M_{cr0} \Big/ \sqrt{\left(1 - \frac{EI_y}{EI_x}\right)\left(1 - \frac{GJ}{2EI_x}\left(1 + \frac{\pi^2 EI_w}{GJL^2}\right)\right)}. \quad (8)$$

In Eqs. (7), (8), M_{cr0} is identical to the one proposed by Timoshenko [4], see Eq. (2).

Equation (8) is almost identical to the ones published in [11–13]. The only difference is the appearance of a "2" in the denominator of the $GJ/2EI_x$ term. The authors mention this slight difference, but do not explain or discuss. It is to mention, that later, in Trahair's [17] book, Eq. (8) is presented. (It is to note that, in [16], an M_{cr} formula is proposed for mono-symmetric cross-sections. It is not clear how it is obtained, but it is clearly different from the one derived by Roberts and Azizian [14] for mono-symmetric sections.)

In [16], the analytical and numerical results are compared to those obtained from experiments. The test-based critical values are determined by the so-called Southwell-plot technique. Looking at the results, it is fair to say that:

1. the linearized formula is clearly incorrect,
2. the experimental results are perhaps somewhat closer to the numerical ones if the prebuckling deflections are considered,
3. but the experimental results are not convincing regarding the effect of prebuckling deflections.

Though in [18], the LTB problem is not discussed, it is worth mentioning here because the paper expresses some criticism regarding the mathematical background of the derivations in [15].

Andrade and Camotim [19] discuss the LTB of prismatic and tapered beams, both with and without the effect of prebuckling deflections. The developed and utilized formulation includes some approximations. It is suggested that the prebuckling effect can be taken into consideration by the $1/\sqrt{1 - I_y/I_x}$ factor, same as in [1, 9].

Machado and Cortínez [20] also investigated doubly-symmetric beams considering the effect of prebuckling deflections. The novelty in this research is the consideration of transverse shear deformations, both along the major and minor axes, assuming laminated material. Variational principles are used, and closed-form solutions are provided. As a special case, if the shear deformations are neglected, the solution is identical to that given in [15]. Various transverse load cases are investigated, including the load height effect on simply supported single-span beams and cantilevers. The general observation is that the shear deformations decrease the critical loads, while the prebuckling deflections increase it. (Note, the effect of various laminations is also discussed.)

Mohri and Potier-Ferry [21] studied doubly- and mono-symmetric beams under various loading conditions, including transverse loading with varying load application heights. In [21], D.E.-s considering the effect of prebuckling deflections are developed and solved. In the basic case, the same solution as in [12] is obtained. It is also commented that in engineering practice it is acceptable to account for the prebuckling effect by the $1/\sqrt{1 - I_y/I_x}$ factor.

Torkamani and Roberts [22] derived new energy equations for flexural-torsional and lateral-torsional buckling of thin-walled beam-columns. The equations are remarkably similar to those in [15]. There are a few differences, however, in how the nonlinear displacements of an arbitrary cross-section point are approximated. These differences are not discussed or explained. Some numerical examples are presented, one is related to LTB, but without special attention to the effect of prebuckling deflections.

Mohri et al. [23] published another article on the same topic. The theory is developed for general open cross-sections, using variational principles and D.E.-s, with the focus being on monosymmetric I- and T-shaped sections. When considering doubly symmetric sections, it is again suggested that the prebuckling effect can be considered by the $1/\sqrt{1-I_y/I_x}$ factor.

An updated version of the weak formulation of the lateral buckling problem is published by Attard and Kim [24]. The kinematic assumptions are like those presented in earlier papers, and the novelty is the consideration of hyperelastic materials. The derived general formulae are utilized to re-derive an M_{cr} formula for the basic case, which is found to be identical to the one presented in [12]. (Also, they try to derive the formula for mono-symmetric cross-sections; in their results, the derivation leads to a cubic equation from which M_{cr} cannot be expressed in closed format. However, this formula is clearly different from that in [14] or in [16].)

The topic is discussed by Erkmen and Attard [25], considering the shear deformations (similarly as discussed earlier in [20]). As for the analytical solution, the earlier formula for the basic doubly-symmetric case is repeated, where the effect of shear is said to be nonexistent. However, numerical (finite element) solutions are also provided and compared with results from the analytical ones. It is noted that "in order to induce bifurcation type post-buckling behavior, an initial small horizontal load is applied in the nonlinear analysis" ([25]:p.921). The numerical results, hence, were obtained by nonlinear incremental analysis, not eigen-value analysis. It is declared that the analytical closed-form solution "is a lower bound to the results based on the nonlinear analysis procedure" ([25]:p.922). In other words, the authors declare that the effect of prebuckling deformations is even larger than what is predicted by the analytical formulae.

The question is discussed again by Mohri et al. [26], but the discussion and conclusions are rather similar to those of [23].

The LTB behavior of U-shaped sections (i.e., unlipped channel) is discussed by Beyer et al. [27]. The energy method is used, and the effect of prebuckling deflections is considered. The focus of the paper is on minor-axis bending. For major-axis bending (which is similar to a doubly symmetrical section), the prebuckling effect is considered by the $1/\sqrt{1-I_y/I_x}$ factor.

In the next related study, conducted by Pezeshky and Mohareb [28], the focus is on the distortional deformations, though shear deformations are optionally considered, too. Variational principles are employed, but closed-form solutions are not provided. From the numerical results it can be deduced that the shear deformations have small effect, but the distortional deformations noticeably reduce the M_{cr}/M_{cr0} ratio.

Finally, the most recent relevant paper is published by Su et al. [29]. Their paper unquestionably discusses the effect of prebuckling deformations, but the subject is different from classic structures in structural engineering. The studied structure is the so-called "serpentine interconnect", which is a beam with a serpentine-shaped axis. The prebuckling deflections are huge compared to classic structures, accordingly, the effect of prebuckling deformations is drastic. Some analytical solutions and numerical examples are shown. As a special case, the classic LTB problem is considered, to which a closed-form solution is derived. The beam is assumed to have a rectangular cross-section, but not necessarily narrow rectangular. The warping effect is not directly considered. The obtained closed-form solution for the critical moment is identical to the one first derived in [7], with the only difference being that the torsion constant is different. The method is developed primarily for numerical solutions. It is worth mentioning that some of the numerical results predict higher values than the analytical solution, suggesting that the analytical solution is not perfectly precise.

