

THEORY OF THE RAILWAY MOVEMENT GEOMETRY AND ITS USE IN PRACTICE

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Summary

The investigation of the movement taking place on the railway track, the kinematic geometry of this phenomenon as well as the determination of the *track geometry adequate to the movement* of the rail vehicles are dealt with.

In the field of the fixed guideway system it is the *rail track which primarily determines the movement*, therefore, both from the point of view of theory and practice it is the movement which should be taken for basis as the most important determining factor in developing the track geometry.

1. Function and method of the kinematic geometry

The *task of the railway kinematic geometry* is, in knowledge of the state and characteristics of the motion, to investigate and determine the geometric structure of the track by which the geometric form of the track is meant and this will be referred to in the following with the simple term *geometry*.

In the course of the kinematic or movement-geometric examination the movement is treated separately and independently from the factors generating it. The problems belonging to the subject matter of the dynamics (or kinetics) of the transport, that is, determining the state of motion (i.e. the uniform, accelerated or decelerated phases of motion) induced by active or passive forces are not dealt with here.

In this study the concepts and aspects of the classic mechanics are used. Thus,

movement is examined in
space and in
time,

the characteristics of the movement being defined as the functions of time.

The basis of the movement-geometric aspect is the *knowledge of the kinematic law of movement*. The motion of the point, or the system of points (i.e. the solid body) is unequivocally determined in case where the position of the

point is known at any moment in short, the law of motion of the system of points is familiar.

Considering the state of motion, the theory of the geometry of movement in accordance with reality permits the examination of the state not only of the steady motion of constant value but, also of that of an *optional kind of motion of a general character*.

The geometrical elements of a rail track of spatial alignment are defined by the *curvature* (G), or by the *change in curvature* (dG/dl). According to their hierarchy of order in the geometry of motion, track elements of

a) constant curvature, i.e.

b) changing curvature

might be distinguished (Fig. 1).

The track element of constant curvature is the *tangent section* (the curvature of which is, according to definition, zero), and the *circular arc section* with a constant curvature of the inverse value of the radius of the circle); a track element of a higher order is the *transition curve* the curvature of which varies with the length of the curve.

In the course of the study, at *tangent sections*, in general, no problems arise, they deserve some special attention only in drawing up the trace, namely in determining their length between curves in plane, and their slopes in profile.

A careful investigation is needed for a suitable formation of the curvilinear section, and within this, especially that of the transition curves where the *geometry of the track is defined by the task to be fulfilled*. In this paper special attention is given to the movements of higher speed with a view to adequate by

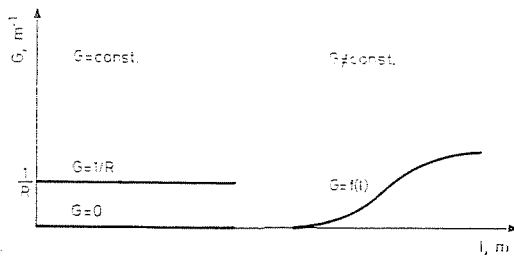


Fig. 1.

developing the rail track, and, in the first line the curvilinear sections. To express the interdependence between the movement and the geometry of the track and for the kinematic characterization of the motion, the *vectors which describe the time-dependent change of the movement* (i.e. the speed vector v , the acceleration vector a , the h vector and the m vector) are used.

During investigating, while making use of the movement geometry, the general *engineering way of looking* is taken as a basis. Correspondingly, knowing

the permissible kinematic load (acceleration, \mathbf{h} vector) partly the structure of the track geometry will be determined, and partly stresses will be calculated and compared to the permissible (i.e. the limiting) stresses. Taking all the

Fundamental cases of movement-geometric examinations

Fundamental cases		Given	Desired
A	Geometric dimensioning	1. Movement state ($v; a_t$) 2. Limiting stresses $ \bar{a}_n ; \bar{h}_n $	Geometrical structure
B	Geometric checking (Examination of the consistency of a given track geometry)	1. Geometrical structure 2. Limiting stresses $ \bar{a}_n ; \bar{h}_n $	1. $ \bar{a} ; (\bar{a} \leq \bar{a}_n)$ 2. $ \bar{h} ; (\bar{h} \leq \bar{h}_n)$
C	Valuating comparison of different geometries (Optimizing problem)	1. Geometrical structures 2. Limiting stresses $ \bar{a}_n ; \bar{h}_n $	1. $ \bar{a}_{min} $ 2. $ \bar{h}_{min} $

Fig. 2.

mentioned into account, according to the examination of the movement geometry, the following basic cases are to be distinguished (Fig. 2):

A) *geometric dimensioning*:

knowing the state of given motion and the limiting stresses, the selection of the geometric structure and the determination of its dimensions,

B) *checking the track geometry in a given case*:

determining the adequacy of an existing track length of given geometry and dimensions in case of a given motion state and limiting stresses,

C) *comparison and estimation of the different geometries*:

determination of the order of sequence and *optimization* of the geometric structures available for selection.

The movement-geometric examinations always have double objectives, namely the determination of the acceleration (\mathbf{a}) vector, and

the characteristic of 3rd order motion (\mathbf{h} vector),

and knowing these, to establish the *design state* and thereafter to solve the movement-geometric problem.

2. Vector equation of the rail track as a spatial curve

In the following, if no distinctive remark is made, by the term "track", in accordance with the designing practices, in general, the *centre line of the track* including the two rails is meant. (The centre line of the track fits in general the midpoints of the gauge, however, in case of a gauge-widening in the curves it lies at a distance of half the value of the normal gauge to the outer rail.)

The centre line of the track is, in reality, of spatial trace, the geometric elements of which are the tangents, the circular arcs and the transition curves.

In examining the track as a *curve in space*, one makes use of the knowledge of the *differential* geometry which applies the analytic method in geometry. Since the calculations are carried out in the first line by applying the differential calculus, one *assumes that the functions* entering in the equations are *continuous* and, in accordance with the character of the problem, can be derived continuously.

In examining the track as a space curve and the motion on the track as a phenomenon proceeding in time t , the position of the points of the track is described by the vector-scalar function

$$\boxed{\mathbf{r} = \mathbf{r}(t)} \quad (2.1)$$

where the position vector \mathbf{r} is directed from the starting point Q (from the origin of the system of coordinates x, y, z involved in our investigation) to the investigated point $P(x, y, z)$ (Fig. 3).

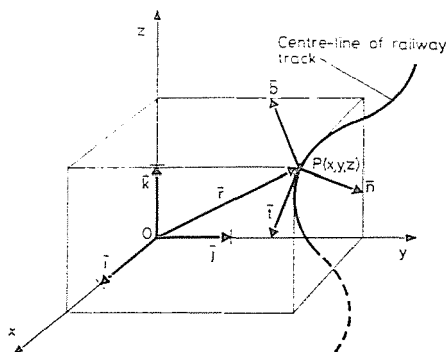


Fig. 3.

During the change of the scalar parameter t the end point of the position vector moves along the space curve describing the track.

Since the parameter t is a scalar variable, to each possible value of which a vector is coordinated, the motion of the point taking place on the rail track, is described by the *vector-scalar* characterized in this way.

Be the orthogonal coordinates of position vector \mathbf{r} , x , y and z functions of time parameter t . In this case, the *vector equation* of the track as a space curve is as follows

$$\boxed{\mathbf{r} = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}} \quad (2.2)$$

which is the expression by coordinates of the *vector function* (2.1). In the vector equation, i , j , k designate the x , y and z unit vectors (Fig. 3).

The equations determining the track described in the form

$$x = x(t), \quad y = y(t) \quad \text{and} \quad z = z(t) \quad (2.3)$$

represent the *scalar system of equations* of the space curve of the track, the three scalar equations of which are equivalent with the vector equation (2.2).

3. Vector-kinematics of the point

In the course of the kinematic study of the movement on the track the way to determine the law of motion, the vectors characterizing the motion (the speed and acceleration vectors, vector- h , vector- m) and the attending trihedron is dealt with. Since in the course of the motion-geometric investigation the motives generating the movement are not discussed, in the following the concepts of the geometric point and the material point are considered as identic.

3.1 Law of motion of the movement of a point on the track

The law of motion of a point moving along the space line of a track is determined by the vector-scalar function

$$\mathbf{r} = \mathbf{r}(t).$$

In the form of coordinates

$$\mathbf{r} = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}$$

wherein:

x, y, z — coordinates of position vector \mathbf{r}

$\mathbf{i}, \mathbf{j}, \mathbf{k}$ — unit vectors of directions x, y , and z , respectively (Fig. 3).

