COMPUTATION OF THE TWO-DIMENSIONAL FLOW DEVELOPING IN THE INTERNAL LAKE OF KIS-BALATON

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Abstract

The hydraulical phenomena in nature are three-dimensional (3D) and varying in time. In most practical cases it is sufficient to consider those phenomena as two-dimensional (2D) in the horizontal plane and varying in time. These phenomena occur primarily where the acceleration along the vertical direction is negligible compared to the gravitational one. Disregarding the variation of hydraulical parameters along the vertical direction we obtain a homogeneous horizontal flow. This phenomenon is described by the relatively simple Reynolds equations. In most practical cases the variation of velocity along the vertical direction has to be taken into account. The integral equations in which the depth average is taken into account are approaching the three-dimensional phenomena rather well.

In our study we performed the result of our several years' research in the field of mathematical modelling of two-dimensional hydrodynamical and transport processes.

The mathematical background was introduced in *paragraph 1.* We showed how the two-dimensional equations (7)—(9) can be achieved from Reynolds equations, valid for the time — averaged mean hydraulic characteristics at a certain point of a three-dimensional turbulent flow. After deriving a closed set of equations, which can be solved, we introduced the numerical solution method of the implicit finite differences in four steps and in alternative direction.

We showed the calibration of the model to prove the accuracy of the results in *paragraph 2.* That is why we compared the results of our model with laboratory measurements. We simulated the phenomenon forming at the tailbay of a hydropower station and compared the flow patterns of the mathematical and physical models (Figs 2 and 3). We proved, that our model and its computer program are suitable for the computation of flows that may be assumed as two-dimensional in the horizontal plane.

In *paragraph 3* the application of the model for the Internal Lake of Kis-Balaton is shown. We computed the near steady flow pattern, which forms when both the water intake and outlet was 10 m^3/s (Fig. 4).

Sensitivity tests were performed for both the velocity coefficient, C (Fig. 5) and the eddy viscosity, ν. We established that neither the determination of C nor that of ν needs field measurements. Satisfactory accuracy can be achieved if one uses the values and formulae based on former experiences, laboratory experiments or data found in some publications for the estimation of C and ν.

In *paragraph 4* we briefly performed the application of the transport-diffusion equation for the Internal Lake of Kis-Balaton. With the help of an example it is shown, that we can compute the concentration of pollutants in a two dimensional space, varying in time (Figs 6 and 7).

1. Mathematical background

Basic Equations

Our initial equations can be derived from the well-known Reynolds equations. The Reynolds equations are valid for the time-averaged mean values of the turbulent flow at a certain point. If the phenomenon is considered as two

dimensional in the horizontal plane, the variation in depth should be approached by mean values. Disregarding the derivation we here give the result, the basic equations for the computation of an open-channel non-steady, depth-averaged two-dimensional, single-layer, turbulent flow.
Continuity equation:

$$\frac{\partial h}{\partial t}+\frac{\partial (uh)}{\partial x}+\frac{\partial (vh)}{\partial y}=0 \tag{1}$$

Momentum equations:
in direction x:

$$\frac{\partial (uh)}{\partial t}+\frac{\partial (u^2h)}{\partial x}+\frac{\partial (uvh)}{\partial y}+\text{gh}\frac{\partial}{\partial x}(z_0+h)+\frac{\tau_{bx}}{\varrho}-\int_0^h\left(\frac{\partial\overline{(u')^2}}{\partial x}+\frac{\partial\overline{(u'v')}}{\partial y}\right)dz+\frac{\partial}{\partial x}\int_0^h(\Delta u)^2\,dz+\frac{\partial}{\partial y}\int_0^h(\Delta u\cdot\Delta v)\cdot dz=0 \tag{2}$$

in direction y:

$$\frac{\partial (vh)}{\partial t}+\frac{\partial (v^2h)}{\partial y}+\frac{\partial (uvh)}{\partial x}+gh\frac{\partial}{\partial y}(z_0+h)+\frac{\tau_{by}}{\varrho}-\int_0^h\left(\frac{\partial\overline{(v')^2}}{\partial y}+\frac{\partial(u'v')}{\partial y}\right)dz+\frac{\partial}{\partial y}\int_0^h(\Delta v)^2\,dz+\frac{\partial}{\partial x}\int_0^h(\Delta v\cdot\Delta u)\,dz=0 \tag{3}$$

