NEW ASPECTS IN THE OPTIMAL DESIGN OF FREE GEODETICAL NETWORKS

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Abstract

The first part of the paper seeks new possibilities to select the criteria-system of first order network design, studies the circumstances that may indicate that both first and second order designs be realized with the aid of the criterian matrix.

The second part of the paper investigates the possibilities of a dynamic fitting on the criterion matrix while strictly adhering to algebraic and geodetic requirements. It also emphasizes the importance of complex network design from the point of view of establishing the necessary isotropy.

The renewal of network design can be connected to the work of Grafarend (1973) [2]. Both in this and in some later works he suggests to realize first order network design through minimizing one of the invariants of the variance-covariance matrix of the coordinates. Several authors (e.g. Sárközy, 1977, Provorov, 1976, etc.) [5, 7] indicated that in geodetical networks for engineering control it is more expedient to undertake optimation concerning the invariants of the variance-covariance matrix of functions of the coordinates selected from the users side. On the one hand, in such a case the requirements given by a concrete task can be enforced much better and on the other, the method has a high number of computation advantages (the matrix is a regular one, its dimension is much less than that of the matrix of variance-covariance of the coordinates, etc.). In the majority of geodetical tasks for engineers the solution brings suitable results and this most probably explains while in literature the subject matter of first order design is treated but seldom.

There are, however, cases when local networks are used in a versatile way that not all functions can be foreseen from which a minimum function mean error is required. In such instances former practice realized designing by minimizing one of the invariants (trace, determinant, etc.) of the variance-covariance matrix of coordinates. The most common criterion, the generalized variance (the determinant of the matrix), however, is nought in case of free networks and thus the design work ought to be carried out on basis of criteria on a lower level of generality (e.g. Sárközy 1977/b, Sárközy 1980, etc.) [8, 10].

As the first it was Sárközy (1980) [10] (in connection with complex network design) and then Koch (1981) [4] (in connection with partial first order design) who thought that the criterion matrix approximation of a *given struc-*

ture should be endeavoured by changing the shape of the network. Koch (1981) [4] tries to realize by a *lesser displacement* of *individual* points, that absolute and relative error-ellipses be approximately circular and, at the same time, their dimensions decrease as far as possible. As a solution he applies convex quadratic programing that is traceable to the simplex method. Sárközy (1980) [10] improves the so-called simplex design by initiating the penalty function for the first and second order design of the complex. Let us note that in case of criterion matrixes depending on the side length the method can only be applied if the side lengths evolving as a consequence of point transposition (or relative side length changes) are small, as the method does not take into consideration changes in the criterion matrix due to changes of the coordinate.

To be able to solve the general task of first order design with the aid of the criterion matrix, let us investigate the problems of applying the criterion matrix in the second order design of free networks. Such a step is mainly justified by the fact that in literature a number of questionable solutions are found in the field of second order design by the correction of which a well-founded method can be had for the generalization of the first order design task.

In the course of accuracy design of the networks, when endeavouring an optimation of the general error-image we start from the basic equation

$$(\mathbf{A}'\,\mathbf{PA})^{+} = \mathbf{Q} \tag{1}$$

In expression (1) **A** is the so-called shape-matrix figuring in the adjustment by parametric method which is a function of the coordinates of the network and the type of measurements realized between its points (distance measurement, angular measurement, direction measurement), and thus in the course of first order design the coordinate changes are reflected in the changes of this matrix; **P** is the diagonal matrix summarizing the planned weights of measurements, in the task of second order design matrix **P** is the unknown quantity; $\mathbf{Q} = \frac{1}{\sigma^2}\mathbf{Cr}$ is the weight coefficient matrix to be derivated from the previously given criterion matrix. The criterion-matrixes are, in general, the functions of *distances* between the points, in certain cases also these of the coordinate differences (Wimmer, (1981) [14], Sárközy (1984) [11]). It follows from the above that in case of generally expected criterion matrix models also the right side of equation (1) changes (in the function of coordinate changes) in the course of first order design processes.

If value $\sigma^2 = 1$ is selected (entailing that also the numerator is selected as a unit in the elements of the weight matrix),

$$(\mathbf{A}'\mathbf{PA})^{+} = \mathbf{Q} = \mathbf{Cr} \tag{2}$$

In his thesis, Wimmer (1981) [14] summarizes well the solution possibilities of the task in the case of second order design. However, the known solutions cannot eliminate completely two difficulties: one of these is to construct the right criterion matrix, while the other is to express the unknown **P** from the pseudo-inverse.