The point can be displaced in three optional directions in space, and so the number of scalar equations determining the coordinates, identical with the degree of freedom of the motion of the point, is also three: $x(t)$, $y(t)$, $z(t)$.

3.2 The attending trihedron

In the course of movement geometric investigations, when defining the characteristic vectors of the movement, the role of the *attending trihedron* is of special significance.

The attending trihedron is defined by three preferred orientation unit vectors couple by couple perpendicular to each other. These unit vectors are: the tangent oriented unit vector \mathbf{t} , the principal-normal oriented unit vector \mathbf{n} and the binormal oriented unit vector \mathbf{b} .

In the system of coordinates x, y, z the position of the attending trihedron changes with the motion of the point on the space curve (Fig. 3).

1 The tangent oriented unit vector

The tangent vector of the space curve, i.e. the velocity of the motion is determined by the first derivative of the position vector with respect to time.

$$\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}.$$

The tangent directed unit vector is

$$\mathbf{t} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}. \quad (3.1)$$

The tangent oriented unit vector can also be determined directly by substituting the arc-length-function $t = t(a)$ into equation (2.1) of the space curve, as the tangent oriented unit vector \mathbf{t} is given by the derivative with respect to the arc length of the position vector

$$\mathbf{t} = \frac{d\mathbf{r}}{ds}. \quad (3.2)$$

The tangent vector of the curve related to the arc-length parameter is always a unit vector, namely

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt} \frac{dt}{ds} = \dot{\mathbf{r}} \frac{1}{\frac{ds}{dt}} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \mathbf{t}$$

2 The principal-normal directed unit vector

Unit vector \mathbf{n} of the direction of the principal normal is perpendicular to the tangent oriented unit vector \mathbf{t} and lies on the osculatory plane associated with point P of the space curve (Fig. 4).

The osculatory plane can be interpreted according to Fig. 4 as follows: consider three different points P_1, P_2, P_3 not fitting to the very same line, which determine a plane in all positions. If points P_1, P_2, P_3 tend towards point

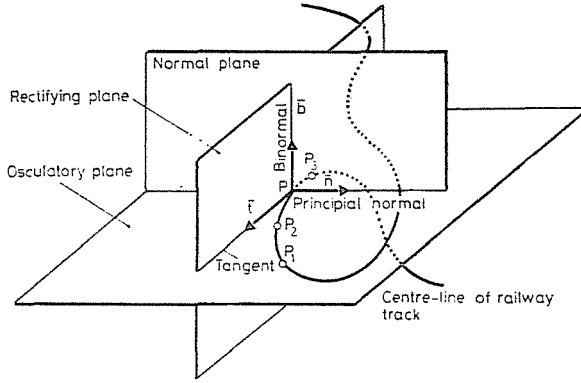


Fig. 4.

P of the space curve then, the limiting position of the plane determined by points P_1, P_2, P_3 is called the osculatory plane of the space curve at point P .

Therefore, the vector

$$\frac{dt}{ds} = \frac{d^2\mathbf{r}}{ds^2} \tag{3.3}$$

falls on the line of the principal normal, i.e. perpendicularly to the line of the tangent, and is the first derivative with respect to the length of arc of the tangent unit vector, i.e. the second derivative vector with respect to the length of arc of the position vector. The absolute value of this vector is

$$\left| \frac{dt}{ds} \right| = G = \frac{1}{\rho} \tag{3.4}$$

with $G =$ curvature of track (m^{-1});

$\rho =$ radius of curvature (m).

The formula of the unit vector \mathbf{n} which fits to the principal normal direction directed towards the centre of curvature is as follows:

$$\mathbf{n} = \frac{\frac{dt}{ds}}{\left| \frac{dt}{ds} \right|} = \frac{1}{G} \frac{dt}{ds} = \rho \frac{dt}{ds} \tag{3.5}$$

3 Unit vector of binormal direction

The unit vector of binormal direction is perpendicular to unit vectors \mathbf{t} and \mathbf{n} of tangential and principal normal directions, respectively, constituting with them a right-spin system (i.e. looking from the end point of \mathbf{b} , the sense

of the shorter rotation transmitting \mathbf{t} into \mathbf{n} is a counterclockwise rotation) (Fig. 4.)

The unit vector of binormal direction is:

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} \quad (3.6)$$

4 The three unit vectors of the attending trihedron determine three planes (Fig. 4), namely

unit vectors \mathbf{t} and \mathbf{n} determine the osculatory plane,
 unit vectors \mathbf{n} and \mathbf{b} determine the normal plane and
 unit vectors \mathbf{t} and \mathbf{b} determine the rectifying plane.

3.3 Kinematic motion characterizing vectors.

For a kinematic representation of the motion taking place on the track, the *motion characteristics* describing the variation of motion in time, i.e. representing the interconnection of the motion and the geometry are used.

At an early stage of the development of railway transport — due to low speed — to describe the motion, a knowledge of the pairs of values of distance-time, i.e., the *speed* were sufficient.

However, later on, in the course of further railway transport development, due to a continuous increase in speed a characterization of the motion on the rail track, an adequate development of the track became inevitably more and more sophisticated, and besides velocity, as critical designing factor, also *acceleration* presented itself as a critical concept in defining the motion on the track.

Nowadays, however, on tracks suitable for high (120 . . . 200 km/h) or very high speed (over 200 km/h) in consequence of a speed surpassing all ever applied the characterization of the motion, i.e. the convenient determination of the geometry of the track inevitably requires the establishment of *kinematic characteristics of motion of a higher order* expressing and suggesting a sophisticated consideration of the variation of motion in time.

The position of a point carrying out a motion on the rail track of curvilinear space is determined by the position vector $\mathbf{r} = \mathbf{r}(t)$ in the knowledge of which it can be established that the most significant kinematic characteristics of the motion are in the following sequence:

speed vector designated with the symbol \mathbf{v} ; (m/s);
acceleration vector, designated with \mathbf{a} ; (m/s²);
third order or h vector, \mathbf{h} (m/s³).

3.4 The speed vector

A significant characteristic of the movement taking place on the track is the *speed* which is a vectorial quantity. Its symbol is v , the unit value m/s or in practical railway use: V km/h.

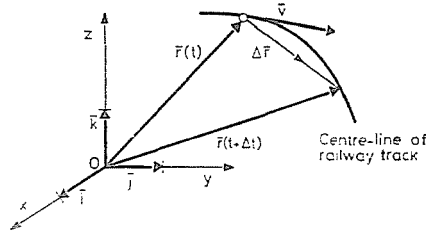


Fig. 5.

By applying the displacement vector (Fig. 5)

$$\Delta \mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$$

its mathematical formula becomes as follows:

$$\mathbf{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} \quad (3.7)$$

i.e. the *speed vector* is the first *vector derived with respect to time of the vector-scalar function* describing the track curve which is defined as the limiting value of the derivative.

Substituting the length of the curvilinear section of the track s and applying formula (3.2) results in:

$$\boxed{\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \cdot \frac{ds}{dt} = \mathbf{t} \cdot |\mathbf{v}| = \mathbf{t} |\dot{\mathbf{r}}|} \quad (3.8)$$

From this expression it follows that the speed vector is tangent-directed and its magnitude, as a scalar quantity, is given by the first derivative of the travel with respect to time, in the knowledge of the route-time function.

Knowing the form of the position vector expressed by coordinates (2.2) the formula given in a system of orthogonal coordinates is as follows

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \quad (3.9)$$

with the speed components

$$v_x = \dot{x}(t), \quad v_y = \dot{y}(t) \quad \text{és} \quad v_z = \dot{z}(t).$$

The magnitude of the speed vector is given by the formula

$$v = |\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} \quad (3.10)$$

and the angles formed with the coordinate-axe (i.e. the direction cosines) by formulae

$$\begin{aligned} \cos \alpha_v &= \cos(\mathbf{v}, x) = \frac{\dot{x}}{|\mathbf{v}|} \\ \cos \beta_v &= \cos(\mathbf{v}, y) = \frac{\dot{y}}{|\mathbf{v}|} \quad \text{és} \\ \cos \gamma_v &= \cos(\mathbf{v}, z) = \frac{\dot{z}}{|\mathbf{v}|} \end{aligned} \quad (3.11)$$

3.5 The acceleration vector

The significant characteristic of movement, more significant from the point of the geometric design of the track than that of speed, is the *acceleration vector*. While speed characterizes the displacement, acceleration is the characteristic parameter of the variation of speed. Its symbol is \mathbf{a} , and the measuring unit m/s^2 .