where:
- u and v are the time-and-depth-averaged velocities in direction x and y, respectively;
- u' an v' are the pulsation velocities;
- h the depth;
- z_0 the bottom level above any reference level;
- τ_{bx} and τ_{by} the bottom friction stresses in direction x and y, respectively;
- the upper dash refer to the time averaged mean values;
- $\Delta u=u-\bar{u}$, $\Delta v=v-\bar{v}$.

In the derivation of the equations (1—3) we have applied numerous common approaches (Breusers 1984, Flokstra 1977).

The momentum changes induced by the pulsation and the excess-stresses due to velocity distribution varying along the horizontal and vertical direction are expressed in the last three terms of the equation. These terms may have great importance in case of an intensively varying velocity distribution along the longitudinal or cross-sections.

For the sake of closing the equations we applied the hypothesis introduced by Boussinesq (1877). Using the hypothesis known also as Reynolds analogy, for the Reynolds stresses the

$$\tau_{i,j} = -\varrho\, \overline{u_i' u_j'} = \varrho \nu_{i,j} \frac{\partial \bar{u}_i}{\partial x_i} \tag{4}$$

form is obtained, where $\tau_{i,j}$ is the turbulent viscosity (Abraham 1982—83). This analogy is applied for the two last terms of the equations (2) and (3). Supposing that the same viscosity coefficient is valid for the stresses in both directions, we get forms

$$-\int_0^h \left(\frac{\partial (u')^2}{\partial x} + \frac{\partial \overline{(u'v')}}{\partial y}\right) dz + \frac{\partial}{\partial x}\int_0^h (\Delta u \cdot \Delta u)\, dz + \frac{\partial}{\partial y}\int_0^h (\Delta u \cdot \Delta v)\, dz = -\nu \left(\frac{\partial^2 (uh)}{\partial x^2} + \frac{\partial^2 (uh)}{\partial y^2}\right) \text{ and}$$

$$-\int_0^h \left(\frac{\partial (v')^2}{\partial y} + \frac{\partial \overline{(u'v')}}{\partial x}\right) dz + \frac{\partial}{\partial y}\int_0^h (\Delta v \cdot \Delta v)\, dz + \frac{\partial}{\partial x}\int_0^h (\Delta v \cdot \Delta u)\, dz = -\nu \left(\frac{\partial^2 (vh)}{\partial y^2} + \frac{\partial^2 (vh)}{\partial x^2}\right) \tag{5}$$

where: ν is the eddy viscosity, which involves the effects of the turbulent integration viscosity as well.

In most practical cases the boundary conditions are given by the discharge varying in time, thus the equations are rearranged for flow per unit width:

$$\text{in direction } x \qquad p = uh$$
$$\text{in direction } y \qquad q = vh \tag{6}$$

By the introduction of (5), (6) and the well-known τ_{bx}, τ_{by} bottom friction stress forms, we obtain the equations of the depth-averaged, open channel, $2D$, turbulent flow:

the continuity equation:

$$\frac{\partial h}{\partial t} + \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} = 0 \tag{7}$$

the momentum equations:

in direction x:

$$\frac{\partial p}{\partial t}+\frac{\partial}{\partial x}\left(\frac{p^2}{h}\right)+\frac{\partial}{\partial y}(pv)+gh\frac{\partial Z}{\partial x}+\frac{gp(p^2+q^2)^{1/2}}{C^2h^2}-$$

$$-\nu\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)=0 \quad (8)$$

in direction y:

$$\frac{\partial q}{\partial t}+\frac{\partial}{\partial y}\left(\frac{q^2}{h}\right)+\frac{\partial}{\partial x}(qu)+gh\frac{\partial Z}{\partial y}+\frac{gq(p^2+q^2)^{1/2}}{C^2h^2}-$$

$$-\nu\left(\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial y^2}\right)=0 \quad (9)$$

where $Z = z_0 + h$.

