PERMUTATIONS WITH A GIVEN NUMBER OF INVERSIONS

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Abstract

In this paper a kind of generalized Pascal triangle is constructed whose k'th entry in its n'th row equals the number of permutations of degree n having exactly k inversions. Let P_n^k be the number of the n-degree permutations having exactly k inversions. Then

$$P_n^k = 0, \text{ if } k \begin{matrix} > \binom{n}{2} + 1, \\ < 0 \end{matrix}$$

so it is presented an algorithm which needs polynomial time only:

$$P_n^k = P_{n-1}^k + \ldots + P_{n-1}^{k-n+1}.$$

Finally it is given a method that the n'th row of our GPT contains $1 + \binom{n}{2}$ (non-zero) entries and the computation of the n'th row requires roughly n^4 operations.

The trivial algorithm determining the number of permutations of n letters having a given number of inversions works in exponential time. That is the trivial algorithm consisting of the cheking the number of inversions in every permutation of degree n requires a time exponentially depending on n.

Here we present another algorithm which needs polynomial time only. Our algorithm consists of the contruction of a "Generalized Pascal triangle" whose k'th entry in its n'th row equals the number of permutations of degree n having exactly k inversions.

Let f and g be number-theorical functions whose values are also natural numbers. We present an infinite matrix, called the *Generalized Pascal triangle* (shortly: GPT) *for the pair* (f, g) by the following rules:

1. The entries of the matrix will be indexed by pairs (i, j) with i natural and j arbitrary integer numbers. For such an entry, we write $[f, g]_i^j$ where the lower index indicates the row, and the upper one the column of the matrix containing the considered entry.

2. $[f, g]_1^0 = 1$, and $[f, g]_1^j = 0$ if $j \neq 0$.

This rule expresses how to fill in the first row of our matrix.

The next rules express how do the following rows depend on the number-theorical functions f and g.

3. If, for any, there exist exactly m_i non-zero entries in the i'th row, then there exist exactly $m_i + f(i)$ non-zero entries in the $i+1$'th row, namely $[f, g]_{i+1}^j \neq 0$ for $j = -m_i - f_i + 1, -m_i - f_i + 3, \ldots, m_i + f_i - 1$.

4. If $[f, g]_{i+1}^j \neq 0$ then $[f, g]_{i+1}^j = \sum_{k=j-g(i)}^{j+g(i)} [f, g]_i^k$.

This rule formulates how many and which entries of the i'th row have to be summed up for obtaining the entries of the $i+1$'th row.

The GPT in the case when $f = g = 1$ turns into the common Pascal triangle consisting of the binomial coefficients

Really by the given rules the matrix will have the following entries.

...	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	...
1							1							
2						1		1						
3					1		2		1					
4				1		3		3		1				
5			1		4		6		4		1			

Another special case of our notion of GPT appears in Vilenkin's popular book in combinatorics where the case $f = g = m-1$ (i.e., both f and g are constant) is treated: the resulting entries give the number of n-digits numbers written in m-ary system with sum of digits k.

...	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6 ...
1							1						
2					1		1		1				
3			1		2		3		2		1		
4	1		3		6		7		6		3		1
5...	4		10		16		19		16		10		4

This table displays the case $m = 3$. (The Pascal triangle was the case $m = 2$.) Let P_n^k be the number of n-degree permutations having exactly k inversions. Then we'd write the following

Theorem: For every natural number n, *the non-zero entries of the n'th row of the GPT* for $f(\mathrm{n}) = g(\mathrm{n}) = n$, *are (from left to right)*

$$P_n^0, \ldots, P_n^{\binom{n}{2}}.$$

where f and g are identical functions.

The first five rows of the GPT in the theorem are displayed here:

...	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6 ...
1							1						
2						1		1					
3				1		2		2		1			
4	1		3		5		6		5		3		1
5...	9		15		20		22		20		15		9 ...

Proof: First we remark that our GPT has

$$1 + \sum_{i=1}^{n-1} f(i) = 1 + \binom{n}{2}$$

non-zero entries has in its n'th row.

The number of inversion of n-degree permutations takes on also $1 + \binom{n}{2}$ values. Define the numbers a_i^j $(i = 1, 2, \ldots; j = \ldots, -1, 0, 1, \ldots)$ as follows:

$$a_i^j = P_i^k \text{ if } j = -\binom{i}{2} + 2k, \text{ else } a_i^j = 0 \left(0 = k = \binom{i}{2}\right). \tag{1}$$

Observe that $[\iota, \iota]_i^j \neq 0$ if and only if $a_i^j \neq 0$. We prove that

$$[\iota, \iota] = a_i^j \tag{2}$$

for every i and j. In view of (1), this will prove the theorem. We have $[\iota, \iota]_1^0 = a_1^0 = 1$. Suppose that (2) is valid for $i = n - 1$. Consider a_n^j; if it equals 0, then $[\iota, \iota]_n^j = 0$. Otherwise there exists a $k \in \left\{0, \ldots, \binom{i}{2}\right\}$ such that $a_n^j = P_n^k$ and $j = -\binom{n}{2} + 2k$. By part 4, in the definition of a GPT, it is enough to prove

$$a_n^j = \sum_{t=j-n+1}^{j+n-1} a_{n-1}^t. \tag{3}$$

Now for some $t = j + r\big(-(n-1) \leq r \leq n - 1\big)$ let $a_{n-1}^t \neq 0$, i.e., $a_{n-1}^t = P_{n-1}^l$, where $t = -\binom{n-1}{2} + 2l$. Comparing the distinct expressions for t, we obtain

$$-\binom{n}{2} + 2k + r = -\binom{n-1}{2} - n + 1 + 2k + r = -\binom{n-1}{2} + 2l$$

whence $k - n + 1 \leq l \leq k$ follows. Thus (3) may be rewritten in the form

$$P_n^k = P_{n-1}^k + \ldots + P_{n-1}^{k-n+1}. \tag{4}$$

The following observation implies (4): all permutations of $\{1, 2, \ldots, n\}$ with k inversions may be obtained (and each of them only once) if we take all permutations of $\{1, \ldots, n-1\}$ having at most k and at least $k - n + 1$ inversions, and insert the element n into each permutation so that the new permutation (of $\{1, \ldots, n\}$) had exactly k inversions. So, the theorem is proved.

What is the time required by this algorithm?

Since the n'th row of our GPT contains $1 + \binom{n}{2}$ (non-zero) entries and each of them may be got by $n - 1$ additions (from the entries of the preceding row), the computation of the n'th row from the $n - 1$'th one requires roughly n^3 operations, thus the computation of the n'th row requires summarily roughly n^4 operations.

References

1. VILENKIN, N. Ya.: Combinatorics (in Russian), Fizmatgiz, Moscow, 1969.

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