The acceleration vector of the moving point may be determined on the basis of Fig. 6 and formula (3.8), by using the following equation

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{\mathbf{r}} = \frac{d}{dt}(|\dot{\mathbf{r}}| \mathbf{t}) = \frac{dv}{dt} \mathbf{t} + v \frac{d\mathbf{t}}{dt} \quad (3.12)$$

As is seen, the acceleration vector is the first derived vector of the speed vector with respect to time and the *second derived vector of the position vector*

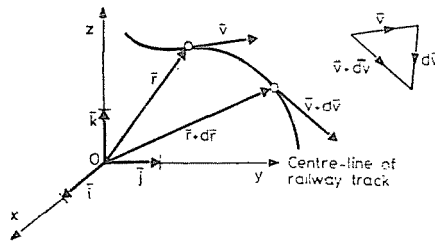


Fig. 6.

with respect to time. The components of the acceleration vector entering in the formula (3.12) are perpendicular to each other, namely the derivation of equation $t^2 = 1$ results in

$$\mathbf{t} \cdot \frac{d\mathbf{t}}{dt} = 0$$

from which follows that vectors \mathbf{t} and $d\mathbf{t}/dt$ are perpendicular to each other. One of the components is *tangent-directed*, and the other is expressed by the relationship (established by making use of relation (3—5)):

$$v \frac{d\mathbf{t}}{dt} = v \frac{d\mathbf{t}}{ds} \frac{ds}{dt} = v^2 \frac{d\mathbf{t}}{ds} = v^2 G \mathbf{n} \quad (3.13)$$

i.e. its direction agrees with that of the *principal normal*. Since the tangent and the principal normal determine the osculatory plane of the curve, the acceleration vector lies on the osculatory plane of the track.

In the formula

$$\mathbf{a} = \frac{dv}{dt} \mathbf{t} + v^2 G \mathbf{n} = \mathbf{a}_t + \mathbf{a}_n \quad (3.14)$$

of the acceleration vector the *tangent directed acceleration* measures the change of the magnitude of speed and its sense corresponds to the sign of dv . In case of an accelerating motion dv is positive, i.e. the sense of the tangent oriented acceleration is the same as that of the speed vector while in case of a decelerating motion it is opposed to that.

The normal directed component of the acceleration vector, the *normal acceleration* measures the change in the direction of the speed vector. Since it is of a principal-normal direction, independently of the direction of the speed vector, it always points towards the centre of curvature (in agreement with the direction of the unit vector \mathbf{n}) wherefore it is called a *centripetal acceleration*.

It ensues from the above that the acceleration vector lies in the osculatory plane on the side of the tangent where also the track is to be found. At the point of inflection where the tangent intersects the travel of the moving point only tangential acceleration results, namely, since $G = 0$ is true, the acceleration of normal direction is equal to zero. And in addition, it is to be seen that *only the acceleration of a point travelling along a straight line might be equal to zero*, since at a curved travel necessarily a normal directed acceleration is induced in every case by the change of direction of the speed vector.

Given the knowledge of the coordinate-formula (2.2) of the position vector \mathbf{r} , the acceleration vector is given by the formula

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} \quad (3.15)$$

the magnitude of its components lying along the lines of the coordinate axes are as follows

$$a_x = \ddot{x}(t); \quad a_y = \ddot{y}(t) \quad \text{and} \quad a_z = \ddot{z}(t).$$

The equation of the acceleration vector is

$$a = |\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \quad (3.16)$$

and the direction cosines (their angles with the axes of the coordinate) by equations:

$$\begin{aligned} \cos \alpha_a &= \cos(\mathbf{a}, x) = \frac{\ddot{x}}{|\mathbf{a}|} \\ \cos \beta_a &= \cos(\mathbf{a}, y) = \frac{\ddot{y}}{|\mathbf{a}|} \quad \text{and} \\ \cos \gamma_a &= \cos(\mathbf{a}, z) = \frac{\ddot{z}}{|\mathbf{a}|} \end{aligned} \quad (3.17)$$

The acceleration vector is inclined at an angle

$$\operatorname{tg} \lambda = \frac{|\mathbf{a}_n|}{|\mathbf{a}_t|} \quad (3.18)$$

to the direction of the tangent.

The values of the so-called lateral acceleration (a_0 , m/s²) developing in the normal plane permissible on the lines of some foreign railways are indicated in *column 2 of Annexe 1*, established on the basis of the prescriptions of the railway in question.

Table 1 contains the values of the *lateral acceleration* established on the basis of the entries of *Annexe 1*, the measurements performed and the results of theoretical investigations which serve as basis for the geometric examinations and for the determination of the numerical values.

From among the kinematic characteristics of a higher order the *characteristics of the third order or h vector* is in the first line the geometrical determinant of the curvilinear railway tracks serving for very high speeds, and also the generator and measure of physiological effects. Its designation is **h**, the measuring unit m/s³.

Table 1
Permissible values of the unbalanced radial acceleration

Characteristic of the change of curvature	The magnitude of unbalanced radial acceleration
In case of a gradual change of curvature (e.g. circular curve with transition)	0.65
In case of a discontinuous change of curvature (e.g. circular curve without transition)	0.35

The \mathbf{h} vector gives an exact information on the variation of acceleration, therefore, by making use of the relationship (3.12) the following may be written

$$\mathbf{h} = \frac{d\mathbf{a}}{dt} = \ddot{\mathbf{r}} = \frac{d}{dt} \left(\frac{dv}{dt} \mathbf{t} + v \frac{d\mathbf{t}}{dt} \right) = \frac{d^2v}{dt^2} \mathbf{t} + 2 \frac{dv}{dt} \frac{d\mathbf{t}}{dt} + v \frac{d^2\mathbf{t}}{dt^2} \quad (3.19)$$

i.e. the third order \mathbf{h} vector is the first derived vector with respect to the time of the acceleration vector and the third derived vector with respect to the time of the position vector.

In the formula (3.19)

$$1. \quad \frac{dv}{dt} = a_t \quad \text{m/s}^2, \quad (3.20)$$

$$2. \quad \frac{d^2v}{dt^2} = \frac{da_t}{dt} \quad \text{m/s}^3, \quad (3.21)$$

3. according to (3.13)

$$\frac{d\mathbf{t}}{dt} = vG\mathbf{n} \quad \text{s}^{-1}, \quad (3.22)$$

$$4. \quad \frac{d^2\mathbf{t}}{dt^2} = \frac{dt}{dt} (vG\mathbf{n}) = \frac{dv}{dt} G\mathbf{n} + v \left(\frac{dG}{dt} \mathbf{n} + G \frac{d\mathbf{n}}{dt} \right) \quad (3.23)$$

where dG/dt — derivative of the curvature with respect to time,

dn/dt — derivative of the unit vector \mathbf{n} with respect to time.

Let us express the derived vector

$$\frac{d\mathbf{n}}{dt} = \frac{d\mathbf{n}}{ds} \frac{ds}{dt} = \mathbf{n}'v$$

with the aid of the unit vectors of the attending trihedron. Since the unit vector \mathbf{n} (of constant length) is of a variable direction it is true that

$$\mathbf{n}' \perp \mathbf{n}$$

it can be written that

$$\mathbf{n}' = a_{21}\mathbf{t} + a_{23}\mathbf{b} \quad (3.24)$$

For determining the coefficients let us multiply equation (3.24) in a scalar way with unit vectors \mathbf{t} and \mathbf{b} . Then $\mathbf{n}'\mathbf{t} = a_{21}$, however, the derivation of $\mathbf{n} \cdot \mathbf{t} = 0$ gives $\mathbf{n}'\mathbf{t} = -\mathbf{n}t'$, and from the formula (1.3.13) $\mathbf{t}' = G\mathbf{n}$, therefore

$$a_{21} = -G$$

and in the second case $\mathbf{n}' \cdot \mathbf{b} = a_{23}$ def. T

where G — curvature of track, m^{-1} and

T — torsion of track, m^{-1} .