The main observations are summarized as follows:

- Most of the research considers simply supported beams with forked supports, and open cross sections. Cantilever beams, clamped beams, and closed sections occur in very few papers, but other cases are not discussed at all.
- There seems to be a consensus that the prebuckling deflection has a positive effect on LTB, i.e., the prebuckling deflection increases the critical moment. Moreover, there is an agreement that the increase is primarily influenced by the ratio of the weak to strong axis moments of inertia.
- Most of the papers agree that the increase due to prebuckling deflections can approximately be expressed

by the $1/\sqrt{1-I_y/I_x}$ factor. More precise formulae are given in several papers, and there are small discrepancies between these formulae. In a few papers, it is suggested that the available analytical formula underestimates the real critical moment.

- In most of the papers, it is implicitly assumed or explicitly stated that the provided formula to consider the prebuckling effect is valid in general. There is one single paper in which it is suggested that the critical moment increase is affected by the supports.
- The discrepancies between the papers are not limited to differences of he provided closed-form solutions for the critical moment with prebuckling effect, differences can also be observed in the underlying basic mechanical-mathematical formulae. In certain papers, criticism can be found regarding the content of other papers.
- Experimental work specifically devoted to the effect of prebuckling deflection is rather scarce; the existing experimental results are not convincing.
- In the literature there is hardly any attempt to use general numerical methods such as the shell finite element method, to verify the analytical results or the developed specific numerical formulations.

3 Analytical solutions
3.1 Overview
3.1.1 Shape functions
In this study, the energy method is used. The total potential is expressed in terms of displacements. Thus, the displacements must be assumed. In case of LTB, the secondary displacements are the lateral translation and twisting rotation, u and φ. The shape functions must satisfy the boundary conditions. In the case of forked supports, the assumed displacement functions are simple half sinewaves:

$$u(z)=u_m \sin\frac{\pi z}{L}$$
$$\varphi(z)=\varphi_m \sin\frac{\pi z}{L} \tag{9}$$

where u_m and φ_m are the displacement amplitudes, and L is the beam length, see Fig. 2. The primary (prebuckling) displacement is the in-plane deflection due to loading. Although it is not included in the energy method solution, it has an influence on the strains and curvatures, which influence is disregarded in classic LTB solutions. The primary displacements can be expressed by classic equations of the strength of materials. For example, in the basic

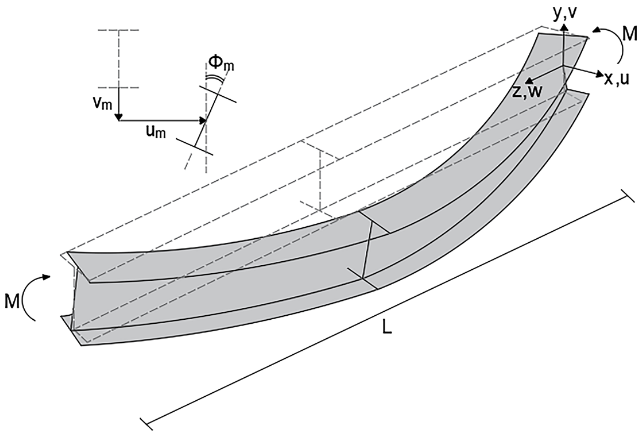

Fig. 2 Coordinate system, displacements

case the beam is simply supported at both ends and subjected to uniform moment along the length, accordingly, the primary displacement of the beam's system line can be described by a quadratic function:

$$v(z)=\frac{v_m 4z(L-z)}{L^2} \quad \text{with } v_m=\frac{M_x L^2}{8EI_x}, \tag{10}$$

where v_m is the maximum vertical displacement, and M_x is the applied uniform bending moment.

3.1.2 Total potential
The total potential (Π) is expressed as the sum the strain energy (S) and the work (W) of the stresses on the nonlinear strains:

$$\Pi = S + W, \tag{11}$$

where

$$W=\int_0^L\int_A \varepsilon_z \frac{M_x}{I_x} y dA dz \tag{12}$$

$$S=\frac{1}{2}\int_0^L\int_A\left(EI_y\kappa_y^{\,2}+EI_w\kappa_{zd}^{\,2}+GJ\kappa_z^{\,2}\right)dA dz. \tag{13}$$

In the strain energy expression, κ_y is the curvature in the lateral direction, (i.e., the rate of change of the tangent of the system line in the lateral direction,) κ_z is the rate of change of the twist angle, and κ_{zd} is the rate of change of κ_z. In the work expression, M_x is the applied moment, ε_z is the nonlinear longitudinal normal strain due to displacements. Both the ε_z longitudinal strain and the κ curvatures must be expressed on the deformed geometry, which requires the transformation between the deformed and undeformed coordinate systems. In the relevant literature, multiple solutions can be found, as will be discussed later.

3.1.3 Curvatures

The $\boldsymbol{T}_R$ rotation matrix can be obtained by the u, v and φ displacement functions (see Section 3.2.1), and once obtained, the curvatures (on the deformed geometry of the beam) can be expressed. The expressions are, see [12–16]:

$$\begin{aligned}\kappa_x &= l_z \frac{dl_y}{ds} + m_z \frac{dm_y}{ds} + n_z \frac{dn_y}{ds} \\ \kappa_y &= l_x \frac{dl_z}{ds} + m_x \frac{dm_z}{ds} + n_x \frac{dn_z}{ds} \\ \kappa_z &= l_y \frac{dl_x}{ds} + m_y \frac{dm_x}{ds} + n_y \frac{dn_x}{ds},\end{aligned} \tag{14}$$

where the elements of $\boldsymbol{T}_R$ are direction cosines:

$$\boldsymbol{T}_R = \begin{bmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix}. \tag{15}$$

The derivation with respect to length s can be approximated by the derivation with respect to the longitudinal coordinate z. It is also to note that the actual expressions for the curvatures are fairly long, and approximations are necessary, as will be discussed later.

