SECOND-ORDER DESIGN OF GEODETIC NETWORKS BY MEANS OF A STATISTICALLY PERFECTLY ISOTROPIC AND HOMOGENEOUS MEAN ERROR MATRIX

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Abstract

Goals of second-order design of geodetic networks are considered, together with mathematical conditions of developing the proper homogeneous, perfectly isotropic variance-covariance matrices. In case of free networks, perfect isotropy will be shown to be possible exclusively for indefinite network scales.

An adjustment method providing for perfect isotropy even in case of telemetry will be suggested, and so will be a simple measurement method to be combined with the suggested adjustment method for automatically safeguarding perfect isotropy.

An optimization algorithm will be presented, to determine a system of measurement weight values resulting in a statistically homogeneous, perfectly isotropic weight coefficient matrix.

Features of the optimum network

Second-order design of geodetic networks is expected to determine the optimum weight distribution in knowledge of positions of network points and of measurements possible in the network. Optimum measurement weights are generally understood as weights yielding a network variance-covarance matrix possibly best approximating the ideal mean error pattern corresponding to the purpose of the network. In case of geodetic networks for universal purposes it is the best if absolute error ellipses degenerate to circles equal in size. Such networks were called statistically homogeneous and isotropic by Grafarend. According to Baarda, local networks advisably have, in addition to the above, circles for relative error ellipses, that is, the network is perfectly homogeneous.

Variance-covariance matrix being scalar product of the weight coefficient, $\mathbf{M} = \mathbf{m}_0^2 \cdot \mathbf{Q}$, in the following it suffices to examine the structure of the weight coefficient matrix. Mean error m_0 of the unit weight can be chosen to cope with the required network accuracy, respecting the weight proportions, by properly determining the measurement mean error values. For elements of the perfectly isotropic weight coefficient matrix $\mathbf{Q}$

$$\begin{aligned} q_{y_i y_j} &= q_{x_i x_j} \\ q_{y_i x_j} &= -q_{y_j x_i} \end{aligned} \tag{1}$$

hold. Provided matrix $\mathbf{Q}$ is also homogeneous, still it is:

$$q_{y_i y_i} = 1 \tag{2}$$

Partitioning the weight coefficient matrix to four parts, separating values belonging to different coordinates:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{YY} & \mathbf{Q}_{YX} \\ \mathbf{Q}_{XY} & \mathbf{Q}_{XX} \end{bmatrix} \tag{3}$$

Relationships for perfect isotropy become:

$$\begin{aligned} \mathbf{Q}_{YY} &= \mathbf{Q}_{XX} \\ \mathbf{Q}_{YX} &= \mathbf{Q}'_{XY} = -\mathbf{Q}_{XY} \end{aligned} \tag{4}$$

Weight coefficient matrices for free networks are always singular. Depending on whether measurements determine the network size or not, the matrix defect number may be $d = 3$ or $d = 4$. Eigenvectors for zero eigenvalues of the singular weight coefficient matrix

$$\begin{aligned} u'_1 &= [1, \quad 1, \quad \ldots, \quad 1; \quad 0, \quad 0, \quad \ldots, \quad 0] \\ u'_2 &= [0, \quad 0, \quad \ldots, \quad 0; \quad 1, \quad 1, \quad \ldots, \quad 1] \\ u'_3 &= [x_1, \quad x_2, \quad \ldots, \quad x_n; \quad -y_1, \quad -y_2, \quad \ldots, \quad -y_n] \\ u'_4 &= [y_1, \quad y_2, \quad \ldots, \quad y_n; \quad x_1, \quad x_2, \quad \ldots, \quad x_n] \end{aligned}$$

where n is the number of points in the network, y_i and x_i are their centroidal coordinates. Vector u_4 has a zero eigenvalue only if the network size is not determined by measurement. Due to its characteristic equation and zero eigenvalues, weight coefficient matrix meets equations

$$\mathbf{Q}u_i = s_i u_i = O$$

These equations written for vectors u_3 and u_4 in partitioned form according to (3) are:

$$\mathbf{Q}u_3 = \begin{bmatrix} \mathbf{Q}_{YY} & \mathbf{Q}_{YX} \\ \mathbf{Q}_{XY} & \mathbf{Q}_{XX} \end{bmatrix} \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_{YY}\, x - \mathbf{Q}_{YX}\, y \\ \mathbf{Q}_{XY}\, x - \mathbf{Q}_{XX}\, y \end{bmatrix} = \begin{bmatrix} O \\ O \end{bmatrix} \tag{5}$$