Hereafter

$$\mathbf{n} \cdot \mathbf{v} = (T\mathbf{b} - G\mathbf{t}) \cdot \mathbf{v} \quad (3.25)$$

and the expression (3.23) can be written as follows

$$\frac{d^2\mathbf{t}}{dt^2} = a_t G \mathbf{n} + v \left[\frac{dG}{dt} \mathbf{n} + Gv(T\mathbf{b} - G\mathbf{t}) \right] \quad (3.26)$$

By making use of the relationships (3.19), (3.20), (3.21), (3.22) and (3.26), the formula of the \mathbf{h} vector may be written, in the form

$$\mathbf{h} = \left(\frac{da_t}{dt} - v^2 G^2 \right) \mathbf{t} + \left(3va_t G + v^2 \frac{dG}{dt} \right) \mathbf{n} + v^3 G T \mathbf{b} \quad (3.27)$$

Summarizing the symbols entering in formula (3.27):

- v — value of speed, m/s;
- a_t — value of tangent-directed acceleration, m/s²;
- G — curvature of track, m⁻¹;
- T — torsion of track, m⁻¹;
- dG/dt — derivative of track curvature with respect to time, m⁻¹s⁻¹.

From formula (3.27) it is to be seen that \mathbf{h} in case of a space track, *rises from the osculatory plane of the track*, and is composed of three orthogonal components. The magnitude of the components can be calculated from five relations which are the definite functions of the speed, acceleration, the derivative of the acceleration with respect to time, the curvature, the derivative of the curvature with respect to time and torsion.

From formula (3.27) of the \mathbf{h} vector the following ensue:

— In case of a constant speed motion or of an accelerating motion the \mathbf{h} vector inclines from the rectification plane of the track (i.e. from the plane determined by the tangent and the binormal) towards the centre of curvature (see Fig. 7 demonstrating the spatial position of the \mathbf{h} vector, the acceleration vector and the speed vector).

— In case of a movement along tangents with constant speed or with acceleration, the magnitude of the \mathbf{h} -vector is always zero.

Since in most of the movement-geometric examinations the effect of the \mathbf{h} factor is decisive (see Chapter 6), the examination of the \mathbf{h} vector dwindling to zero in case of a curvilinear track ($G \neq 0$) is of particular significance.

By making use of the formula of the position-vector \mathbf{r} (2.2) expressed by using coordinates, the formula of the \mathbf{h} vector reads

$$\mathbf{h} = \frac{d\mathbf{a}}{dt} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} \quad (3.28)$$

Table 2
Permissible magnitudes of the h vector

Character of change of curvature	Magnitude of the 3rd order characteristic
In case of a continuous, breakless change of curvature (function dG/dl)	0.5
In case of a change of curvature with break (function dG/dl)	0.4
In case of a discontinuous change of curvature (function dG/dl)	0.3
In case of the circular curve and the straight line joining without transition	0.2

4. Vector kinematics of rigid bodies

In the following the *movement geometry of a rigid body* will be investigated by taking into account the general statements made when discussing the vector kinematics in the preceding chapter.

In the investigations the rigid body is considered as a *system of points*, that is, a population of mass points.

A characteristic feature of the rigid body is that the spacings of its points do not change in the course of its motion. However, in reality in case of *material bodies* the spacings of the points can vary under the effect of an external force but, in the course of the kinematic examination of the railway track one may use the geometric abstraction of the *rigid body*, since in the course of investigation the incidental changes in the dimensions of the moving body are negligible compared with the dimensions of the rigid body.

4.1 Law of motion of the rigid body (i.e. system of material points)

The *motion of the system of points* (the rigid body) from the point of kinematics is unequivocally definite if one knows the position in space (i.e. the space coordinates) of the system of points at any moment, in short, the law of motion of the system of points (the rigid body).

The law of motion of the rigid body as read from Fig. 8 can be written as follows

$$\mathbf{r}(\mathbf{w}, t) = \mathbf{r}_A(t) + \varrho(t, \xi, \eta, \zeta) = \mathbf{r}_A(t) + \varrho(t, \mathbf{w}) \quad (4.1)$$

which is a *vector-vector function* and describes the motion of the rigid body in the way of a two-variable function of variables t and \mathbf{w} .

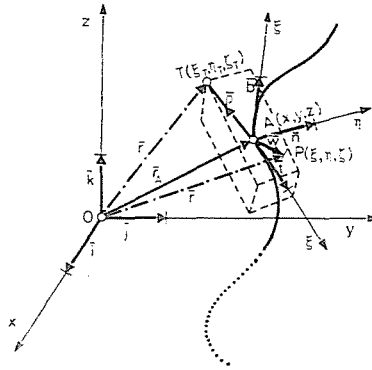


Fig. 8.

The symbols entering in formula (4.1) have the following meanings:

- $r_A(t)$ — position vector of selected point $A(x, y, z)$ (incidentally centre of gravity) of a rigid body (vehicle) in the system of coordinates x, y, z (briefly “object-vector”);
- $w(t)$ — vector directed from a selected point of the rigid body (vehicle) A towards optional point P of the vehicle examined, the position vector in system ξ, η, ζ (Fig. 8);
- $r(w, t)$ — position vector of an optional point of the rigid body (vehicle) in the system of coordinates x, y, z (briefly “image vector”).

Written in form of coordinates, the law of motion of the rigid body reads:

$$r(w, t) = [x_A(t)\mathbf{i} + y_A(t)\mathbf{j} + z_A(t)\mathbf{k}] + \varrho[\xi\mathbf{t}(t) + \eta\mathbf{n}(t) + \zeta\mathbf{b}(t)] \quad (4.2)$$

(From formula (4.2) it is to be seen that the degree of freedom of the rigid body is six.)

Stabilization of the variable w gives the law of motion of a selected point of the rigid body (vehicle), ($\xi = \text{const}$; $\eta = \text{const}$; $\zeta = \text{const}$), consequently in case of point T (see Fig. 8):

$$r_T(w, t) = [x_A(t)\mathbf{i} + y_A(t)\mathbf{j} + z_A(t)\mathbf{k}] + [\xi_T\mathbf{t}(t) + \eta_T\mathbf{n}(t) + \zeta_T\mathbf{b}(t)], \quad (4.3)$$

while stabilization of the time-parameter t permits the *determination of the positions of all of the points of the rigid body at a given moment of time ($t = \text{const.}$; $\xi \neq \text{const.}$; $\eta \neq \text{const.}$; $\zeta \neq \text{const.}$).*

4.2 Vectors characterizing the motion

For a kinematic characterization of the motion taking place on the railway track of spatial trace, the *vectors characterizing the motion*, i.e. describing its progression in time are used which, at the same time express the *relation between the motion and geometry*.

vehicle, which may be the centre of gravity of it, is \mathbf{r}_A . The components of the vector ρ oriented from point A towards the critical point T examined are as follows:

- in the tangential direction: a.t,
- in a cross direction (perpendicular to the track centre-line, along the line connecting the running faces of the rails):

$$b \frac{\Delta \mathbf{r}}{|\Delta \mathbf{r}|} \quad \text{and}$$

- perpendicularly to the plane placed on the tangent, i.e. cross direction

$$c \left(\mathbf{t} \times \frac{\Delta \mathbf{r}}{|\Delta \mathbf{r}|} \right).$$

where:

- a, b, c — longitudinal measurements established from dimensions of the vehicle, m ;
- \mathbf{t} — unit vector of tangent direction $\left(\mathbf{t} = \frac{d\mathbf{r}}{dl} \right)$;
- $\Delta \mathbf{r}$ — difference-vector $\mathbf{r}_A - \mathbf{r}_b$ at the cross section of the track examined (Fig. 9).

Knowing the position vector (“object vector”) and the vector ρ the position vector of the critical point of the vehicle T examined (“image vector”) can be given in the system of coordinates x, y, z with the aid of formula

$$\mathbf{r} = \mathbf{r}_A + \rho$$

and in the knowledge of the above equation the vectors characterizing the motion can be established. The differences between the values of the vectors characteristic to the motion at points T and A which furnish explanation to the practical use and significance of the examination of the motion of a rigid body or a point can also be determined.