3.1.4 Longitudinal normal strain

To calculate the work of the loads/stresses, strains are needed. Assuming that there are longitudinal stresses only (as usual in any beam-model-based solution), only the longitudinal normal strain is needed, which is derived from the translations. According to the Green-Lagrange strain tensor, the strain can be expressed as:

$$\varepsilon_z = \frac{\partial w_{xy}}{\partial z} + \frac{1}{2}\left(\left(\frac{\partial u_{xy}}{\partial z}\right)^2 + \left(\frac{\partial v_{xy}}{\partial z}\right)^2 + \left(\frac{\partial w_{xy}}{\partial z}\right)^2\right) \tag{16}$$

where u_{xy}, v_{xy}, and w_{xy} are the translation at an arbitrary cross-section point. As it is typical in classic buckling solutions, the $(\partial w_{xy}/\partial z)^2$ nonlinear term is neglected. All translations must be interpreted on the deflected geometry. According to e.g., [15], the translations of an arbitrary cross-section point (at the x, y position, with sectoral coordinate ω) can be expressed as:

$$\begin{bmatrix} u_{xy} \\ v_{xy} \\ w_{xy} \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \boldsymbol{T}_R \begin{bmatrix} x \\ y \\ -\omega\kappa_z \end{bmatrix} - \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \tag{17}$$

where u, v, and w are the translations at the centroid (and due to double symmetry, the shear center and centroid coincide). $\boldsymbol{T}_R$ is substituted into Eq. (17) to calculate u_{xy}, v_{xy}, and w_{xy}, then they are substituted into the strain expression Eq. (16). Without further simplifications, the final formula is extremely long (with many dozens of terms). However, many of these terms are zero if the cross-section is doubly-symmetric, due to the fact that y is measured from the centroid. As a result, the integral of all the terms in ε_z that are independent of y or contain y^2 are equal to zero. In other words, only the terms that are linearly dependent on y are necessary to consider. With this, the expression for ε_z is greatly simplified, but still might be long, hence, some approximations might be reasonable.

3.1.5 Equation system, critical load

The expressions for the curvatures and longitudinal strain can be substituted into the total potential formula. After the integrations, the total potential is expressed in terms of the displacement parameters, i.e., the displacement amplitudes u_m and φ_m. Note, v_m is not an independent displacement parameter, since it is defined by M_x, see Eq. (10). According to the theorem of stationarity of total potential, equilibrium exists if the total potential is stationary, i.e.:

$$\frac{\partial \Pi}{\partial u_m} = 0 \quad \frac{\partial \Pi}{\partial \varphi_m} = 0\,. \tag{18}$$

The above expressions form a system of two equations. Applying certain simplifications (which will be discussed later), the equations are linear, and can be written in matrix format as:

$$[\boldsymbol{C}]\begin{bmatrix} u_m \\ \varphi_m \end{bmatrix} = 0\,, \tag{19}$$

where $\boldsymbol{C}$ is a 2 × 2 coefficient matrix dependent on M_x. A nontrivial solution of the homogeneous system of linear equation exists if the coefficient matrix is singular, i.e., its determinant equals to zero.

$$\det(\boldsymbol{C}) = 0 \tag{20}$$

This condition can be satisfied if M_x takes specific value(s), which is (are) the critical moment(s) M_{cr}. To be able to solve the det($\boldsymbol{C}$) = 0 equation, further approximations might be necessary, as discussed in Section 3.2.

3.2 Variants and approximations in the derivation

3.2.1 Transformation matrix

Using the rotational angles α, β and φ about the x, y and z axis, respectively, the rotation matrix can be expressed. If the rotations are large and no approximations are introduced, it is relatively easy to define the transformation matrix by the sines and cosines of the rotational angles; in this case however, the order of how the rotations around the three axes occur matters. On the other hand, if the rotations are (very) small, the cosines can be taken as 1, and the sines can be approximated by the value of the angle; leading to the $\boldsymbol{T}_R$ transformation matrix being simple and independent of the order of the rotations. However, to solve the LTB problem with prebuckling deflections, moderately large rotations must be assumed. Essentially, the sine and cosine of an angle are approximated using the Taylor series expansion up to quadratic terms.

There are two ways in the relevant papers to express the transformation matrix. In [12], it can be understood that the rotations around x, y and z are applied one by one, then the sine and cosine terms are approximated by Taylor series, and in the transformation matrix, the linear terms and some quadratic terms are kept. Regarding the quadratic terms, the approximation is based on the logic that the secondary displacements are infinitesimally small (hence β and φ are small), but the primary displacement is moderately large (hence α is moderately large). Accordingly, only the quadratic terms associated with α are kept. The resulting transformation matrix is Eq. (23). Papers [15] and [18] present a different version of $\boldsymbol{T}_R$. Though [15] and [18] use different mathematical apparatus, the same transformation matrix is derived when expressed by the angles, as Eq. (24).

The rotation angles must be expressed using the displacement functions. The angle about the longitudinal axis is directly given by the φ function. For α and β, there are two alternatives in the literature. The simplest approximation, used in [12], is Eq. (21):

$$\begin{aligned} \alpha &= -\frac{dv}{dz} = -v' \\ \beta &= \frac{du}{dz} = u'. \end{aligned} \tag{21}$$

In [15] and [18], however, α and β are approximated more accurately as:

$$\begin{aligned} \alpha &= -\frac{dv}{dz} + \frac{1}{2}\varphi\frac{du}{dz} = -v' + \frac{1}{2}\varphi u' \\ \beta &= \frac{du}{dz} + \varphi\frac{dv}{dz} = u' + \frac{1}{2}\varphi v'. \end{aligned} \tag{22}$$

To obtain the necessary transformation matrix, Eq. (23) or Eq. (24) must be substituted into either Eq. (21) or Eq. (22). In [12], Eq. (21) is substituted into Eq. (23) which leads to a transformation matrix as Eq. (25).

