FINITE ELEMENTS FOR TWO-DIMENSIONAL FREE SURFACE FLOW

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Abstract

The paper presents an application of the finite element method for two-dimensional shallow water circulation problem under land type boundary condition. The extended Galerkin finite element method is applied for finitization in space. To discretize time, a comparison between Heun's and Hamming's methods is given by a numerical example.

Introduction

In recent years, solving the shallow water equations several variants of the finite element method have been presented [6], [7].

Some cases were based on Ritz's method. For example, Fix [4] presented the finite elements of the shallow water wave problem using potential functions. In certain finite element applications Galerkin's method was considered the most convenient tool for formulating finite models. Grotkop [5] presented the method of discretizing both space and time functions following Galerkin's method. Cullen [3] employed a leap-frog scheme to discretize the time function. Galerkin's method using isoparametric finite elements was employed by Taylor Davis [13], in which the Runge-Kutta scheme was recommended for the integration in time. Oden and Weleford [2] combined the quadratic triangular elements with a 4th order Runge-Kutta method for time integration. Kawahara [8] applied the shallow water equations for "tsunami" wave propagation in Japan. "Tsunami" is a sudden rise in sea level created by an earthquake deep at the sea bottom, and this kind of generated tide waves are propagated towards the coast. The finite element method for the space functions of velocity and tide elevation was based on the conventional Galerkin method. To discretize time, an explicit method, Lax-Wendroff finite difference method was employed which is the predictor step of the well-known Heun method.

The variant of the finite element method given in this paper is based on the shallow water equations presented by Brebbia [1]. The prescribed boundary conditions are of a land type. The essence of the solution is that the extended

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Galerkin finite element method [11] has been used for space discretization. That is, the boundary conditions will be satisfied only in an average way. The interpolation functions are based on three node triangular finite elements. For numerical integration in time two methods have been employed, Heun's and Hamming's methods [12]. In [9], it was discussed how the use of corrector steps can influence the stability of the solution. Starting with the explicit method used by Kawahara, as predictor step, it was completed with the corrector step resulting the well-known Heun method. Going further, this time the Heun method is compared with Hamming's method, which is perhaps the most widely used predictor-corrector scheme to solve initial value problems.

As the paper refers to a certain stage of the research the Coriolis force, the wind effect and the eddy viscosity term for the time being are neglected, for simplicity. Taking into consideration these neglected terms gradually, this finite element solution will be a useful tool for the simulation of two-dimensional free surface flow in lakes, reservoirs, settling pools, irrigation basins, around hydraulic structures etc.

Basic equations

Neglecting the Coriolis forces, wind effects and the eddy viscosity terms the basic equations are the following continuity and momentum ones:

$$\frac{\partial H}{\partial t} + \frac{\partial q_1}{\partial X_1} + \frac{\partial q_2}{\partial X_2} = 0 \tag{1}$$

$$\left.\begin{aligned} &\frac{\partial q_1}{\partial t} + \frac{\partial}{\partial X_1}\left(\frac{q_1^2}{H}\right) + \frac{\partial}{\partial X_2}\left(\frac{q_1 q_2}{H}\right) - B_1 = 0 \\ &\frac{\partial q_2}{\partial t} + \frac{\partial}{\partial X_1}\left(\frac{q_1 q_2}{H}\right) + \frac{\partial}{\partial X_2}\left(\frac{q_1^2}{H}\right) - B_2 = 0 \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} B_1 &= -gH\frac{\partial(Z_0 + H)}{\partial X_1} - \frac{g}{C^2}\frac{q_1(q_1^2 + q_2^2)^{1/2}}{H^2} \\ B_2 &= -gH\frac{\partial(Z_0 + H)}{\partial X_2} - \frac{g}{C^2}\frac{q_2(q_1^2 + q_2^2)^{1/2}}{H^2} \end{aligned}\right\} \tag{3}$$

where (Fig. 1)

q_1 — the momentum flux in the direction of X_1,
q_2 — the momentum flux in the direction of X_2
H — water depth,
Z_0 — height of the bottom,
g — gravity acceleration (9.81 m/s²),
C — the friction factor.