4.3 Examination of curves without cant ($m = 0$)

Numerical data:

$$V = 160 \text{ km/h, } \left(v = \frac{160}{3.6} \text{ m/s} \right),$$

$$R = 4000 \text{ m, } (L = 88 \text{ m}),$$

$$a = 12.25 \text{ m,}$$

$$b = 1.43 \text{ m,}$$

$$c = 2.025 \text{ m.}$$

b) Examination of the speed vector

The formula of the speed vector at the critical point of vehicle T (Fig. 10) is:

$$\begin{aligned} \mathbf{v}_T = \frac{d\mathbf{r}}{dt} = & \left[v \cos \frac{v}{R} t - \frac{av}{R} \sin \frac{v}{R} t + \frac{bv}{R} \cos \frac{v}{R} t \right] \mathbf{i} + \\ & + \left[v \sin \frac{v}{R} t + \frac{av}{R} \cos \frac{v}{R} t + \frac{bv}{R} \sin \frac{v}{R} t \right] \mathbf{j} + 0 \mathbf{k} \end{aligned} \quad (4.7)$$

At the moment $t = 0$, the magnitude of the speed vector can be obtained with the aid of formula

$$|\mathbf{v}_T| = \sqrt{\left(v + \frac{bv}{R}\right)^2 + \left(\frac{av}{R}\right)^2}$$

and by replacement of the numerical data

$$|\mathbf{v}_T| = \sqrt{\left(\frac{160}{3.6} + \frac{1.43 \cdot 160}{3.6 \cdot 4000}\right)^2 + \left(\frac{12.25 \cdot 160}{3.6 \cdot 4000}\right)^2} = 160.058 \text{ km/h}$$

The magnitude of the speed vector at point A is given by the formula

$$\mathbf{v}_A = \frac{d\mathbf{r}_A}{dt} = \left(v \cos \frac{v}{R} t\right) \mathbf{i} + \left(v \sin \frac{v}{R} t\right) \mathbf{j} + 0 \mathbf{k} \quad (4.8)$$

The numerical value of the speed vector at point $A(t=0)$ is $V = 160 \text{ 000 km/h}$. The difference is $V = 0.058 \text{ km/h}$ which in comparison with point A means a change of 0.036 per cent.

c) Examination of the acceleration vector

At the point of vehicle examined one obtains

$$\begin{aligned} \mathbf{a}_T = \frac{d\mathbf{v}}{dt} = & \left[-\frac{v^2}{R} \sin \frac{v}{R} t - \frac{av^2}{R^2} \cos \frac{v}{R} t - \frac{bv^2}{R^2} \sin \frac{v}{R} t \right] \mathbf{i} + \\ & + \left[\frac{v^2}{R} \cos \frac{v}{R} t - \frac{av^2}{R^2} \sin \frac{v}{R} t + \frac{bv^2}{R^2} \cos \frac{v}{R} t \right] \mathbf{j} + 0 \mathbf{k} \end{aligned} \quad (4.9)$$

Substituting the numerical value into formula (4.9) gives, for the acceleration ($t = 0$):

$$|\mathbf{a}_T| = \sqrt{\frac{a^2 v^4}{R^4} + \left(\frac{v^2}{R} + \frac{bv^2}{R^2}\right)^2} = 0.4940 \text{ m/s}^2$$

The acceleration vector at point A is given by the equation

$$a_A = \frac{d\mathbf{v}_A}{dt} = -\left(\frac{v^2}{R} \sin \frac{v}{R} t\right) \mathbf{i} + \left(\frac{v^2}{R} \cos \frac{v}{R} t\right) \mathbf{j} + 0 \mathbf{k} \quad (4.10)$$

The numerical value of the acceleration at point A ($t = 0$) is as follows

$$|a_A| = \frac{v^2}{R} = 0.49383 \text{ m/s}^2$$

The difference Δa is 0.00017 m/s^2 (0.034 per cent). (For this examination the plane x, y of the system of coordinates is assumed to be horizontal.)

d) *Examination of the \mathbf{h} vector*

At the point of the vehicle examined the value of the \mathbf{h} vector can be obtained by using the formula:

$$\begin{aligned} \mathbf{h}_T = \frac{d\mathbf{a}}{dt} = & \left[-\frac{v^3}{R^2} \cos \frac{v}{R} t + \frac{av^3}{R^3} \sin \frac{v}{R} t - \frac{bv^3}{R^3} \cos \frac{v}{R} t \right] \mathbf{i} + \\ & + \left[-\frac{v^3}{R^2} \sin \frac{v}{R} t - \frac{av^3}{R^3} \cos \frac{v}{R} t - \frac{bv^3}{R^3} \sin \frac{v}{R} t \right] \mathbf{j} + 0 \mathbf{k} \end{aligned} \quad (4.11)$$

Replacement of the numerical data gives the numerical magnitude of the \mathbf{h} vector ($t = 0$):

$$|\mathbf{h}_T| = \sqrt{\left(\frac{v^3}{R^2} + \frac{bv^3}{R^3}\right)^2 + \frac{a^2v^6}{R^6}} = 0.005489 \text{ m/s}^3$$

The \mathbf{h} vector at point A can be calculated by applying the formula:

$$\mathbf{h}_A = \frac{d\mathbf{a}_A}{dt} = -\left(\frac{v^3}{R^2} \cos \frac{v}{R} t\right) \mathbf{i} - \left(\frac{v^3}{R^2} \sin \frac{v}{R} t\right) \mathbf{j} + 0 \mathbf{k} \quad (4.12)$$

The numerical value of the \mathbf{h} vector at point A ($t = 0$) is as follows:

$$|\mathbf{h}_A| = \frac{v^3}{R^2} = 0.005487 \text{ m/s}^3$$

The difference is

$$\Delta h = 0.000002 \text{ m/m}^3 \quad (\text{i.e. } 0.036 \text{ per cent}).$$

4.4 Examination of curves with a cant ($m \neq 0$)

Numerical data: $V = 160$ km/h,
 $R = 1300$ m,
 $m_R = 0.133$ m
 $a = 12.25$ m
 $b = 1.43$ m
 $c = 2.025$ m

a) Determination of vectors \mathbf{r}_A , \mathbf{q} , and \mathbf{r}

Considering Fig. 9 the following formulae can be written:

$$\mathbf{r}_A = \left(R \sin \frac{v}{R} t \right) \mathbf{i} + R \left(1 - \cos \frac{v}{R} t \right) \mathbf{j} + \frac{m_R}{2} \mathbf{k} \quad (4.13)$$

$$\mathbf{r}_b = \left[\left(R - \frac{3}{4} \right) \sin \frac{v}{R} t \right] \mathbf{i} + \left[\left(R - \frac{3}{4} \right) \left(1 - \cos \frac{v}{R} t \right) + \frac{3}{4} \right] \mathbf{j} + 0 \mathbf{k}$$

$$\Delta \mathbf{r} = \mathbf{r}_A - \mathbf{r}_b = \left(\frac{3}{4} \sin \frac{v}{R} t \right) \mathbf{i} - \left(\frac{3}{4} \cos \frac{v}{R} t \right) \mathbf{j} + \frac{m_R}{2} \mathbf{k}$$

$$\left(|\Delta \mathbf{r}| \approx \frac{3}{4} \text{ m} \right)$$

$$\mathbf{q} = a \mathbf{t} + b \frac{\Delta \mathbf{r}}{|\Delta \mathbf{r}|} - c \left(\mathbf{t} \times \frac{\Delta \mathbf{r}}{|\Delta \mathbf{r}|} \right) \quad (4.14)$$

from which one obtains by substitution of $vt = l$

$$a \mathbf{t} = a \frac{d\mathbf{r}_A}{dl} = \left(a \cos \frac{l}{R} \right) \mathbf{i} + \left(a \sin \frac{l}{R} \right) \mathbf{j} + 0 \mathbf{k}$$

$$b \cdot \frac{\Delta \mathbf{r}}{|\Delta \mathbf{r}|} = \left(b \cdot \sin \frac{l}{R} \right) \mathbf{i} - \left(b \cdot \cos \frac{l}{R} \right) \mathbf{j} + b \frac{2m_R}{3} \mathbf{k}$$

$$-c \left(\mathbf{t} \times \frac{\Delta \mathbf{r}}{|\Delta \mathbf{r}|} \right) = -c \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \frac{l}{R} & \sin \frac{l}{R} & 0 \\ \sin \frac{l}{R} & -\cos \frac{l}{R} & \frac{2m_R}{3} \end{bmatrix} =$$

$$= - \left(c \frac{2m_R}{3} \sin \frac{l}{R} \right) \mathbf{i} + \left(c \frac{2m_R}{3} \cos \frac{l}{R} \right) \mathbf{j} + c \mathbf{k}$$

the formula of the image vector \mathbf{r} is:

$$\begin{aligned} \mathbf{r} = \mathbf{r}_A + \varrho = & \left[R \sin \frac{v}{R} t + a \cos \frac{v}{R} t + b \sin \frac{v}{R} t - c \frac{2m_R}{3} \sin \frac{v}{R} t \right] \mathbf{i} + \\ & + \left[R \left(1 - \cos \frac{v}{R} t \right) + a \sin \frac{v}{R} t - b \cos \frac{v}{R} t + c \frac{2m_R}{3} \cos \frac{v}{R} t \right] \mathbf{j} + \\ & + \left[\frac{m_R}{2} + b \frac{2m_R}{3} + c \right] \mathbf{k} \end{aligned} \quad (4.15)$$

b) *Examination of the speed vector*

At the critical point of vehicle T the speed vector is

$$\begin{aligned} \mathbf{v}_T = \frac{d\mathbf{r}}{dt} = & \left[v \cos \frac{v}{R} t - \frac{av}{R} \sin \frac{v}{R} t + \frac{bv}{R} \cos \frac{v}{R} t - \frac{2cm_R v}{3R} \cos \frac{v}{R} t \right] \mathbf{i} + \\ & + \left[v \sin \frac{v}{R} t + \frac{av}{R} \cos \frac{v}{R} t + \frac{bv}{R} \sin \frac{v}{R} t - \frac{2cm_R v}{3R} \sin \frac{v}{R} t \right] \mathbf{j} + 0 \mathbf{k} \end{aligned} \quad (4.16)$$