$$\mathbf{Q}u_4 = \begin{bmatrix} \mathbf{Q}_{YY} & \mathbf{Q}_{YX} \\ \mathbf{Q}_{XY} & \mathbf{Q}_{XX} \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_{YY}\, y + \mathbf{Q}_{YX}\, x \\ \mathbf{Q}_{XY}\, y + \mathbf{Q}_{XX}\, x \end{bmatrix} = \begin{bmatrix} O \\ O \end{bmatrix} \tag{6}$$

Substituting Eqs (4) for perfect isotropy:

$$\mathbf{Q}u_3 = \begin{bmatrix} \mathbf{Q}_{YY}x - \mathbf{Q}_{YX}y \\ -\mathbf{Q}_{YX}x - \mathbf{Q}_{YY}y \end{bmatrix} = \begin{bmatrix} O \\ O \end{bmatrix} \tag{7}$$

$$\mathbf{Q}u_4 = \begin{bmatrix} \mathbf{Q}_{YY}y + \mathbf{Q}_{YX}x \\ -\mathbf{Q}_{YX}y + \mathbf{Q}_{YY}x \end{bmatrix} = \begin{bmatrix} O \\ O \end{bmatrix} \tag{8}$$

Equations (7) and (8) show fulfilment of (7) for a perfectly isotropic matrix **Q** to require also (8) to be met, hence number of defects of a perfectly isotropic weight coefficient matrix cannot be $d = 3$, that is, the network cannot have a definite size.

Demonstrably, coefficient matrix **N** of the normal equation system — pseudo-inverted of a perfectly isotropic weight coefficient matrix **Q** — partitioned similarly to (3):

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_{YY} & \mathbf{N}_{YX} \\ \mathbf{N}_{XY} & \mathbf{N}_{XX} \end{bmatrix}$$

also meets relationships:

$$\begin{aligned} \mathbf{N}_{YY} &= \mathbf{N}_{XX} \\ \mathbf{N}_{YX} &= \mathbf{N}'_{XY} = -\mathbf{N}_{XY} \end{aligned} \tag{9}$$

These relationships linear by weight permit much simplified methods to be applied in second-order network design compared to those applied in actul practice.

Measurement types admissible for perfect isotropy

Previously it has been stated that a perfectly isotropic network is only subject to shape determination — rather than size determination — measurements. That is, either perfect isotropy or telemetry should be renounced of. Telemetry cannot be dispensed with, partly since it is indispensable for determining the network size, and partly, it would be unreasonable to shun recent, up-to-date, high-precision telemeters. Again, it would be a pity to renounce of perfect isotropy, with all its advantages for the theory of errors.

This contradiction is dissolved by altering the adjustment method of geodetic networks so as to separately adjusting the network shape and size, possible by separating the functions of telemetry to determine shape and size. In adjusting the network shape, measured distances are involved in calculations as scales, together with sightings and goniometry. This stage of calculations exclusively for shape determination permits to achieve perfect isotropy. Adjustment should involve one or more scale factors depriving telemetry from the function of size determination. Now, the intermediary equation for one distance measurement:

$$d_{ij} + v_{ij} = c_k \sqrt{(y_j - y_i)^2 + (x_j - x_i)^2} \tag{10}$$

where d_{ij} is the measured distance; v_{ij} the correction; and c_k the scale factor introduced as a new unknown.

The second stage of computations is to reckon with the function of telemetry to determine sizes, so that in knowledge of the network shape, its scale gets determined. This computation takes advantage of the unit value of measured scale factors after correction for the atmosphere.

If in course of adjusting the network shape, a single scale factor is introduced for all the telemetry, adjustment differs from the traditional procedure only in that the network is assigned the scale only afterwards.

It is, however, more expedient to introduce one scale factor for each standpoint. Effective non-unit factors of distance measurements are primarily due to the inadequate knowledge of atmospheric conditions. Refractive index of air for carrier waves can be calculated from temperature, barometric pressure, partial humidity values. Under average measurement conditions, mean error of refractive index determination is $\pm(1-5)$ mm/km. Even if the residual error of atmospheric measurement corrections is different for each measurement, it can be assumed to be similar for measurements made from the same standpoint at a slight delay, partly because of the similar atmospheric conditions, and partly of the common or much correlated atmospheric measurements.