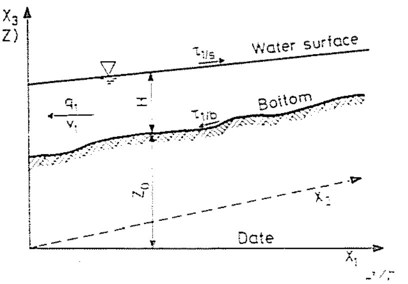


Fig. 1. Geometrical notation for shallow water equations

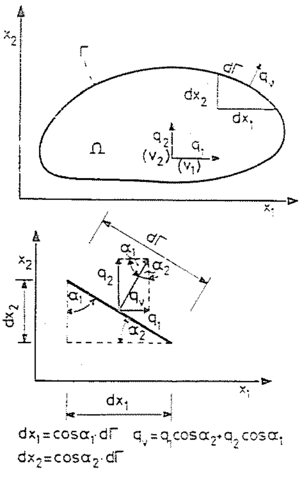


Fig. 2. Boundary definitons

Initial conditions of this system of equations:

$$\left.\begin{aligned} H(X_1, X_2, O) &= H^0 \\ q_1(X_1, X_2, O) &= q_1^0 \\ q_2(X_1, X_2, O) &= q_2^0 \end{aligned}\right\} \tag{4}$$

Boundary conditions are introduced by the normal momentum flux, in other terms the land type boundary is prescribed (Fig. 2):

$$q_v = \begin{cases} \hat{q}_v(t) = 0 & \text{on } \Gamma_1 \\ \hat{q}_v(t) \neq 0 & \text{on } \Gamma_2 \end{cases} \tag{5}$$

$\Gamma_1 U \Gamma_2 = \Gamma$
the whole boundary.

The extended Galerkin method

In order to formulate the finite element model we can write the equations (1), (2) plus boundary conditions (5) in the following way using the extended Galerkin method:

$$\iint_{\Omega} \left[\frac{\partial H}{\partial t} + \frac{\partial q_1}{\partial X_1} + \frac{\partial q_2}{\partial X_2}\right] H \, d\Omega = \int_{\Gamma} (q_\nu - \hat{q}_\nu) \, H \, d\Gamma \tag{6}$$

$$\left.\begin{aligned} &\iint_{\Omega} \left[\frac{\partial q_1}{\partial t} + \frac{\partial}{\partial X_1}\left(\frac{q_1^2}{H}\right) + \frac{\partial}{\partial X_2}\left(\frac{q_1 q_2}{H}\right) - B_1\right] q_1 d\Omega = 0 \\ &\iint_{\Omega} \left[\frac{\partial q_2}{\partial t} + \frac{\partial}{\partial X_1}\left(\frac{q_1 q_2}{H}\right) + \frac{\partial}{\partial X_2}\left(\frac{q_2^2}{H}\right) - B_2\right] q_2 d\Omega = 0 \end{aligned}\right\} \tag{7}$$

The finite element equations

The same interpolation functions are applied for q_1, q_2 and H. They are for one element:

$$\left.\begin{aligned} q_1^e &= \sum_{i=1}^{n} q_{1i} \, \varphi_i = \Phi^T \cdot q_1^n \\ q_2^e &= \sum_{i=1}^{n} q_{2i} \, \varphi_i = \Phi^T \cdot q_2^n \\ H^e &= \sum_{i=1}^{n} H_i \, \varphi_i = \Phi^T \, H^n \end{aligned}\right\} \tag{8}$$

Substituting these values into the momentum and continuity equations (6), (7), we can obtain:

$$\left.\begin{aligned} \mathbf{M}^e \frac{\partial q_1^n}{\partial t} - F_1^e = 0 \\ \mathbf{M}^e \frac{\partial q_2^n}{\partial t} - F_2^e = 0 \end{aligned}\right\} \tag{9}$$