Equations (7)—(9) can be solved numerically. As a result of the computation we can obtain the values of the functions

$$\begin{aligned} Z &= Z(x, y, t) \\ p &= p(x, y, t) \\ q &= q(x, y, t) \end{aligned} \quad (10)$$

in discrete points.

Nonlinear Dissipation and Eddy Viscosity

The accuracy and stability of the solution is influenced by the discretisation of the domain.

Difficulties are caused by the nonlinear convective terms, because their discretisation is possible with numerical errors, only.

This numerical diffusion causes the amplitude-and-phase error, and the uncertain, stability too. According to Hirt's stability condition the sum of numerical diffusion has to be positive (Vreugdenhil—Voogt 1975). By increasing the positive eddy viscosity (ν) and reducing the negative numerical one, this criteria is fulfilled.

At present there is no suitably accurate form for the computation of the eddy viscosity. The values of ν computed by approximating formulae of different authors may show extreme deviations (Rátky 1986).

Numerical Solution

There are several methods for the numerical solution of equations (7—9). We have chosen the *implicit finite difference, four-step, alternative direction solution* method, because of its numerous advantages (Abbot 1979, Peyer—Taylor 1983, Stelling 1984).

By the method of finite differences the domain is covered by a Δx, Δy and Δt size grid. The functions are determined in each intersection of the grid. In each grid point only one unknown value is assumed. The derivatives are approached by differential quotients. On Figure 1 as an example for the x-direction computation (X-sweep), the simultaneously considered points are shown.

The four step scheme divides one time period into four parts (Abbot 1979). First equations (7) and (9) are solved for the total domain along the decreasing direction of x, then equations (7), (8) in the decreasing direction of y, on the third level the direction x equations but in the increasing direction of y, and at last, in the fourth step equations (7), (9) are solved in the increasing direction of x. In this way for the solution of the set of equations, if the initial and boundary conditions are known and suitably rearranged, the well-known special Gauss-elimination (double sweep method) may be applied.

2. Testing and calibration of the mathematical model

The reliability of the model and the computer program has been checked by a series of tests. During the development of the model and the program we have undertaken more than 30 tests. The results of the tests, due to their volume, are not given here, they can be found in different papers (Rátky—Suryadi-Barmawi 1984, Rátky 1985).

These tests proved that the mathematical model and its computer program works well. The results were in accordance with the simplifying and limiting conditions introduced during the derivation and also with our view concerning hydraulics. To achieve an approximation of the physical phenomenon with suitable accuracy, calibration must be done, too.

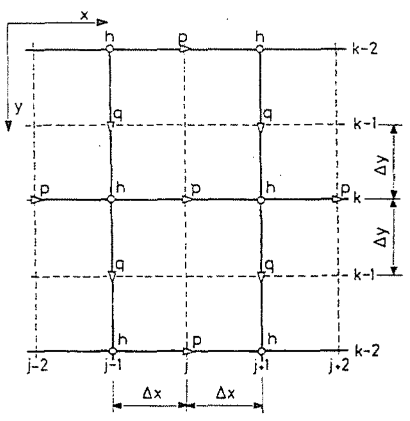

Fig. 1. Grid points, taken into consideration in equation (8)

As we did not have suitable data of the Kis-Balaton for this, we calibrated our model with the phenomenon formed in a tailbay of a hydropower station.

For calibration we applied results $u = u(x, y)$ and $v = v(x, y)$ gained with physical models.

The stream lines of the physical and mathematical models having the same geometrical data, initial and boundary conditions are shown in Figs 2 and 3. In the figures the separation pier and the tailbay are illustrated (at the computation: $C = 60$ m$^{1/2}$/s, $\nu = 5$ m^2/s). The stream lines of the two figures coincidence well. The results of the sensitivity tests of the velocity coefficient (C) and the eddy viscosity can be found in Literature (Rátky 1986).

All these proved, that the developed mathematical model and also its computer program are suitable to compute the problems, which can be considered as two dimensional in the horizontal plane.