Hence, this calculation method is also argued for by the possibility to reduce such systematic errors.

Measurement method providing for perfect isotropy

Let us consider the function of sightings and distance measurements in forming the coefficient matrix of the normal equation system of shape adjustment. Partial derivatives of sightings and distance measurements with respect to unknowns are:

$$\frac{\partial I_{ij}}{\partial y_j} = -\frac{\partial I_{ij}}{\partial y_i} = \frac{x_j - x_i}{s_{ij}^2} = a_{ij}$$

$$\frac{\partial I_{ij}}{\partial x_j} = -\frac{\partial I_{ij}}{\partial x_i} = \frac{y_j - y_i}{s_{ij}^2} = b_{ij}, \quad \frac{\partial I_{ij}}{\partial z_i} = -1$$

$$\frac{\partial d_{ij}}{\partial y_j} = -\frac{\partial d_{ij}}{\partial y_i} = \frac{y_j - y_i}{s_{ij}} = -s_{ij} b_{ij}$$

$$\frac{\partial d_{ij}}{\partial x_j} = -\frac{\partial d_{ij}}{\partial x_i} = \frac{x_j - x_i}{s_{ij}} = s_{ij} \; a_{ij}, \quad \frac{\partial d_{ij}}{\partial c_i} = s_{ij}$$

where I_{ij} is the sighting value between points i and j; d_{ij} and s_{ij} are measured and calculated distance, resp.; z_i is the orientation constant; and c_i the scale factor. Comprising partial derivatives into a vector each, and introducing notations:

$$f_{ij} = \frac{\partial I_{ij}}{\partial y}, \quad g_{ij} = \frac{\partial I_{ij}}{\partial x} \tag{11}$$

partial derivatives of distance measurements with respect to coordinates are

$$\frac{\partial d_{ij}}{\partial y} = -s_{ij} f_{ij}, \quad \frac{\partial d_{ij}}{\partial x} = s_{ij} g_{ij}$$

Only weight coefficient matrices of coordinates being needed as a rule, orientation constants and scale factors being independent, they may be eliminated by means of Schreiber's equations. Let us write coefficient matrix N of the normal equation system:

$$\mathbf{N} = \Sigma p^I_{ij} \begin{bmatrix} f_{ij} f'_{ij} & f_{ij} g'_{ij} \\ g_{ij} f'_{ij} & g_{ij} g'_{ii} \end{bmatrix} + \Sigma p^d_{ij} s^2_{ij} \begin{bmatrix} g_{ij} g'_{ij} & - g_{ij} f'_{ij} \\ -f_{ij} g'_{ij} & f_{ij} f'_{ij} \end{bmatrix} + \mathbf{N}_z + \mathbf{N}_c \tag{12}$$

where $\mathbf{N}_z$ and $\mathbf{N}_c$ are Schreiber's equation corrections belonging to orientation constants, and to scale factors, respectively. Comparing Eq. (12) with Eq. (9), perfect isotropy is seen to arise if sightings and distance measurements are made in the same sides of the network, and if weights of sightings and of distance measurements on the same side are related as:

$$p^I_{ij} = p^d_{ij} s^2_{ij} \tag{13}$$

Also the ratio of mean errors can be expressed from (13):

$$\frac{m^I_{ij}}{\varrho} = \frac{m^d_{ij}}{s_{ij}}$$

Measurements planned according to this coupling, respecting weight ratios between measurement pairs, automatically provide for perfect isotropy.

Safeguarding homogeneity

The described measurement method, although automatically provides for perfect isotropy, misses homogeneity. The measurement method is, however not unambiguous for a given network. Sighting-distance measurement pairs are selected, and respecting weight proportions, also weight values may be differently selected. Recommendation of group weights for sightings is though a limitation, group weights remain arbitrary. Properly selecting these free parameters yields a way to meet the condition of homogeneity.

Selecting a measurement layout provides for as many free unknowns as there are standpoints. The number of conditions equals that of network points:

$$q_{ii} = 1.$$

This set of equations can be solved as a rule, but it being other than linear, the solution cannot be obtained by a direct method. In experimental computations, the following iteration method worked: Assuming arbitrary (e.g. equal) values for group weights, matrix $\mathbf{Q}$ is calculated. In places of these preliminary values, tangential hyperplanes of conditions are written, and solution of the obtained linear equation system, that is, intersection of tangential planes, is considered as the subsequent approximation. This procedure is repeated until homogeneity is approximated to the needed degree.