$$\mathbf{M}^e \frac{\partial H^n}{\partial t} - F_H^e = 0 \tag{10}$$

where

$$\left.\begin{aligned} \mathbf{M}^e &= \iint_{\Omega^e} \Phi\Phi^T d\Omega \\ F_1^e &= \iint_{\Omega^e} (\Phi B_1 - \Phi A_1) d\Omega \\ F_2^e &= \iint_{\Omega^e} (\Phi B_2 - \Phi A_2)\, d\Omega \\ F_H^e &= \iint_{\Omega^e} (\Phi_{1}, q_1 + \Phi_{2}, q_2) d\Omega - \int_{\Gamma^e} \Phi q_\nu d\Gamma \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} A_1 &= \frac{\partial}{\partial X_1}\left(\frac{q_1^2}{H}\right) + \frac{\partial}{\partial X_2}\left(\frac{q_1 q_2}{H}\right) \\ A_2 &= \frac{\partial}{\partial X_1}\left(\frac{q_1 q_2}{H}\right) + \frac{\partial}{\partial X_2}\left(\frac{q_2^2}{H}\right) \end{aligned}\right\} \tag{12}$$

For the approximation of A_1, A_2, B_1, B_2 the same interpolation functions are used:

$$\left.\begin{aligned} A_1 &= \Phi^T A_1^n \\ A_2 &= \Phi^T A_2^n \\ B_1 &= \Phi^T B_1^n \\ B_2 &= \Phi^T B_2^n \end{aligned}\right\} \tag{13}$$

We can assemble [10], (9), (10) for the entire continuum:

$$\left.\begin{aligned} \mathbf{M}\frac{\partial q_1}{\partial t} &= F_1 \\ \mathbf{M}\frac{\partial q_2}{\partial t} &= F_2 \end{aligned}\right\} \tag{14}$$

$$\mathbf{M}\frac{\partial H}{\partial t} = F_H \tag{15}$$

This time we used three node triangle elements and linear interpolation functions.

Numerical integration in time

To solve equations (14), (15) two predictor-corrector iterative schemes are attempted, e.g. [12]. In the first case Heun's method is applied. The following steps are to be carried out:

predictor

$$\left.\begin{aligned}\tilde{q}_{1i+1} &= q_{1i-1} + 2 \cdot \Delta t \cdot q'_{1i}\\ \tilde{q}_{2i+1} &= q_{2i-1} + 2 \cdot \Delta t \cdot q'_{2i}\\ \widetilde{H}_{i+1} &= H_{i-1} + 2 \cdot \Delta t \cdot H'_{i}\end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned}\bar{q}'_{1i+1} &= \mathbf{M}^{-1} \cdot F_1 (\tilde{q}_1, \tilde{q}_2, \widetilde{H})_{i+1}\\ \bar{q}'_{2i+1} &= \mathbf{M}^{-1} \cdot F_2(\tilde{q}_1, \tilde{q}_2, \widetilde{H})_{i+1}\\ \overline{H}'_{i+1} &= \mathbf{M}^{-1} \cdot F_H (\tilde{q}_1, \tilde{q}_2, \widetilde{H})_{i+1}\end{aligned}\right\} \tag{17}$$

(of course by solving a system of linear equations instead of inverting $\mathbf{M}$)
corrector

$$\left.\begin{aligned}q_{1i+1} &= q_{1i} + \frac{\Delta t}{2} \cdot (q'_{1i} + \bar{q}'_{1\,i+1})\\ q_{2\,i+1} &= q_{2i} + \frac{\Delta t}{2} \cdot (q'_{2i} + \bar{q}'_{2\,i+1})\\ H_{i+1} &= H_i + \frac{\Delta t}{2} \cdot (H'_i + \overline{H}'_{i+1})\end{aligned}\right\} \tag{18}$$

The second, Hamming's method consists of the following steps:
predictor

$$\left.\begin{aligned}\tilde{q}_{1i+1} &= q_{1\,i-3} + \frac{4 \cdot \Delta t}{3} (2 \cdot q'_{1i-2} - q'_{1i-1} + 2 \cdot q'_{1i})\\ \tilde{q}_{2i+1} &= q_{2i-3} + \frac{4 \cdot \Delta t}{3} (2 \cdot q'_{2i-2} - q'_{2i-1} + 2 \cdot q'_{2i})\\ \widetilde{H}_{i+1} &= H_{i-3} + \frac{4 \cdot \Delta t}{3} (2 \cdot H'_{i-2} - H'_{i-1} + 2 \cdot H'_{i})\end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned}\bar{q}_{1i+1} &= \tilde{q}_{1i+1} + \frac{112}{121} (C_{1i} - \tilde{q}_{1i})\\ \bar{q}_{2i+1} &= \tilde{q}_{2\,i+1} + \frac{112}{121} (C_{2i} - \tilde{q}_{2i})\\ \overline{H}_{i+1} &= \widetilde{H}_{i+1} + \frac{112}{121} (C_{3i} - \widetilde{H}_{i})\end{aligned}\right\} \tag{20}$$