However, if Eq. (22) is substituted into Eq. (24), it leads to a transformation matrix with entries up to 4th-order terms, and it is reasonable to introduce approximations. If the 4th-order terms are eliminated, the resulting matrix is Eq. (26). In [18], the transformation matrix is essentially similar to Eq. (26), but some 3rd-order terms are eliminated, namely from entries (1,3) and (2,3). The resulting matrix is then Eq. (27). In [15], the transformation matrix is similar to Eq. (26) or Eq. (27), but further simplified as Eq. (28). It can be noticed that the 3rd-order terms are eliminated from (1,3) and (2,3), plus, the entries (1,2) and (3,2) are modified. This modification is not mentioned, hence not commented in the paper. Several further variants of the matrix could be defined, depending on what terms are eliminated or kept. Since it is a widely used engineering approximation to eliminate all the 3rd-order terms, the second-order approximation is provided here as Eq. (29).

$$\boldsymbol{T}_{R,\text{angle}}^{Va} = \begin{bmatrix} 1 & -\varphi & \beta \\ \varphi + \alpha\beta & 1 - \frac{1}{2}\alpha^2 & -\alpha \\ -\beta + \alpha\varphi & \alpha & 1 - \frac{1}{2}\alpha^2 \end{bmatrix} \tag{23}$$

$$\boldsymbol{T}_{R,\text{angle}}^{Pi} = \begin{bmatrix} 1 - \frac{1}{2}\beta^2 - \frac{1}{2}\varphi^2 & -\varphi + \frac{1}{2}\alpha\beta & \beta + \frac{1}{2}\alpha\varphi \\ \varphi + \frac{1}{2}\alpha\beta & 1 - \frac{1}{2}\alpha^2 - \frac{1}{2}\varphi^2 & -\alpha + \frac{1}{2}\beta\varphi \\ -\beta + \frac{1}{2}\alpha\varphi & \alpha + \frac{1}{2}\beta\varphi & 1 - \frac{1}{2}\alpha^2 - \frac{1}{2}\beta^2 \end{bmatrix} \tag{24}$$

$$\boldsymbol{T}_{R}^{Va} = \begin{bmatrix} 1 & -\varphi & u' \\ \varphi - u'v' & 1 - \frac{1}{2}(v')^2 & v' \\ -u' - v'\varphi & -v' & 1 - \frac{1}{2}(v')^2 \end{bmatrix} \tag{25}$$

$$\boldsymbol{T}_R^{3\text{rd}} = \begin{bmatrix} 1-\frac{1}{2}(u')^2-\frac{1}{2}\varphi^2-\frac{1}{2}u'v'\varphi & -\varphi-\frac{1}{2}u'v'+\frac{1}{4}(u')^2\varphi-\frac{1}{4}(v')^2\varphi & u'+\frac{1}{4}u'\varphi^2 \\ \varphi-\frac{1}{2}u'v'+\frac{1}{4}(u')^2\varphi-\frac{1}{4}(v')^2\varphi & 1-\frac{1}{2}(v')^2-\frac{1}{2}\varphi^2+\frac{1}{2}u'v'\varphi & v'+\frac{1}{4}v'\varphi^2 \\ -u'-v'\varphi+\frac{1}{4}u'\varphi^2 & -v'+u'\varphi+\frac{1}{4}v'\varphi^2 & 1-\frac{1}{2}(u')^2-\frac{1}{2}(v')^2 \end{bmatrix} \tag{26}$$

$$\boldsymbol{T}_R^{To} = \begin{bmatrix} 1-\frac{1}{2}(u')^2-\frac{1}{2}\varphi^2-\frac{1}{2}u'v'\varphi & -\varphi-\frac{1}{2}u'v'+\frac{1}{4}(u')^2\varphi-\frac{1}{4}(v')^2\varphi & u' \\ \varphi-\frac{1}{2}u'v'+\frac{1}{4}(u')^2\varphi-\frac{1}{4}(v')^2\varphi & 1-\frac{1}{2}(v')^2-\frac{1}{2}\varphi^2+\frac{1}{2}u'v'\varphi & v' \\ -u'-v'\varphi+\frac{1}{4}u'\varphi^2 & -v'+u'\varphi+\frac{1}{4}v'\varphi^2 & 1-\frac{1}{2}(u')^2-\frac{1}{2}(v')^2 \end{bmatrix} \tag{27}$$

$$\boldsymbol{T}_R^{Pi} = \begin{bmatrix} 1-\frac{1}{2}(u')^2-\frac{1}{2}\varphi^2-\frac{1}{2}u'v'\varphi & -\varphi-\frac{1}{2}u'v'+\frac{1}{2}(u')^2\varphi & u' \\ \varphi-\frac{1}{2}u'v'-\frac{1}{2}(v')^2\varphi & 1-\frac{1}{2}(v')^2-\frac{1}{2}\varphi^2+\frac{1}{2}u'v'\varphi & v' \\ -u'-v'\varphi+\frac{1}{2}u'\varphi^2 & -v'+u'\varphi+\frac{1}{2}v'\varphi^2 & 1-\frac{1}{2}(u')^2-\frac{1}{2}(v')^2 \end{bmatrix} \tag{28}$$

$$\boldsymbol{T}_R^{2\text{nd}} = \begin{bmatrix} 1-\frac{1}{2}(u')^2-\frac{1}{2}\varphi^2 & -\varphi-\frac{1}{2}u'v' & u' \\ \varphi-\frac{1}{2}u'v' & 1-\frac{1}{2}(v')^2-\frac{1}{2}\varphi^2 & v' \\ -u'-v'\varphi & -v'+u'\varphi & 1-\frac{1}{2}(u')^2-\frac{1}{2}(v')^2 \end{bmatrix} \tag{29}$$

3.2.2 Curvatures

Using one of the above transformation matrices and considering Eqs. (14) and (15), the curvatures can be expressed in terms of the displacement functions. The obtained formulae are long. For example, using $\boldsymbol{T}_R^{Pi}$, the curvature formulae have 9, 7 and 11 terms for κ_x, κ_y, and κ_z, respectively. Most of the terms are higher-order.