The numerical value of the speed vector for $t = 0$ will be

$$|\mathbf{v}_T| = \sqrt{\left(v + \frac{bv}{R} - \frac{2cm_R v}{3R} \right)^2 + \frac{a^2 v^2}{R^2}} = 160.073 \text{ km/h}$$

The speed vector at point A :

$$\mathbf{v}_A = \frac{d\mathbf{r}_A}{dt} = \left(v \cos \frac{v}{R} t \right) \mathbf{i} + \left(v \sin \frac{v}{R} t \right) \mathbf{j} + 0 \mathbf{k} \quad (4.17)$$

At point A the magnitude of speed ($t = 0$) is 160 000 km/h, the difference $\Delta v = 0.073$ km/h which corresponds to 0.046 per cent.

c) *Examination of the acceleration vector*

At the point of vehicle examined we have:

$$\begin{aligned} \mathbf{a}_T = \frac{d\mathbf{v}}{dt} = & \left[-\frac{v^2}{R} \sin \frac{v}{R} t - \frac{av^2}{R^2} \cos \frac{v}{R} t - \frac{bv^2}{R^2} \sin \frac{v}{R} t + \frac{2cm_R v^2}{3R^2} \sin \frac{v}{R} t \right] \mathbf{i} + \\ & + \left[\frac{v^2}{R} \cos \frac{v}{R} t - \frac{av^2}{R^2} \sin \frac{v}{R} t + \frac{bv^2}{R^2} \cos \frac{v}{R} t - \frac{2cm_R v^2}{3R^2} \cos \frac{v}{R} t \right] \mathbf{j} + 0 \mathbf{k} \end{aligned} \quad (4.18)$$

By substituting the numerical data ($t = 0$) and by considering the gravitational acceleration one obtains:

$$|\mathbf{a}_T| = \sqrt{\frac{a^2 v^4}{R^4} + \left(\frac{v^2}{R} + \frac{bv^2}{R^2} - \frac{2cm_R v^2}{3R^2} \right)^2} - \frac{m_R \cdot 9.81}{1.5} = 0.6520 \text{ m/s}^2$$

The formula of the acceleration vector at point A is:

$$\mathbf{a}_A = \frac{d\mathbf{v}_A}{dt} = -\left(\frac{v^2}{R} \sin \frac{v}{R} t\right) \mathbf{i} + \left(\frac{v^2}{R} \cos \frac{v}{R} t\right) \mathbf{j} + 0 \mathbf{k} \quad (4.19)$$

The numerical value of the acceleration vector at point A ($t = 0$) taking the gravitational acceleration into account will be:

$$|a_A| = \frac{v^2}{R} - \frac{m_R 9.81}{1.5} = 0.6496 \text{ m/s}^2$$

The difference is $\Delta a = 0.0024 \text{ m/s}^2$, i.e. 0.37 p.c.

d) *Examination of the h vector*

For the critical point of the vehicle:

$$\begin{aligned} \mathbf{h}_T = \frac{d\mathbf{a}}{dt} = & \left[-\frac{v^3}{R^2} \cos \frac{v}{R} t + \frac{av^3}{R^3} \sin \frac{v}{R} t - \frac{bv^3}{R^3} \cos \frac{v}{R} t + \frac{2cm_R v^3}{3R^3} \cos \frac{v}{R} t \right] \mathbf{i} + \\ & + \left[-\frac{v^3}{R^2} \sin \frac{v}{R} t - \frac{av^3}{R^3} \cos \frac{v}{R} t - \frac{bv^3}{R^3} \sin \frac{v}{R} t + \frac{2cm_R v^3}{3R^3} \sin \frac{v}{R} t \right] \mathbf{j} + 0 \mathbf{k} \quad (4.20) \end{aligned}$$

The numerical value of the \mathbf{h} vector for $t = 0$ is:

$$|\mathbf{h}_T| = \sqrt{\left(-\frac{v^3}{R^2} - \frac{bv^3}{R^3} + \frac{2cm_R v^3}{3R^3}\right)^2 + \frac{a^2 v^6}{R^6}} = 0.05197 \text{ m/s}^3$$

The formula of the \mathbf{h} vector at point A is as follows:

$$\mathbf{h}_A = \frac{d\mathbf{a}_A}{dt} = -\left(\frac{v^3}{R^2} \cos \frac{v}{R} t\right) \mathbf{i} - \left(\frac{v^3}{R^2} \sin \frac{v}{R} t\right) \mathbf{j} + 0 \mathbf{k} \quad (4.21)$$

and the numerical value of the \mathbf{h} vector, at point $A(t = 0)$ will be:

$$|h_A| = \frac{v^3}{R^2} = 0.05195 \text{ m/m}^3, \text{ with a difference of}$$

$$\Delta h = 0.00002 \text{ m/s}^3 \quad (\text{i.e. } 0.038 \text{ p.c.})$$

4.5 Examination of the cosine-transition curve without a cant ($m = 0$)

Numerical data: $V = 160 \text{ km/h}$,

$$L = 88 \text{ m}, \quad (R = 4000 \text{ m}),$$

$$m = 0,$$

$$a = 12.25 \text{ m}$$

$$b = 1.421 \text{ m},$$

$$c = 2.025 \text{ m}.$$

a) *Determination of vectors \mathbf{r}_A , ρ , and \mathbf{r}*

By rationally considering Fig. 10 and taking for basis the approximate formulae of marking from the cosine transition curve and by replacement of $l = vt$ one obtains:

$$\mathbf{r}_A = vt \mathbf{i} + \left[\frac{v^2 t^2}{4R} - \frac{L^2}{2\pi^2 R} \left(1 - \cos \frac{\pi v}{L} t \right) \right] \mathbf{j} + 0 \mathbf{k} \quad (4.22)$$

$$\rho = a \mathbf{t} - b \mathbf{n} + c \mathbf{b}, \quad (4.23)$$

carrying out the substitution $vt = l$ gives

$$a \mathbf{t} = a \frac{d\mathbf{r}_A}{dl} = a \mathbf{i} + \left(\frac{al}{2R} - \frac{aL}{2\pi R} \sin \frac{\pi}{L} l \right) \mathbf{j} + 0 \mathbf{k}$$

$$-b \mathbf{n} = -b \frac{1}{G} \frac{d\mathbf{t}}{dl} = 0 \mathbf{i} - b \mathbf{j} + 0 \mathbf{k}$$

$$c \mathbf{b} = c(\mathbf{t} \times \mathbf{n}) = c \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \frac{l}{2R} - \frac{L}{2\pi R} \sin \frac{\pi l}{L} & 0 \\ 0 & 1 & 0 \end{bmatrix} = c \mathbf{k}$$

The formula of the image vector \mathbf{r} reads:

$$\begin{aligned} \mathbf{r} = \mathbf{r}_A + \rho = (vt + a) \mathbf{i} + \left[\frac{v^2 t^2}{4R} - \frac{L^2}{2\pi^2 R} \left(1 - \cos \frac{\pi v}{L} t \right) + \frac{av}{2R} t - \right. \\ \left. - \frac{aL}{2\pi R} \sin \frac{\pi v}{L} t - b \right] \mathbf{j} + c \mathbf{k} \end{aligned} \quad (4.24)$$

b) *Examination of the speed vector*

The formula of the speed vector at the critical point of the vehicle reads

$$\mathbf{v}_T = \frac{d\mathbf{r}}{dt} = v \mathbf{i} + \left[\frac{v^2 t}{2R} - \frac{Lv}{2\pi R} \sin \frac{\pi v}{L} t + \frac{av}{2R} - \frac{av}{2R} \cos \frac{\pi v}{L} t \right] \mathbf{j} + 0 \mathbf{k} \quad (4.25)$$