corrector

$$\left.\begin{aligned}\check{q}_{1\,i+1} &= \mathbf{M}^{-1}\cdot F_1(\bar{q}_1, \bar{q}_2, \bar{H})_{i+1}\\ \check{q}_{2\,i+1} &= \mathbf{M}^{-1}\cdot F_2(\bar{q}_1, \bar{q}_2, \bar{H})_{i+1}\\ \check{H}_{i+1} &= \mathbf{M}^{-1}\cdot F_H(\bar{q}_1, \bar{q}_2, \bar{H})_{i+1}\end{aligned}\right\} \tag{21}$$

$$\left.\begin{aligned}C_{1\,i+1} &= \frac{1}{8}\left[9\cdot q_{1i} - q_{1i-2} + 3\cdot \Delta t\cdot(\check{q}_{1i+1} + 2\cdot q'_{1i} - q'_{1i-1})\right]\\ C_{2\,i+1} &= \frac{1}{8}\left[9\cdot q_{2i} - q_{2i-2} + 3\cdot \Delta t\,(\check{q}_{2i+1} + 2\cdot q'_{2i} - q'_{2i-1})\right]\\ C_{3i+1} &= \frac{1}{8}\left[9\cdot H_i - H_{i-2} + 3\cdot \Delta t\cdot(\check{H}'_{i+1} + 2H'_i - H'_{i-1})\right]\end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned}q_{1i+1} &= C_{1i+1} - \frac{9}{121}(C_{1i+1} - \check{q}_{1\,i+1})\\ q_{2i+1} &= C_{2i+1} - \frac{9}{121}(C_{2i+1} - \check{q}_{2i+1})\\ H_{i+1} &= C_{3i+1} - \frac{9}{121}(C_{3i+1} - \check{H}_{i+1})\end{aligned}\right\} \tag{23}$$

Example

The example presents the simulation of the two-dimensional problem. The geographic boundary and the bottom topography of the examined domain can be varied arbitrarily and the friction factor is different for each element. In this way the example is able to demonstrate a problem often occurring in hydraulic engineering practice. Flow in lakes, cooling ponds and other water bodies can be approximated similarly to the given example.

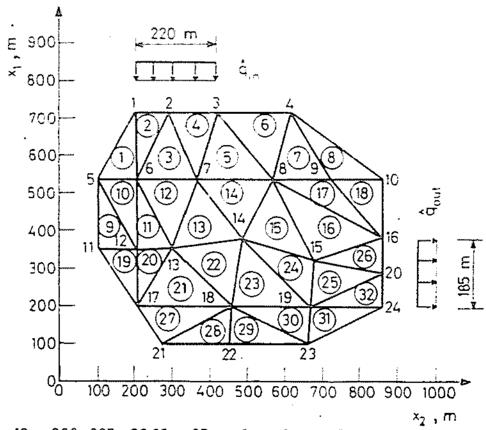
The examined region and the finite element grid are shown in Fig. 3. A grid of 32 elements and 24 nodes has been laid out reflecting the varying bottom topography.

The model is started with a "flat" condition, so the initial momentum fluxes equal to zero. The initial depth can be seen on Fig. 3.

The boundary condition is prescribed between nodes 1—2, 2—3 and 16—20, 20—24 where the normal fluxes are enforced as the function of time. $\check{q}_{\text{in}}$ and $\check{q}_{\text{out}}$ are also shown in Fig. 3.