Several experiments were undertaken to solve both problems. In the case of the criterion matrix the basic difficulty was that the Taylor-Kármán structure matrix proposed by Grafarend (1975) [3] has a full rank and thus, applying this model the equality of the right and left side of (2) it cannot hold itself, even in principle. Schaffrin et alia (1980) [12] applied transformation

$$\mathbf{Cr}^{(\mathrm{sing})} = \mathbf{A}^{-}\mathbf{A}\mathbf{Cr}(\mathbf{A}^{-}\mathbf{A})' \tag{3}$$

to eliminate the contradiction, where $\mathbf{A}^{-}$ is the arbitrary g-inverse of the shape matrix. Similar results can be achieved by different variations of the S transformation (Strang Van Hees (1982) [13]).

According to experience with numerical examples, however, the solutions distort the error-circles characterizing the isotrope structure into ellipses, and so, if we intend to model an isotrope error image even in case of rank identity, some other solution has to be looked for. As a principle let us start from the fact that the planned singular matrix is interpreted as a pseudo-inverse of the normal equation coefficient matrix of the given coordinates. In case our plane free network does not contain linear measures, its weight coefficient matrix has four defects that are the consequences of the following function mean error expressions originating from the adjustment of the free network:

a) The variances of centre of gravity coordinates equal nought

$$q_{y_s y_s} = 0 \tag{4}$$

$$q_{x_s x_s} = 0 \tag{5}$$

where $y_s = \frac{1}{n} \Sigma y_i$ and $x_s = \frac{1}{n} \Sigma x_i$ are the centre of gravity coordinates of the network.

b) The variance of the average point of gravity radius equals nought (this condition comes up only if there is no distance measurement, as then the scale of the network is determined by the average radius).

$$q_{r_s r_s} = 0 \tag{6}$$

where

$$r_S = \sqrt{\frac{1}{n} \Sigma[(x_i - x_s)^2 + (y_i - y_s)^2]}$$

c) The sum total of the moments of coordinate changes to the point of gravity axes is free of error (viz. the axes determined by the preliminary coordinates do not occur rotation while adjustment).

$$q_{MM} = 0 \tag{7}$$

where

$$M = \Sigma[-(y_i - y_s)\, \delta x_i + (x_i - x_s)\, \delta y_i] = 0$$

Let us now turn to point of gravity coordinates $\eta_i = y_i - y_s$; $\xi_i = x_i - x_s$ and form the **F** coefficient matrix of the above functions:

$$\mathbf{F}' = \begin{matrix} 1 & 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 \\ \eta_1 & \eta_2 & \eta_3 & \dots & \eta_n & \xi_1 & \xi_2 & \xi_3 & \dots & \xi_n \\ \xi_1 & \xi_2 & \xi_3 & \dots & \xi_n & -\eta_1 & -\eta_2 & -\eta_3 & \dots & -\eta_n \end{matrix} \tag{8}$$

According to the law of error propagation, considering (4), (5), (6), (7)

$$\mathbf{F}'\ \mathbf{Q}^{(\mathrm{sing})}\ \mathbf{F} = 0 \tag{9}$$

It can be shown that it the columns of **F**' are linearily independent from each other then

$$\mathbf{Q}^{(\mathrm{sing})}\ \mathbf{F} = 0 \tag{10}$$

follows from (9).

But it follows from equation (10) that the matrix lines of F are nothing else the non-normalized eigenvectors belonging to the zero eigenvalues of the $\mathbf{Q}^{(\mathrm{sing})}$ matrix. The other group of requirements needed to form the criterion matrix are "user" demands:

$$q_{y_iy_i} = q_{y_jy_j} = 1 \tag{11}$$

$$q_{x_ix_i} = q_{x_jx_j} = 1 \tag{12}$$

$$q_{y_ix_i} = q_{x_iy_i} = 0 \tag{13}$$

$$q_{y_iy_j} = q_{x_ix_j} \tag{14}$$

$$q_{y_ix_j} = -q_{y_jx_i} \tag{15}$$

It should be noted that conditions (14) and (15) differ from the Taylor-Kármán structure proposed by Grafarend (1974) [3]. Condition (14) is important because when adhering to it, also the relative error ellipses become circles. If we suppose that there is a correlation function relation which is *identical* between the points regarding both coordinates and depends but on the distance, (15) can be satisfied only, if

$$q_{y_ix_j} = -q_{y_jx_i} = 0 \tag{16}$$

(Sárközy, 1980). In this case $n\dfrac{n-1}{2}$ number, $q_{ij} = f(l)$ form covariance has to be determined in consideration of the condition system (10), where n is the number of network points, $f(l)$ the covariance function depending on the mutual distance of the points.