3. Application of the model for the internal lake of Kis-Balaton

We simulated a supposed operation condition with the real geometrical data of the Internal Lake (case) of the Kis-Balaton.

We supposed an initial condition of $Z = 106.5$ mBf constant water level elevation and corresponding to this a static state of $u = 0$, $v = 0$ m/s.

Presumed boundary conditions:

During $t = 200$ s both at the upper intake and the lower outlet sluices the discharge increases until $Q = 10$ m^3/s. Our purpose was to compute the flow pattern of a steady state with a 10 m^3/s intake and outlet, respectively. A near-steady state corresponding the boundaries was formed 1.5 hours after a cold start. Figure 4 shows the flow pattern drawn by the computer in case of $C = 5$ m$^{1/2}$/s velocity coefficient and $\nu = 5$ m^2/s eddy viscosity. In the absence of suitable data for calibration we can establish that the flow pattern corresponds our experiences and our view as to hydraulics.

In the absence of data for the distribution of the velocity coefficient in space, we supposed a constant C in the whole area. To decide if the model needed more accurate data we performed a sensitivity test. With a constant $\nu = 5$ m^2/s eddy viscosity we used computations varying the velocity coefficient. The differences in the flow patterns can hardly be noticed. Figure 5 shows the distribution of the flow per unit width in direction y, in a 1200 m cross-section from the intake sluice, computed with different velocity coefficients. The average values of the cross-section can also be seen. As it is to be seen, in spite of the relatively great variation of C, the difference in discharge is about $\pm 10\%$. Because of the low velocity, the influence of friction is not

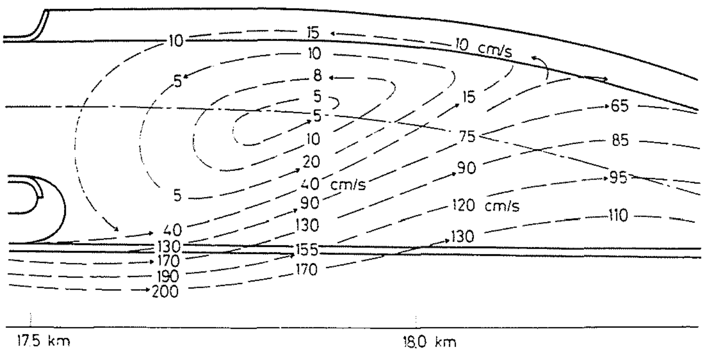

Fig. 2. Flow pattern in the tailbay of a hydropower station gained by laboratory experiments