Time integration is carried out using both Heun's and Hamming's methods. In both cases within each time step the predictor is followed by only

Node	x_1	x_2	z_0	K	v_{1^0}	v_{2^0}	H^0
1	720	200	100.00	20	0	0	1.00
2	720	295	99.80	35	0	0	1.20
3	720	420	100.00	20	0	0	1.00
4	720	615	100.00	20	0	0	1.00
5	540	100	100.00	20	0	0	1.00
6	540	200	99.95	25	0	0	1.05
7	540	360	99.798	35	0	0	1.202
8	540	565	99.94	25	0	0	1.06
9	540	720	99.97	25	0	0	1.03
10	540	860	100.00	20	0	0	1.00
11	350	100	100.00	20	0	0	1.00
12	350	200	99.97	25	0	0	1.03
13	360	300	99.94	25	0	0	1.06
14	380	485	99.795	35	0	0	1.205
15	325	670	99.793	35	0	0	1.207
16	385	860	100.00	20	0	0	1.00
17	200	200	100.00	20	0	0	1.00
18	200	455	99.94	25	0	0	1.06
19	200	665	99.93	25	0	0	1,07
20	290	860	99.79	35	0	0	1,21
21	100	275	100.00	20	0	0	1.00
22	100	450	100.00	20	0	0	1.00
23	100	655	100.00	20	0	0	1.08
24	100	860	100.00	20	0	0	1.00



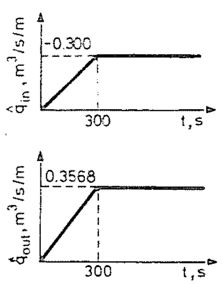


Fig. 3. Finite idealization of the two-dimensional problem, initial conditions, boundary condition

one corrector. After some preliminary tests, the computed time step has been decided as $\Delta t = 5$ s.

The results, using Heun's method for time integration, are illustrated in Figs 4, 5. Solution by both integration schemes are compared in Figs 6, 7. The velocity distribution calculated by Hamming's method is shown in Fig. 8.

It can be seen that the results of the two time integration method are well in agreement and to get a nearly identical accuracy the same number of function evaluation has been needed.

One time step has required about about 25—30 s running time.

The computations have been accomplished by the TPA 1148 computer of the Civil Engineering Faculty of the Technical University of Budapest.