As the condition system (10) is equivalent with equation $6n$ in case of distance measured networks, and with equation $8n$ in case of direction measured networks the degree of freedom of the function $f(l)$ is

$$\gamma = n\frac{n-1}{2} - 6n, \text{ and/or} \tag{17}$$

$$\gamma = n\frac{n-1}{2} - 8n \tag{18}$$

In the following we will discuss the networks only the scale of which was determined by measuring. In this case a free parameter can occur but if $n > 13$, viz. when there are less points than 13 the system does not only have free parameters but not even the degree of freedom that guarantees fulfillment of our conditions.

As in engineering practice networks with less than 13 points are no rarity, the number of free parameters has to be increased with these networks to be able to design the criterion matrix. If condition (15) suffices and we intend to but approximate condition (16), then we will have a free parameter with a further $n\frac{n-1}{2}$ viz. in case of a network with 7 and/or more points, for the criterion matrix the complete isotropy can be assured (both the absolute and the relative error ellipses degenerate into circles).

The requirement concerning relative error circles can be fulfilled but by accident for networks with less than 7 points, therefore during the design process the approximate fulfillment of condition (15) can be our aim. If the number of points of the network is 4 or 3, we have to dispense with fulfilling condition (14), very precisely.

Summing up the above, constructing the criterion matrix can be guaranteed with the following algorithm:

a) $n > 13$, conditions (10) to (16) and a covariance function (Sárközy (1980) [10], or Wimmer (1981) [14]).

b) $n = 13$, conditions of point a) without a covariance function.

c) $13 > n > 7$ conditions (10) to (15) as well as the objective function

$$\Sigma q^2_{y_i x_j} \rightarrow \text{minimum} \tag{19}$$

d) $n = 7$ conditions of point c) without (19)

e) $7 > n > 5$ conditions from (10) till (14) and the objective function

$$\Sigma(q_{y_i x_j} + q_{y_j x_i})^2 \rightarrow \text{minimum} \tag{20}$$

f) $n = 5$ conditions of point e) without the objective function.

From the above it follows unanimously that if also the *first order* design is realized with the aid of the proposed criterion matrixes, it has to be kept in mind that the condition in the fourth line of (8) is coordinate dependent and in case a) also the selected $f(l)$ covariance function depends on the form of the network.

The task of design can be realized on basis of the target function

$$\sum_{i=1}^{2n} \sum_{j=1}^{2n} (q_{ij} - \tilde{q}_{ij})^2 \rightarrow \text{minimum} \tag{21}$$

with the aid of the gradient projection method (Sárközy, (1976) [6]).

While in the course of first order design the range permitted for changing the coordinates ensure in general sufficiently big coordinate changes to find the global optimum with expression (21) (viz. that the variance-covariance matrix computable from the designed coordinates be in agreement with the criterion matrix), in the process of second order design, fitting to the fixed criterion matrix, due to the relatively small normal equation change that can be achieved with positive weights, leaves much to be demanded. This fact is underlined, among others, by the investigations of Gáspár (1984) [1], in the course of which different networks for analysis of deformation had been designed with a regular TK-structure, and/or by taking into consideration conditions of (10) to (14). The results show that though designs realized with singular criterion matrixes very often give a *near*-isotrope error image, second order design with fixed criterion matrixes in general does not give a satisfying result as to isotropy.

If, therefore, we adher to isotropy it must be permitted in the course of design, that while keeping to conditions concerning isotropy also the criterion matrix may change. Let us visualize the essence of this change for the case of a fixed network, for the sake of simplicity. Let us denote the normal equation coefficient matrix with **N**, and then

$$\mathbf{A'PA} = \mathbf{N} = \mathbf{Q}^{-1}$$

and

$$\mathbf{NQ} = \mathbf{E} \tag{23}$$

where **E** is the unit matrix. If therefore **P** and **Q** are changed in a way that over and above conditions (11)—(15) also condition (23) is kept, the matrix **P**, the result of designing will ensure that the isotropy and homogeneity determined with conditions (11)—(15) should manifest themselves without distortion in the network. Designing is thus possible with the following objective function:

$$\sum_{=1}^{2n} \sum_{=1}^{2n} (\tilde{q}'_i n_j - E_{ij})^2 \rightarrow \text{min} \tag{24}$$

$$p \geq 0 \tag{25}$$

and conditions (11)—(15).