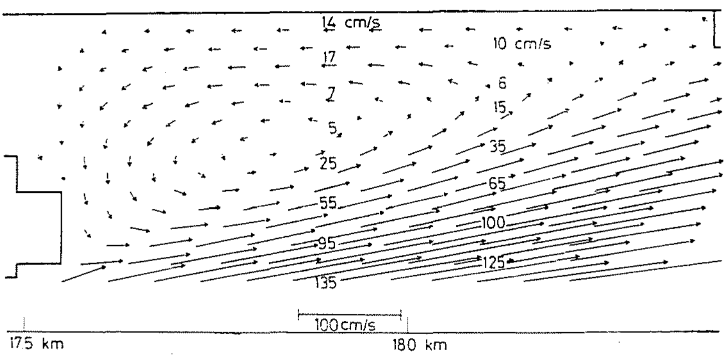

Fig. 3. Computed flow pattern

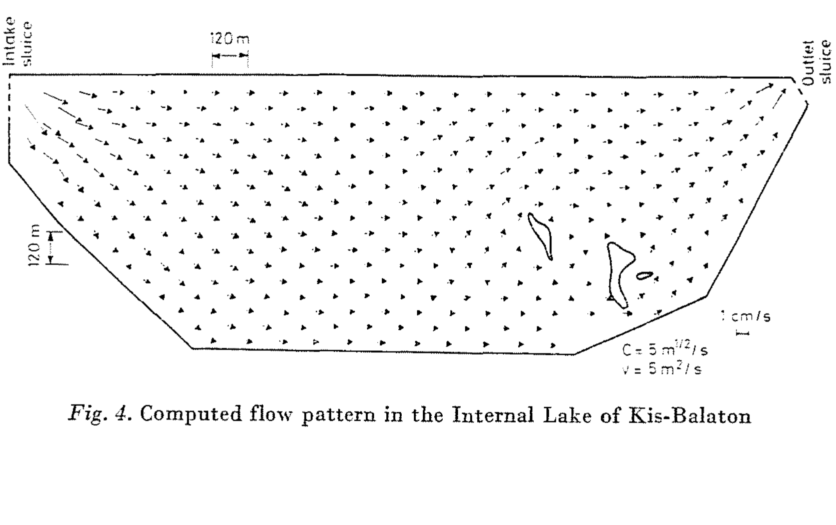

Fig. 4. Computed flow pattern in the Internal Lake of Kis-Balaton

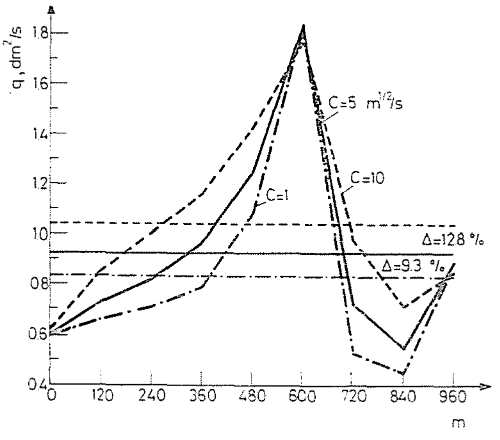

Fig. 5. Distribution of flow per unit width at 1200 m from the intake sluice

decisive, through it is not negligible, either. This effect further decreases if the discharge is lower.

The sensitivity test of the velocity coefficient (though it was not a full test) showed that the model does not need special field measurements to determine the value of C. Satisfactory accuracy can be achieved if the value of C is estimated with the help of former experiences and laboratory experiments or publications. But it is necessary to use a value, which includes the special characteristics of the simulated area and varies along the computational grid.

Concerning the above establishment we have also taken into consideration all the assumptions which the model includes or may include: the uncertainty of the geometrical and operational data, the derivation of the mathematical model, the discretization and the solution.

Besides the velocity coefficient there is one more parameter which ensures the connection between the 3D real phenomenon and the 2D model. This parameter is the eddy viscosity (ν). The formulae to be derived by integrating the pulsation velocity in time, and by integration in depth, cannot be solved as yet. As mentioned, the values received by empirical formulae show a rather important deviation. That is why we also made a sensitivity test to show the influence of the eddy viscosity.

With a constant value of the velocity coefficient (C $=5$ m$^{1/2}$/s) we undertook computation using the following values of eddy viscosity: $\nu = 1$; 5 and 10 m^2/s. We do not give here the flow patterns obtained, as the differences can hardly be seen. We calculated the differences of the mean value of the flow per unit width in several cross sections. The differences were less than 1%. Similarly to the velocity coefficient field measurements are not necessary to determine the value of ν with an accuracy corresponding to the accuracy of the model. Satisfactory results can be achieved by estimation. Because of the negligible difference, it is not necessary to use different values at each grid point or in the different directions.

4. Modelling the dispersion of pollutants

Due to the purpose of the Kis-Balaton it is very important to determine the water quality parameters of the lake. The result of the above hydrodynamical model, the distribution of velocity has a great importance from this point of view.