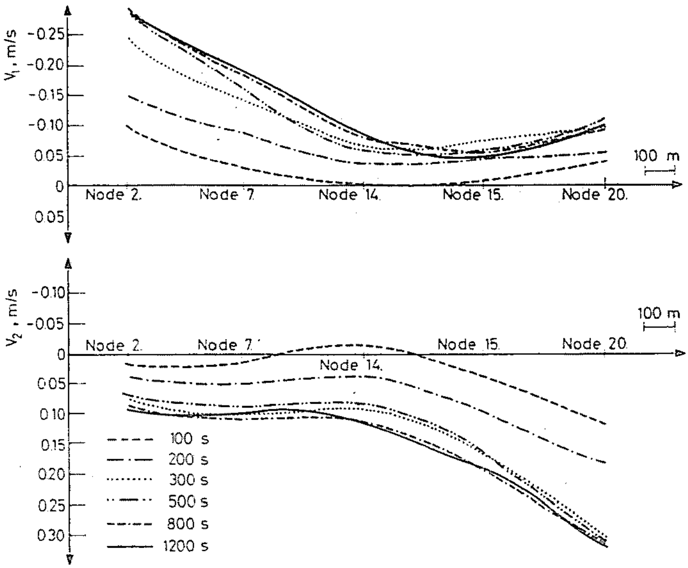


Fig. 4. Computed velocities at nodes 2, 7, 14, 15, 20

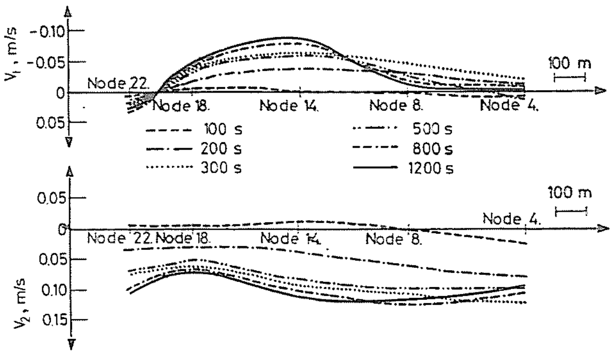


Fig. 5. Computed velocities at nodes 22, 18, 14, 8, 4

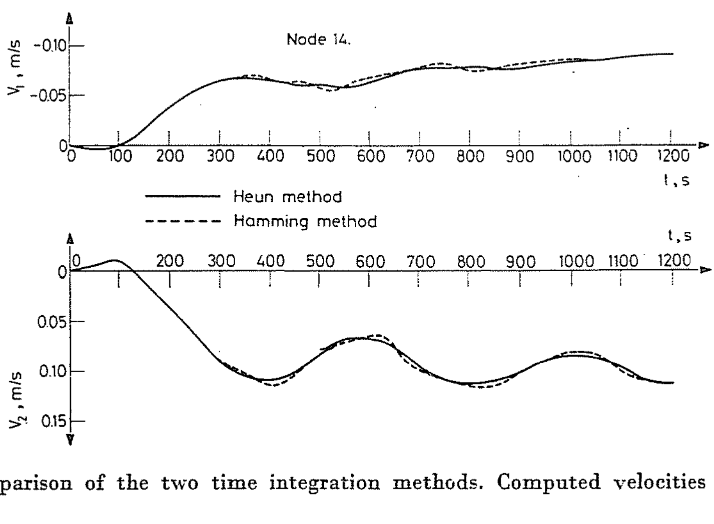


Fig. 6. Comparison of the two time integration methods. Computed velocities at node 14

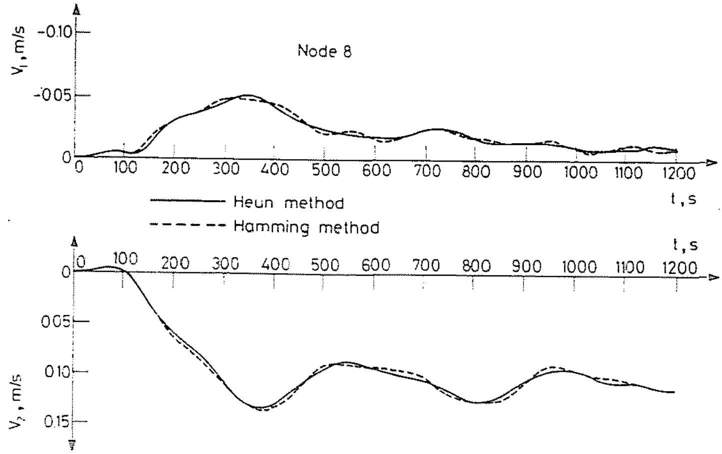


Fig. 7. Comparison, at node 8

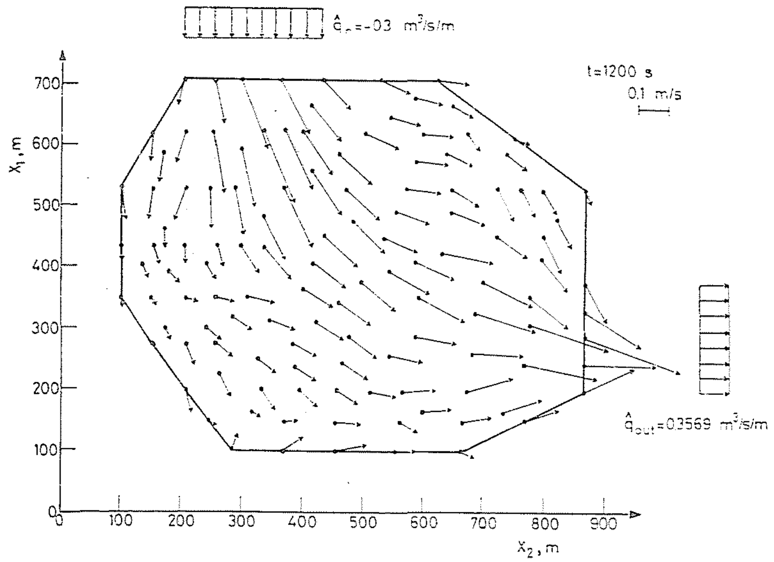


Fig. 8. Velocity distribution at $t = 1200$ s

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