If the linear and quadratic terms are kept, the curvatures are expressed (from almost any of the above-mentioned variants, with the exception of $\boldsymbol{T}_R^{Va}$) as follows:

$$\begin{aligned} \kappa_x &= -v'' + \varphi u'' \\ \kappa_y &= u'' + \varphi v'' \\ \kappa_z^{gen} &= \varphi' - \frac{1}{2}u'v'' + \frac{1}{2}u''v'. \end{aligned} \tag{30}$$

On the other hand, the κ_z curvature obtained from $\boldsymbol{T}_R^{Va}$ is slightly different:

$$\kappa_z^{Va} = \varphi' - u'v''. \tag{31}$$

It is to note that in [13, 14], another equation is used for κ_z (derived differently, not directly from a transformation matrix) as follows:

$$\kappa_z^{Ro} = \varphi' - u'v'' + u''v'. \tag{32}$$

It is to observe that there is agreement in the literature on how to express κ_y, while various variants for κ_z exist. In κ_x, the 2nd-order term is sometimes eliminated, however, this has no effect on the critical moment formula, since κ_x is not directly employed in the derivations.

3.2.3 Longitudinal strain

Following the steps described in Section 3.1.4, the longitudinal normal strain is expressed through the displacements of the beam's system line. The actual expression depends on the considered rotation matrix, but typically has one first-order term, one second-order term, several third-order terms, and several fourth-order terms. According to the logic of the linear buckling analysis, the first-order term should be disregarded. It might also be reasonable to neglect the fourth-order terms. With these eliminations, there is a second-order term and some third-order terms. A few possible expressions are given here, as follows:

- from $\boldsymbol{T}_R^{2\text{nd}}$:

$$\varepsilon_z^{2\text{nd}} = \left(\varphi u'' - \varphi\varphi' v' - \frac{1}{2}(u')^2 v'' - (v')^2 v'' - \frac{1}{2} u' u'' v' \right) y \tag{33}$$

- from $\boldsymbol{T}_R^{3\text{rd}}$ or $\boldsymbol{T}_R^{To}$:

$$\varepsilon_z^{3\text{rd}} = \begin{pmatrix} \varphi u'' + \frac{1}{4}\varphi^2 v'' - \frac{1}{2}(u')^2 v'' - (v')^2 v'' \\ -\frac{1}{2}\varphi\varphi' v' - \frac{1}{2} u' u'' v' \end{pmatrix} y \tag{34}$$

- from $\boldsymbol{T}_R^{Pi}$:

$$\varepsilon_z^{Pi} = \left(\varphi u'' + \frac{1}{2}\varphi^2 v'' - \frac{1}{2}(u')^2 v'' - (v')^2 v'' - \frac{1}{2} u' u'' v' \right) y . \tag{35}$$

In [13–15], ε_z^{Pi} is further simplified by keeping one single 3rd-order term only, as follows:

$$\varepsilon_z^{Pi,\text{simple}} = \left(\varphi u'' + \frac{1}{2}\varphi^2 v'' \right) y . \tag{36}$$

From $\boldsymbol{T}_R^{3\text{rd}}$ or $\boldsymbol{T}_R^{To}$, but with keeping only one 3rd-order term similarly to the previous case:

$$\varepsilon_z^{3\text{rd},\text{simple}} = \left(\varphi u'' + \frac{1}{4}\varphi^2 v'' \right) y . \tag{37}$$

From any $\boldsymbol{T}_R$, if only the single second-order term is kept:

$$\varepsilon_z^{2\text{nd},\text{simple}} = (\varphi u'') y . \tag{38}$$

3.2.4 Approximations due to cross-section characteristics

If the cross-section is open, it is reasonable to introduce approximations (which will be referred to as "open") as follows:

$$\left(\frac{EI_w / L^2}{EI_x} \right)^2 \cong 0 \quad \left(\frac{GJ}{EI_x} \right)^2 \cong 0 \quad \frac{GJ\left(EI_w / L^2\right)}{(EI_x)^2} \cong 0 . \tag{39}$$

The formulae can be further simplified (which option will be referred to as "open-simple") assuming that:

$$\frac{EI_w / L^2}{EI_x} \cong 0 \quad \frac{GJ}{EI_x} \cong 0 . \tag{40}$$

If the cross-section is closed, the warping is negligible, but the Saint-Venant torsion rigidity is significant, hence the following approximation might be used (referred to as option "closed"):

$$\frac{EI_w / L^2}{EI_x} \cong 0 . \tag{41}$$

The formulae can further be simplified (resulting in option "closed-simple") assuming that:

$$\frac{EI_w / L^2}{EI_x} \cong 0 \quad \left(\frac{GJ}{EI_x} \right)^2 \cong 0 . \tag{42}$$

3.2.5 Optional reduction of equation degree

Even if the above-discussed approximations are introduced, the final equation, i.e., Eq. (20), from which the critical moment can be calculated, is of 4th-degree. Since there is no cubic term in the equation, it can still be solved, and a closed-form expression (even if long) can be obtained for M_{cr}. However, in the literature the higher-degree terms are always eliminated and finally the critical moment is calculated from a simplified quadratic equation.

4 Critical moment variants

4.1 Doubly-symmetric open sections

The derivation of the critical moment can be completed as summarized in Section 3.1, but the final result (e.g., final expression for the critical moment) is dependent on various details. The determining factors are as follows: the $\boldsymbol{T}_R$, the curvatures, the longitudinal strain, the assumed stiffness ratios of the cross-section, and the potential elimination of higher-degree M_{cr} terms in the final equation.

The elements of $\boldsymbol{T}_R$ matrix are combined from the displacement functions and their derivatives. The curvatures and the longitudinal normal strain are also expressed by the combination of the displacement functions and their derivatives. In most structural engineering stability problems, when displacement functions and/or derivatives are combined, it is appropriate to eliminate cubic or higher-order terms. However, the literature suggests that to have the critical moment with prebuckling deflections, third-order terms are required, too, in the $\boldsymbol{T}_R$ transformation matrix, and in the ε_z strain, but it is not clear which cubic terms are necessary.

Moreover, in the literature, I-shaped and (narrow) rectangular sections are discussed, and some stiffness values are assumed to be small (compared to others), but – in many cases – without introducing a consistent assumption system. Closed sections with high torsional rigidity (e.g., RHS) are not discussed, therefore, it remains unknown how the assumed stiffness ratios affect the results.