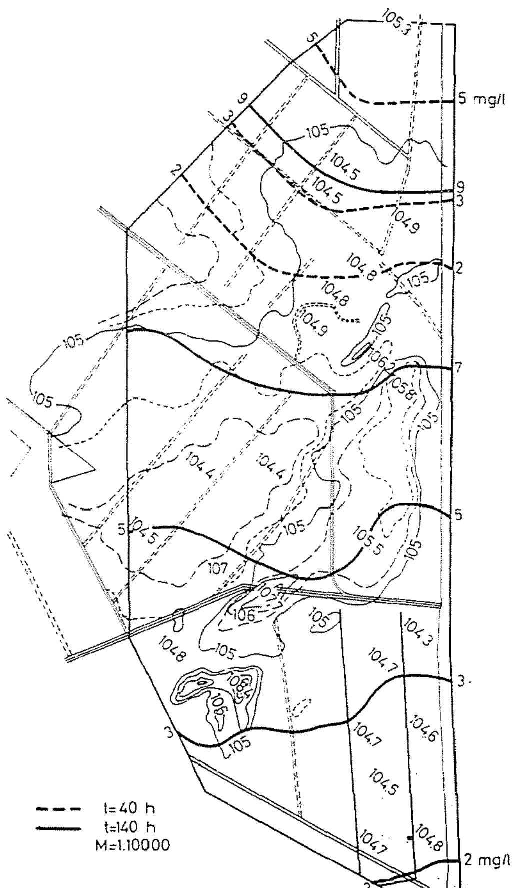

Fig. 6. Isoconcentration curves according to the computation

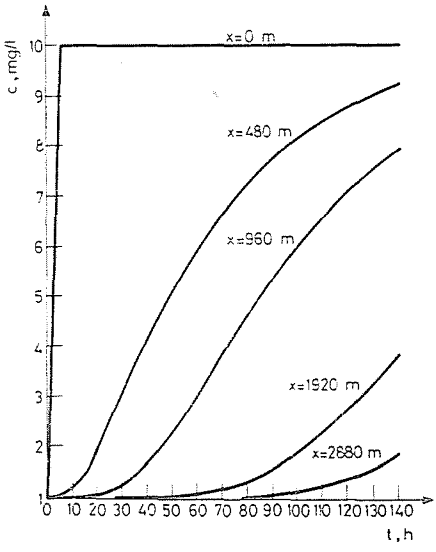

Fig. 7. Concentration as a function of time

In this paragraph we mention a mathematical model for the computation of the concentration of any pollutant or suspended material in two dimensions varying in time. We do not give details, just the results of the first steps of our research.

The governing equation is the two-dimensional, depth-averaged unsteady transport — diffusion equation:

$$h\frac{\partial c}{\partial t}+\frac{\partial}{\partial x}(pc)+\frac{\partial}{\partial y}(qc)-\frac{\partial}{\partial x}\left(hD_x\frac{\partial c}{\partial x}\right)-\frac{\partial}{\partial y}\left(hDy\frac{\partial c}{\partial y}\right)=0 \qquad (11)$$

where C (mg/l) — concentration of the pollutant
D_x, D_y (m^2/s) — turbulent dispersion coefficients, in x and y directions, respectively.

To solve the transport equation, variation of the velocity-distribution and depth must be known. These are given by the hydrodynamical model. For the numerical solution, similarly to the hydrodynamical one, we used implicit finite differences, a four step alternative direction solution method.

As an example we supposed the velocity distribution shown in Fig. 4, and simulated the transport of a pollutant reaching the lake at the upper intake sluice. Due to the constant water intake and outlet, the velocity distribution showed a near-steady state. The natural concentration of the "clean" water was 1 mg/l. Within 5 hours the concentration increased up to 10 mg/l, while the water intake and outlet remained 10 m^3/s. The results can be seen in Fig. 6 indicating identical concentration values (isoconcentration curves) at the 40th and 140th hour. The advancement and spreading of the pollutant can be followed. The reasons of the deformation of the isocentration curves are on the one hand the dominance of the convective term (velocity) in transport and on the other hand, the variation of the bottom level. The results also show how the concentration varies in time at a certain point or cross-section. Figure 7 shows the variation of concentration in time at several points at a different distance of the intake sluice. It can be seen that at the outlet sluice (2880 m) the variation of concentration is very slow,

1% of the pollution	in about 4 days,
5% of the pollution	in about 5 days,
10% of the pollution	in about 6 days

reached the lower sluice.

By the above example one can not draw any conclusion as to the detention time or the operation of the sluices. We only intended to show our transport model. We wanted to indicate that with the help of the hydrodynamical and transport model, the hydraulic characteristics and the water quality parameters of the lake can be computed and the variation of these parameters due to any operation can be forecast, thus an operation condition corresponding the water quality can be worked out.

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