The numerical value of the speed vector at point $L/2 = vt$ is

$$|\mathbf{v}_T| = \sqrt{v^2 + \left(\frac{vL}{4R} - \frac{Lv}{2\pi R} + \frac{av}{2R} \right)^2} = 160.0009 \text{ km/h}$$

The formula of the speed vector at point A reads

$$\mathbf{v}_A = \frac{d\mathbf{r}_A}{dt} = v \mathbf{i} + \left(\frac{v^2}{2R} t - \frac{Lv}{2\pi R} \sin \frac{\pi v}{L} t \right) \mathbf{j} + 0 \mathbf{k} \quad (4.26)$$

The numerical value of the speed vector at point $A(L/2=vt)$ is as follows:

$$v_A = \frac{dr_A}{dt} = \sqrt{v^2 + \left(\frac{vL}{4R} - \frac{Lv}{2\pi R} \right)^2} = 160.0003 \text{ km/h}$$

the difference being

$$\Delta v = |v_T| - |v_A| = 0.0006 \text{ km/h (0.0004 per cent.)}$$

c) *Examination of the acceleration*

The vector of the acceleration at the critical point of the vehicle is given by formula

$$a_T = \frac{dv}{dt} = 0 \mathbf{i} + \left[\frac{v^2}{2R} - \frac{v^2}{2R} \cos \frac{\pi v}{L} t + \frac{a\pi v^2}{2RL} \sin \frac{\pi v}{L} t \right] \mathbf{j} + 0 \mathbf{k} \quad (4.27)$$

The numerical value of the vector of acceleration point $L/2 = vt$ can be obtained by replacement of the actual data into equation (4.27):

$$|a_T| = \frac{v^2}{2R} + \frac{a\pi v^2}{2RL} = 0.3548 \text{ m/s}^2$$

The formula for the determination of the acceleration vector reads as follows:

$$a_A = \frac{dv_A}{dt} = 0 \mathbf{i} + \left(\frac{v^2}{2R} - \frac{v^2}{2R} \cos \frac{\pi v}{L} t \right) \mathbf{j} + 0 \mathbf{k} \quad (4.28)$$

By replacement of the actual data into equation (4.28) one obtains the numerical value of the acceleration vector at point $A(L/2 = vt)$:

$$|a_A| = \frac{d^2r_A}{dt^2} = \frac{v^2}{2R} = 0.2469 \text{ m/s}^2$$

The difference between the magnitudes of the acceleration vectors at the middle of the transition curve is

$$\Delta a = |a_T| - |a_A| = 0.1079 \text{ m/s}^2, \text{ i.e.,}$$

43 per cent.

According to the above calculations the deviation of the accelerations at the midpoint of the transition curve is comparatively significant, however, considering its numerical value (0.1 m/s^2) it does not play a decisive role, as, after the transition curve, the magnitude of the lateral acceleration increases to 0.4938 m/s^2 , as pointed out in the preceding paragraph, where the value of Δa is only 0.00017 m/s^2 , i.e. 0.034 per cent.

d) *Examination of the \mathbf{h} vector*

The formula of the \mathbf{h} vector at the critical point of the vehicle reads as follows:

$$\mathbf{h}_T = \frac{d\mathbf{a}}{dt} = 0 \mathbf{i} + \left[\frac{\pi v^3}{2RL} \sin \frac{\pi v}{L} t + \frac{a\pi^2 v^3}{2RL^2} \cos \frac{\pi v}{L} t \right] \mathbf{j} + 0 \mathbf{k} \quad (4.29)$$

The numerical value of the \mathbf{h} vector at point $L/2=vt$ is (namely, the change in the curvature is the highest in that point of the cosine transition curve:

$$|\mathbf{h}_T| = \frac{\pi v^3}{2RL} = 0.3917 \text{ m/s}^3$$

Formula of the \mathbf{h} vector at point A is

$$\mathbf{h}_A = \frac{d\mathbf{a}_A}{dt} = 0 \mathbf{i} + \left(\frac{\pi v^3}{2RL} \sin \frac{\pi v}{L} t \right) \mathbf{j} + 0 \mathbf{k} \quad (4.30)$$

The numerical value of the \mathbf{h} vector at point A is as follows ($L/2=vt$):

$$|\mathbf{h}_A| = \left| \frac{d^3 \mathbf{r}_A}{dt^3} \right| = \frac{\pi v^3}{2RL} = 0.3917 \text{ m/s}^3,$$

and, in this case, $\Delta h = 0$

4.6 *Examination of the cosine-transition-curve with a cant ($m \neq 0$)*

The numerical data are as follows:

$$\begin{aligned} V &= 160 \text{ km/h,} \\ L &= 278 \text{ m, (} R = 1300 \text{ m),} \\ m_R &= 0.133 \text{ m,} \\ a &= 12.25 \text{ m,} \\ b &= 1.421 \text{ m,} \\ c &= 2.025 \text{ m,} \\ |\Delta \mathbf{r}| &= 3/4 \text{ m.} \end{aligned}$$

a) *Determination of vectors \mathbf{r}_A , ρ and \mathbf{r}*

Respective consideration of Fig. 9 and the approximate formulae of marking the cosine-transition-curve and carrying out the substitution of $l = vt$ results:

$$\mathbf{r}_A = vt \mathbf{i} + \left[\frac{v^2 t^2}{4R} - \frac{L^2}{2\pi^2 R} \left(1 - \cos \frac{\pi v}{L} t \right) \right] \mathbf{j} + \frac{m_R}{4} \left(1 - \cos \frac{\pi v}{L} t \right) \mathbf{k} \quad (4.31)$$

$$\mathbf{r}_b = vt \mathbf{i} + \left[\frac{3}{4} + \frac{v^2 t^2}{4R} - \frac{L}{2\pi^2 R} \left(1 - \cos \frac{\pi v}{L} t \right) \right] \mathbf{j} + 0 \mathbf{k}$$

$$\begin{aligned} \Delta \mathbf{r} = \mathbf{r}_A - \mathbf{r}_b &= 0 \mathbf{i} - \frac{3}{4} \mathbf{j} + \frac{m_R}{4} \left(1 - \cos \frac{\pi}{L} l \right) \mathbf{k} \\ &\left(|\Delta \mathbf{r}| \approx \frac{3}{4} \text{ m} \right) \\ \rho &= at + b \frac{\Delta \mathbf{r}}{|\Delta \mathbf{r}|} - c \left(\mathbf{k} \times \frac{\Delta \mathbf{r}}{|\Delta \mathbf{r}|} \right) \end{aligned} \quad (4.32)$$

carrying out the substitution $vt = l$ gives

$$\begin{aligned} at &= a \frac{d\mathbf{r}_A}{dl} = a \mathbf{i} + \left(\frac{al}{2R} - \frac{aL}{2\pi R} \sin \frac{\pi}{L} l \right) \mathbf{j} + \left(\frac{am_R \pi}{4L} \sin \frac{\pi}{L} l \right) \mathbf{k} \\ b \frac{\Delta \mathbf{r}}{|\Delta \mathbf{r}|} &= 0 \mathbf{i} - b \mathbf{j} + \frac{bm_R}{3} \left(1 - \cos \frac{\pi}{L} l \right) \mathbf{k} \\ -c \left(\mathbf{k} \times \frac{\Delta \mathbf{r}}{|\Delta \mathbf{r}|} \right) &= -c \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \frac{l}{2R} - \frac{L}{2\pi R} \sin \frac{\pi}{L} l & \frac{m_R \pi}{4L} \sin \frac{\pi}{L} l \\ 0 & -1 & \frac{m_R}{3} \left(1 - \cos \frac{\pi}{L} l \right) \end{bmatrix} = \\ &= \left[-\frac{cm_R l}{6R} \left(1 - \cos \frac{\pi}{L} l \right) + \frac{cm_R L}{6\pi R} \sin \frac{\pi}{L} l \left(1 - \cos \frac{\pi}{L} l \right) - \right. \\ &\quad \left. - \frac{cm_R \pi}{4L} \sin \frac{\pi}{L} l \right] \mathbf{i} + \frac{cm_R}{3} \left(1 - \cos \frac{\pi}{L} l \right) \mathbf{j} + c \mathbf{k} \end{aligned}$$