Finally, the M_{cr} formulae in the literature are solutions of quadratic equations. However, this is possible only if the higher-degree M_{cr} terms are eliminated. It is questionable whether the effect of this simplification can always be justified.

The solution for M_{cr} is, therefore, very far from being unambiguous; this explains why various formulae are found in various papers. Actually, several dozens of different M_{cr} formulae could be derived. A few possible formulae are presented here, to demonstrate which options lead to the formulae found in the literature, and how the details of the derivations influence the final results.

First, open cross-sections are considered, therefore "open" and "open-simple" options are employed. The considered derivation variants are summarized in Table 1. Variant (*ref*), used here as a reference, is from the simplest available formula, which is the most frequently shown in available papers, e.g., [1, 19, 21, 23, 27]. Variant (*a*) is the reproduction of the Pi-Trahair formula as in [16]. Variant (*b*) is the reproduction of the formula in [12] and [14]. Variant (*c*) is obtained by applying the geometric approximations proposed by [18]. Variant (*d*) is similar to (*c*), but the simplified formula is employed for the longitudinal strain (similarly to the simplified longitudinal strain formula by Pi and Trahair [16]). Variant (*e*) is obtained by a consistent quadratic approximation in each step (i.e., eliminating the cubic terms systematically). The variants denoted by (+) are included here in order to observe the influence of neglecting or considering the 4th-degree term in the final equation. Accordingly, variant (*a*+) is similar to (*a*), but the final equation is 4th-degree, and variant (*c*+) is similar to (*c*), but the final equation is 4th-degre. The obtained formulae are shown in Eqs. (44)–(49), with M_{cr0} in all of them being:

$$M_{cr0} = \frac{\pi}{L}\sqrt{EI_y\left(GJ + \frac{\pi^2 EI_w}{L^2}\right)} \tag{43}$$

$$M_{cr}^{(ref)} = M_{cr0}\Big/\sqrt{1-\frac{EI_y}{EI_x}} = M_{cr0}\Big/\sqrt{1-\frac{I_y}{I_x}} \tag{44}$$

$$M_{cr}^{(a)} = M_{cr0}\Big/\sqrt{\left(1-\frac{EI_y}{EI_x}-\frac{GJ}{2EI_x}-\frac{L^2 EI_w}{2EI_x L^2}+\frac{GJEI_y}{2\left(EI_x\right)^2}+\frac{\pi^2 EI_w EI_y}{2\left(EI_x\right)^2 L^2}\right)} \tag{45}$$

$$M_{cr}^{(b)} = M_{cr0}\Big/\sqrt{\left(1-\frac{EI_y}{EI_x}-\frac{GJ}{EI_x}-\frac{\pi^2 EI_w}{EI_x L^2}+\frac{GJEI_y}{\left(EI_x\right)^2}+\frac{\pi^2 EI_w EI_y}{\left(EI_x\right)^2 L^2}\right)} \tag{46}$$

$$M_{cr}^{(c)} = M_{cr0}\Big/\sqrt{\left(1-\frac{EI_y}{EI_x}-\frac{2GJ}{EI_x}-\frac{2\pi^2 EI_w}{EI_x L^2}+\frac{GJEI_y}{2\left(EI_x\right)^2}+\frac{\pi^2 EI_w EI_y}{2\left(EI_x\right)^2 L^2}\right)} \tag{47}$$

$$M_{cr}^{(d)} = M_{cr0}\Big/\sqrt{\left(1-\frac{3EI_y}{2EI_x}-\frac{GJ}{2EI_x}-\frac{\pi^2 EI_w}{2EI_x L^2}+\frac{GJEI_y}{2\left(EI_x\right)^2}+\frac{\pi^2 EI_w EI_y}{2\left(EI_x\right)^2 L^2}\right)} \tag{48}$$

$$M_{cr}^{(e)} = M_{cr0}\Big/\sqrt{\left(1-\frac{2EI_y}{EI_x}-\frac{GJ}{2EI_x}-\frac{\pi^2 EI_w}{2EI_x L^2}+\frac{GJEI_y}{2\left(EI_x\right)^2}+\frac{\pi^2 EI_w EI_y}{2\left(EI_x\right)^2 L^2}\right)}. \tag{49}$$

It is to note that in the case of variants (*a*+) and (*c*+), M_{cr} is calculated from a 4-th-degree equation; the obtained formulae can be expressed in closed format, but they are relatively long, thus, not presented here.

It is obvious that the expressions for M_{cr} are dependent on the details of the derivation, leading to different M_{cr} formulae. To be able to evaluate the differences, a simple numerical study is provided. Obviously, the critical moment values are dependent on the cross-section properties, the beam length, and the material constants, however, here the focus is on the effect of the derivation details, therefore, hypothetical cross-sections are used with assumed stiffness ratios. Considering typical doubly-symmetric I-shaped steel sections, it can be observed that GJ/EI_x is around (or

Table 1 Summary of options considered for DSI sections

Variant	Transf. matrix	Curvatures	Nonlinear longit. strain	Cross-section model	Final equation	Equations
(*ref*)	$\boldsymbol{T}_R^{2\text{nd}}$	κ_y and κ_z^{gen}	$\varepsilon_z^{Pi,\text{simple}}$	open simple	quadratic	Eq. (44)
(*a*)	$\boldsymbol{T}_R^{Pi}$	κ_y and κ_z^{gen}	$\varepsilon_z^{Pi,\text{simple}}$	open	quadratic	Eq. (45)
(*b*)	no T_R	κ_y and κ_z^{Ro}	$\varepsilon_z^{Pi,\text{simple}}$	open	quadratic	Eq. (46)
(*c*)	$\boldsymbol{T}_R^{To}$ or $\boldsymbol{T}_R^{2\text{nd}}$	κ_y and κ_z^{gen}	$\varepsilon_z^{3\text{rd}}$	open	quadratic	Eq. (47)
(*d*)	$\boldsymbol{T}_R^{To}$	κ_y and κ_z^{gen}	$\varepsilon_z^{3\text{rd},\text{simple}}$	open	quadratic	Eq. (48)
(*e*)	$\boldsymbol{T}_R^{2\text{nd}}$	κ_y and κ_z^{gen}	$\varepsilon_z^{2\text{nd},\text{simple}}$	open	quadratic	Eq. (49)
(*a*+)	$\boldsymbol{T}_R^{Pi}$	κ_y and κ_z^{gen}	$\varepsilon_z^{Pi,\text{simple}}$	open	4th-degree	-
(*c*+)	$\boldsymbol{T}_R^{To}$	κ_y and κ_z^{gen}	$\varepsilon_z^{3\text{rd}}$	open	4th-degree	-

smaller) than 0.01. Also, $\pi^2 EI_w/L^2/EI_x$ is around (or smaller) than 0.001. By assuming these rigidity ratios, the solution becomes independent of the length and material. The critical moment increase, i.e., the $(M_{cr} - M_{cr0})/M_{cr0}$ values are plotted in Fig. 3 for various, practically relevant I_y/I_x ratios.