As a method of solution the gradient projection method or the method of efficient directions can be applied. Let us see how objective function (24) changes in the case of free networks

$$\mathbf{A'PA} = \mathbf{N} = \mathbf{Q}^{+} \tag{26}$$

where the $^{+}$ symbol indicates the Moore-Penrose inverse. As is known, the pseudo inverse can be defined with the following equations:

$$\mathbf{QNQ} = \mathbf{Q} \tag{27}$$

$$\mathbf{NQN} = \mathbf{N} \tag{28}$$

If (27) is multiplied from the left or the right with N, and (28) with Q, it can be seen that the product of the two matrixes equals the square of the product, viz. the product of the two matrixes is the projector

$$\mathbf{NQ} = \mathbf{Pr} \tag{29}$$

If it is our intention to apply the objective function (24), we have to determine the **Pr** value substituting in our case **E**.

Let us undertake the singular value decomposition of the two matrixes:

$$\mathbf{Q} = \mathbf{SD}_Q\mathbf{S}' \tag{30}$$

$$\mathbf{N} = \mathbf{SD}_N\mathbf{S}' \tag{31}$$

where $\mathbf{D}_Q$ and $\mathbf{D}_N$ are the diagonal matrixes containing eigenvalues, **S** is the so-called modal matrix containing normalized eigenvectors. As the eigenvectors belonging to the zero eigenvalues are present in the equations but formally (30), (31) can be written also with the aid of eigenvectors pertaining to $2n-d = r$ number, non-zero eigenvalues:

$$\mathbf{Q} = \mathbf{S}_r\,\mathbf{D}_r^{-1}\,\mathbf{S}_r' \tag{32}$$

$$\mathbf{N} = \mathbf{S}_r\,\mathbf{D}_r\,\mathbf{S}_r' \tag{33}$$

Let us substitute (32) and (33) into (29):

$$\mathbf{Pr} = \mathbf{S}_r\mathbf{D}_r\mathbf{S}_r'\mathbf{S}_r\,\mathbf{D}_r^{-1}\,\mathbf{S}_r' \tag{34}$$

as

$$\mathbf{S}_r'\mathbf{S}_r = \mathbf{E}r \tag{35}$$

and

$$\mathbf{D}_r\mathbf{D}_r^{-1} = \mathbf{E}r \tag{36}$$

that is

$$\mathbf{Pr} = \mathbf{S}_r\mathbf{S}_r' \tag{37}$$

It is known that

$$\mathbf{SS}^{-1} = \mathbf{SS}' = \mathbf{E} \tag{38}$$

Let us separate in the modal matrix of expression (38) the part-matrixes relating to zero and non-zero eigenvalues

$$\mathbf{E} = \mathbf{S}_r\mathbf{S}_r' + \mathbf{S}_d\mathbf{S}_d' = \mathbf{Pr} + \mathbf{S}_d\,\mathbf{S}_d' \tag{39}$$

and

$$\mathbf{Pr} = \mathbf{E} - \mathbf{S}_d\mathbf{S}_d' \tag{40}$$

Matrix $\mathbf{S}_d'$ is, however, known, it is nothing else but the normalized matrix form of (8) $\mathbf{F}'$, that is

$$s_i' = \frac{f_i'}{+\sqrt{f_i'f_i}} \tag{41}$$

In this way $\mathbf{Pr}$ can be computed and the design task can be realized in consideration of conditions (24) (25) and (11)—(15), with the modification that in (24) $\mathbf{Pr}_{ij}$ should figure instead of E_{ij}.

Though, in principle, the method could also be used for first order design, in practice it would greatly increase the time requirement of computations, and can thus be only suggested, if the permissible point-deviations are so very small that the desired isotropy cannot be achieved with the aid of objective function (21).

Both the problems indicated and the solution possibilities prove more and more convincingly that my former proposition concerning complex network design (Sárközy, 1979) [9] which then had no reaction, can now be realized realistically and became at the same time a task following in a natural mode from the substance of design work. As already shown earlier, the necessity to introduce the dynamic criterion matrix is provoked by the fact that the possibility contained in changing the positive weights is too little to render identical the normal equation coefficient matrix of the designed network with the fixed criterion matrix pseudo-inverse. If, however, also the form of the network is changed for our purposes, then despite the fact that this changes also the criterion matrix itself in a small way, the probability is increased that as a result of designing the desired correspondence be established. For the purposes of complex design an objective function of the type (21) sems the most suitable and after having completed the experimental computations we wish to come back to a detailed introduction of the method.

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