Let us consider now how to establish the set of equations for the tangential planes. Weight coefficient to a coordinate is obtained as:

$$q_{ii} = e_i' \mathbf{Q} e_i \tag{14}$$

where e_i is the i-th unit vector. Its partial derivative with respect to a measurement is obtained from:

$$\frac{\partial q_{ii}}{\partial p_k} = -e_i' \mathbf{Q} a_k a_k' \mathbf{Q} e_i = -(e_i' \mathbf{Q} a_k)^2$$

Knowledge of partial derivatives permits to write the set of tangential plane equations:

$$\sum_{k=1}^{m} \frac{\partial q_{ii}}{\partial p_k} p_k - q_{ii}^{\circ} p_k^{\circ} = 0 \quad (i = 1, 2, \ldots, n) \tag{15}$$

If this set of equations is inconsistent, then it is advisable to apply its pseudoinverted for the solution to be obtained between parallel tangential planes.

This method was found to rather rapidly converge. The exact value is normally obtained after the third iteration. Since this method provides for a statistically homogeneous and perfectly isotropic weight coefficient matrix for a wide range of measurement layouts, several optimum designs coping with the original goal are possible. From among these optimum designs, those where measurement weights little differ are advisably selected, lest excessive repetition numbers are required.

Let us see now, as an example, the outcome of designing a network by this method. The network is of the form seen in Fig. 1.
List of network coordinates:

Point no.	Y	X
1	+189.2	+328.9
2	+409.2	+368.7
3	+577.5	+365.5
4	+608.8	0.0
5	+394.6	−31.0
6	0.0	+3.0

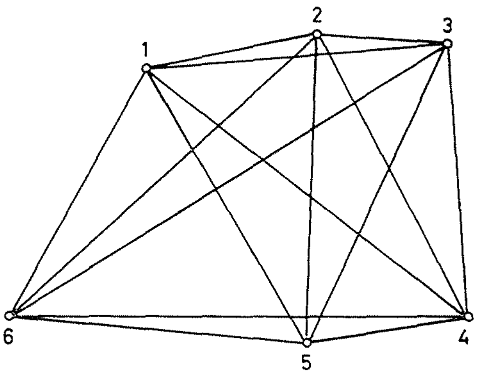

Fig. 1

In designing the network, beside perfect isotropy, unit radii of absolute error circles were required. Final results of computations yielded the measurement weights:

Stand-point	Sighting point	Sighting weight	Telemetry weight
1	2	0.162	0.138
	3	0.162	0.045
	4	0.162	0.024
	5	0.162	0.040
	6	0.162	0.048
2	1	0.145	0.123
	3	0.145	0.217
	6	0.145	0.020
3	1	0.440	0.123
	4	0.440	0.139
	5	0.440	0.098
	6	0.440	0.040
4	1	0.228	0.034
	2	0.228	0.055
	3	0.228	0.072
	5	0.228	0.207
	6	0.228	0.026
5	1	0.143	0.035
	2	0.143	0.038
	3	0.143	0.031
	4	0.143	0.130
	6	0.143	0.038
6	1	0.533	0.159
	2	0.533	0.075
	3	0.533	0.048
	4	0.533	0.061
	5	0.533	0.144

Radii of absolute error circles from design measurement weights were of unit size at an accuracy of 10^{-7}, while radii of relative error circles were:

	Network points	Relative error circles
1	2	1.501
	3	1.615
	4	1.456
	5	1.491
	6	1.672
2	3	1.571
	4	1.615
	5	1.533
	6	1.524
3	4	1.654
	5	1.548
	6	1.340
4	5	1.497
	6	1.515
5	6	1.671

In addition to equal absolute error circles, also radii of relative error circles exhibit slight standard deviation, providing for homogeneous network structure.

Continuity of the computation method involves non-integer repetition numbers calculated from the weights, hence the final plan is obtained after rounding, and selecting a purposeful instrument pair. This problem will be helped by applying integer valued programming methods. Remind, however, that also a measurement plan obtained by rounding off is expedient. In the presented example, error circles calculated from rounded off weights differ from the design values by less than 3%.

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