Since for the given basic case of LTB, the solutions by [16], i.e., (*a*) and by [12], i.e., (*b*) are re-derived by various researchers, it is fair to assume that these solutions are reasonably correct. It can be observed that variant (*c*) results are very similar to those from (*a*) and (*b*).

It is clear that the simplest, so-called reference formula yields nearly the same results. This means that it is reasonable to use "open-simple" option in practical cases (at least in the basic case).

Moreover, the results seem to justify the suggestion from various papers that the effect of prebuckling deflection can be accounted for by the $1/\sqrt{1-I_y/I_x}$ factor. Mathematically, however, this is simply due to the fact that the $EI_w/L^2/EI_x$ and GJ/EI_x rigidity ratios are small for practical I-shaped steel sections.

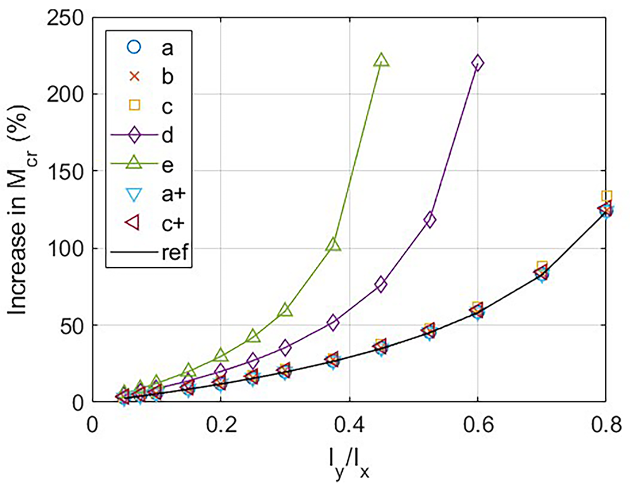

Fig. 3 Open sections: moment increase due to prebuckling deflection

It is clear from the formulae that no real root exists if is large in any variant. In most variants, the $I_y/I_x = 1$ is the point of singularity; while in variants (*d*) and (*e*), the singularity occurs for much smaller value of I_y/I_x. Moreover, for medium I_y/I_x values, variant (*d*) and (*e*) lead to results very different from any other variants. Therefore, variant (*d*) and (*e*) can be judged as incorrect. The results suggest that in these options, some important terms are missing from the displacement approximations, leading to poor approximation(s) of the function(s), which finally leads to poor prediction for the critical moment.

It can be also observed that the 4th-degree moment term in the final equation has very little effect. This is particularly true when comparing (*a*) and (*a*+); though the M_{cr} values are not equal, the difference is extremely small.

4.2 Doubly-symmetric closed sections

Unlike in open sections, the torsion rigidity is significant in closed sections, and this has an effect on the M_{cr} formulae. Two of the above-mentioned variants are therefore re-calculated, using "closed" and "closed-simple" cross-section approximations. Namely: variant (*a*) and variant (*c*) are considered, (*a*1) and (*c*1) being the simplified, (*a*2) and (*c*2) being the more complex ones. Moreover, the effect of eliminating the 4th-degree term in the final equation is illustrated in variants (*a*1) and (*c*1): if the 4th-degree term is kept, the resulting variants are identified as (*a*1+) and (*c*1+), respectively. The characteristics of the variants are summarized in Table 2. The obtained formulae are summarized as Eqs. (50)–(53). In the case of variants (*a*1+) and (*c*1+) M_{cr} is calculated from a 4-th-degree equations; the formulae are long, therefore not presented here.

Table 2 Summary of options considered for RHS sections

Variant	Transf. matrix	Curvatures	Nonlinear longit. strain	Cross-section model	Final equation	Equations
(*a*1)	$\boldsymbol{T}_R^{Pi}$	κ_y and κ_z^{gen}	$\varepsilon_z^{Pi,\text{simple}}$	closed-simple	quadratic	Eq. (50)
(*a*2)	$\boldsymbol{T}_R^{Pi}$	κ_y and κ_z^{gen}	$\varepsilon_z^{Pi,\text{simple}}$	closed	quadratic	Eq. (51)
(*c*1)	$\boldsymbol{T}_R^{To}$ or $\boldsymbol{T}_R^{2\text{nd}}$	κ_y and κ_z^{gen}	$\varepsilon_z^{3\text{rd}}$ or $\varepsilon_z^{2\text{nd}}$	closed-simple	quadratic	Eq. (52)
(*c*2)	$\boldsymbol{T}_R^{To}$ or $\boldsymbol{T}_R^{2\text{nd}}$	κ_y and κ_z^{gen}	$\varepsilon_z^{3\text{rd}}$ or $\varepsilon_z^{2\text{nd}}$	closed	quadratic	Eq. (53)
(*a*1+)	$\boldsymbol{T}_R^{Pi}$	κ_y and κ_z^{gen}	$\varepsilon_z^{Pi,\text{simple}}$	closed-simple	4th-degree	-
(*c*1+)	$\boldsymbol{T}_R^{To}$ or $\boldsymbol{T}_R^{2\text{nd}}$	κ_y and κ_z^{gen}	$\varepsilon_z^{3\text{rd}}$ or $\varepsilon_z^{2\text{nd}}$	closed-simple	4th-degree	-

$$M_{cr}^{(a1)} = M_{cr0} \Big/ \sqrt{1-\frac{EI_y}{EI_x}-\frac{GJ}{2EI_x}+\frac{EI_y GJ}{2(EI_x)^2}} = M_{cr0} \Big/ \sqrt{\left(1-\frac{EI_y}{EI_x}\right)\left(1-\frac{GJ}{2EI_x}\right)} \tag{50}$$

$$M_{cr}^{(a2)} = M_{cr0} \Big/ \sqrt{1-\frac{EI_y}{EI_x}-\frac{GJ}{2EI_x}+\frac{EI_y GJ}{2(EI_x)^2}+\frac{9(GJ)^2}{48(EI_x)^2}-\frac{\pi^2(GJ)^2}{48(EI_x)^2}} \tag{51}$$

$$M_{cr}^{(c1)} = M_{cr0} \Big/ \sqrt{1-\frac{EI_y}{EI_x}-\frac{2GJ}{EI_x}+\frac{EI_y GJ}{2(EI_x)^2}} \tag{52}$$

$$M_{cr}^{(c2)} = M_{cr0} \Big/ \sqrt{1-\frac{EI_y}{EI_x}-\frac{2GJ}{EI_x}+\frac{EI_y GJ}{2(EI_x)^2}+\frac{9(GJ)^2}{48(EI_x)^2}-\frac{\pi^2(GJ)^2}{48(EI_x)^2}} \tag{53}$$

To illustrate the similarities and differences between the variants, numerical values are provided, using hypothetical stiffness properties. In practical RHS sections the torsional stiffness can be approximated as $GJ/EI_x = 0.6(I_y/I_x)^{2/3}$. With this assumption, the moment increase values can be calculated, and the results are shown in Fig. 4. Fig. 4 shows that the effect of *GJ* is non-negligible in the case of closed sections; in fact, it increases the M_{cr}/M_{cr0} ratio. However, usually, the higher-degree terms with *GJ* have very small effect. Though variants (*a*) and (*c*) have been found to be very similar for open cross-sections, they lead to rather different M_{cr}/M_{cr0} ratios for closed sections, particularly if the final equation is quadratic.

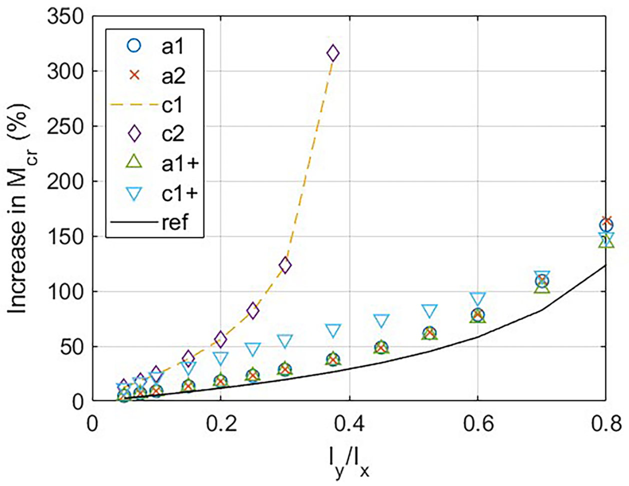

Fig. 4 Closed sections: moment increase due to prebuckling deflection

It is to mention that in all these cases, M_{cr0} is the same as in Eq. (43). The 4th-degree moment term in the final equation has noticeable effect; in variant (*a*) the effect is relatively small, but in variant (*c*) it seems to be absolutely necessary to keep the 4th-degree moment term, otherwise the results look unrealistic.

5 Conclusions

In this paper, analytical solutions for the lateral-torsional buckling of thin-walled beams, considering the effect of prebuckling deformations, were discussed. Doubly symmetric beams with cross sections with low and high torsional rigidities were considered. The critical moment formulae proposed in earlier papers for simply supported beams subjected to uniform moment were re-derived, identifying the crucial decision points which can/will influence the final formula. Then, demonstrative numerical results were presented. The main conclusions are as follows.

In the analytical derivations, the transformation of displacements is necessary due to the 3D rotations of the system line of the beam. Since the rotations are not necessarily small, the transformation can be realized in multiple ways. Moreover, during the derivations, many higher-order terms show up, and some of them are important, while others are not. It is not self-evident which terms should be kept, and which terms can be eliminated; earlier papers show a significant scatter in this regard. Moreover, in certain publications some inconsistencies can be found. All these factors lead to variations in the end results.

Both the derivations presented in this paper and the in-depth study of the literature suggest that approximations should be done carefully since they might lead to erroneous results if done improperly. The results suggest that in the curvatures, up to second-order terms are necessary and enough to consider. In the longitudinal normal strain, however, 3rd-order terms are necessary too. It might be enough to consider selected 3rd-order term(s), but they need to be carefully selected. Regarding the transformation matrix, though in certain cases it is enough to consider the second-order terms only, but in other cases higher-order terms are necessary, too; therefore, considering the 3rd-order terms is recommended. In general, the results suggest that the set of approximations proposed and applied by Pi and Trahair [15, 16] lead to reasonably precise critical moment values for a wide range of lateral-torsional problems.

The derived formulae clearly show that the torsional rigidity have important effect. The suggestion of multiple previous papers that the critical moment increase can be approximated by the $1/\sqrt{1-I_y/I_x}$ ratio can be confirmed in the case of beams with small torsional rigidity (e.g., open thin-walled sections). Furthermore, the prebuckling effect is modified if the torsional rigidity is significant (e.g., closed thin-walled sections), with closed sections experiencing a higher effect of prebuckling deformations. This means that the appropriate closed form solutions can vary depending on several factors, and no single "exact" formula exists.

This paper serves as a first step in a more comprehensive investigation of the problem. Although it gives a good overview for the derivation of analytical solutions, other methods, such as numerical investigations, are necessary to validate which formulae are suitable for different cases. This is why further studies of the topic of prebuckling deformations effect on the critical moment of LTB of beams considering other factors and cases, such as the effect of end supports, intermediate supports, as well numerical investigations using beam and shell FEM elements are conducted and will be presented in future papers.

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