The image vector \mathbf{r} can be expressed as follows

$$\begin{aligned} \mathbf{r} = \mathbf{r}_A + \rho &= \left[vt + a - \frac{cm_R vt}{6R} \left(1 - \cos \frac{\pi v}{L} t \right) + \right. \\ &+ \left. \frac{cm_R L}{6\pi R} \sin \frac{\pi v}{L} t \left(1 - \cos \frac{\pi v}{L} t \right) - \frac{cm_R \pi}{4L} \sin \frac{\pi v}{L} t \right] \mathbf{i} + \\ &+ \left[\frac{v^2 t^2}{4R} - \frac{L^2}{2\pi^2 R} \left(1 - \cos \frac{\pi v}{L} t \right) - b + \frac{avt}{2R} - \frac{aL}{2\pi R} \sin \frac{\pi v}{L} t + \right. \\ &+ \left. \frac{cm_R}{3} \left(1 - \cos \frac{\pi v}{L} t \right) \right] \mathbf{j} + \left[\frac{m_R}{4} \left(1 - \cos \frac{\pi v}{L} t \right) + \right. \\ &+ \left. \frac{bm_R}{3} \left(1 - \cos \frac{\pi v}{L} t \right) + \frac{am_R \pi}{4L} \sin \frac{\pi v}{L} t + c \right] \mathbf{k} \end{aligned} \quad (4.33)$$

b) *Examination of the speed vector*

The speed vector at the critical point of the vehicle

$$\begin{aligned}
 \mathbf{v}_T = \frac{d\mathbf{r}}{dt} = & \left[v - \frac{cm_R v}{6R} \left(1 - 2 \cos \frac{\pi v}{L} t \right) - \frac{cm_R \pi v^2 t}{6RL} \sin \frac{\pi v}{L} t - \right. \\
 & - \frac{cm_R v}{6R} \cos^2 \left(\frac{\pi v}{L} t \right) + \frac{cm_R v}{6R} \sin^2 \left(\frac{\pi v}{L} t \right) - \frac{cm_R \pi^2 v}{4L^2} \cos \frac{\pi v}{L} t \left. \right] \mathbf{i} + \\
 & + \left[\frac{v^2 t}{2R} - \frac{Lv}{2\pi R} \sin \frac{\pi v}{L} t + \frac{av}{2R} \left(1 - \cos \frac{\pi v}{L} t \right) + \right. \\
 & + \frac{cm_R \pi v}{3L} \sin \frac{\pi v}{L} t \left. \right] \mathbf{j} + \left[\left(\frac{m_R \pi v}{4L} + \frac{bm_R \pi v}{3L} \right) \sin \frac{\pi v}{L} t + \right. \\
 & \left. + \frac{am_R \pi^2 v}{4L^2} \cos \frac{\pi v}{L} t \right] \mathbf{k} \quad (4.34)
 \end{aligned}$$

The numerical value of the speed vector at point $L/2 = vt$ is as follows:

$$\begin{aligned}
 |\mathbf{v}_T| = & \left[\left(v - \frac{cm_R \pi v}{12R} \right)^2 + \left(\frac{Lv}{4R} - \frac{Lv}{2\pi R} + \frac{av}{2R} + \frac{cm_R \pi v}{3L} \right)^2 + \right. \\
 & \left. + \left(\frac{m_R \pi v}{4L} + \frac{bm_R \pi v}{3L} \right)^2 \right]^{1/2} = 160.042 \text{ km/h}
 \end{aligned}$$

The expression of the speed vector at point $A (L/2 = vt)$ reads

$$\mathbf{v}_A = \frac{d\mathbf{r}_A}{dt} = v \mathbf{i} + \left(\frac{v^2}{2R} t - \frac{Lv}{2\pi R} \sin \frac{\pi v}{L} t \right) \mathbf{j} + \left(\frac{m_R \pi v}{4L} \sin \frac{\pi v}{L} t \right) \mathbf{k} \quad (4.35)$$

The numerical value of the speed vector at point $A (L/2 = vt)$ is $(L/2 = vt)$:

$$|\mathbf{v}_A| = \left| \frac{d\mathbf{r}_A}{dt} \right| = \left[v^2 + \left(\frac{Lv}{4R} - \frac{Lv}{2\pi R} \right)^2 + \left(\frac{m_R \pi v}{4L} \right)^2 \right]^{1/2} = 160.030 \text{ km/h}$$

The difference in values of the speed vector is:

$$\mathbf{v} = |\mathbf{v}_T| - |\mathbf{v}_A| = 0.012 \text{ km/h (0.01\%)}$$

c) *Examination of the acceleration vector*

The expression of the acceleration vector at the critical point of the vehicle is

$$\begin{aligned}
 \mathbf{a}_T = \frac{d\mathbf{v}}{dt} = & \left[-\frac{cm_R \pi v^2}{2RL} \sin \frac{\pi v}{L} t - \right. \\
 & - \frac{cm_R \pi^2 v^3}{6RL^2} t \cdot \cos \frac{\pi v}{L} t + \frac{2cm_R \pi v^2}{3RL} \left(\sin \frac{\pi v}{L} t \right) \cos \frac{\pi v}{L} t +
 \end{aligned}$$

$$\begin{aligned}
& + \frac{cm_R\pi^3v^2}{4L^3} \sin \frac{\pi v}{L} t \Big] \mathbf{i} + \left[\frac{v^2}{2R} \left(1 - \cos \frac{\pi v}{L} t \right) + \right. \\
& \quad \left. + \frac{a\pi v^2}{2RL} \sin \frac{\pi v}{L} t + \frac{cm_R\pi^2v^2}{3L^2} \cos \frac{\pi v}{L} t \right] \mathbf{j} + \\
& + \left[\left(\frac{m_R\pi^2v^2}{4L^2} + \frac{bm_R\pi^2v^2}{3L^2} \cos \frac{\pi v}{L} t - \frac{am_R\pi^3v^2}{4L^3} \sin \frac{\pi v}{L} t \right) \right] \mathbf{k} \quad (4.36)
\end{aligned}$$

The value of the acceleration vector and point $L/2 = vt$ can be calculated by using the numerical data and the value of the acceleration due to gravity as follows:

$$\begin{aligned}
|\mathbf{a}_T| &= \left[\left(-\frac{cm_R\pi v^2}{2RL} + \frac{cm_R\pi^3v^2}{4L^3} \right)^2 + \left(\frac{v^2}{2R} + \frac{a\pi v^2}{2RL} \right)^2 + \right. \\
& \quad \left. + \left(\frac{am_R\pi^3v^2}{4L^3} \right)^2 \right]^{1/2} - \frac{m_R \cdot 9.81}{2 \cdot 1.5} = 0.4300 \text{ m/s}^2
\end{aligned}$$

The acceleration vector is to be calculated by the formula

$$\begin{aligned}
\mathbf{a}_A &= \frac{d\mathbf{v}_A}{dt} = 0 \mathbf{i} + \left(\frac{v^2}{2R} - \frac{v^2}{2R} \cos \frac{\pi v}{L} t \right) \mathbf{j} + \\
& \quad + \left(\frac{m_R\pi^2v^2}{4L^2} \cos \frac{\pi v}{L} t \right) \mathbf{k} \quad (4.37)
\end{aligned}$$

The magnitude of the acceleration vector at point A ($L/2 = vt$) is obtained by taking into account the acceleration due to gravity:

$$|\mathbf{a}_A| = \left| \frac{d^2\mathbf{r}_A}{dt^2} \right| - \frac{m_R \cdot 9.81}{2 \cdot 1.5} = \frac{v^2}{2R} - \frac{m_R - 9.81}{2 \cdot 1.5} = 0.3248 \text{ m/s}^2$$

The difference of the magnitudes of the acceleration vectors is to be found as follows:

$$\Delta \mathbf{a} = |\mathbf{a}_T| - |\mathbf{a}_A| = 0.1052 \text{ m/s}^2 \text{ (32\%)}$$

The acceleration vector is comparatively significant, however, considering its magnitude is not highly decisive since after the transition curve, in the circular arc of a radius $R = 1300$ m, as seen from paragraph 4.4.3, the lateral acceleration attains the value 0.65 m/s^2 but the value Δa is only 0.0024 m/s^2 (i.e. 0.37 per cent).

d) Examination of the \mathbf{h} vector

The \mathbf{h} vector, at the critical point of the vehicle, is to be found by using formula:

$$\begin{aligned}
\mathbf{h}_T = \frac{d\mathbf{a}}{dt} = & \left[-\frac{2cm_R\pi^2v^3}{3RL^2} \cos \frac{\pi v}{L} t + \frac{cm_R\pi^4v^3}{4L^4} \cos \frac{\pi v}{L} t + \right. \\
& \left. + \frac{cm_R\pi^3v^4}{6RL^3} t \sin \frac{\pi v}{L} t - \frac{2cm_R\pi^2v^3}{3RL^2} \sin^2 \left(\frac{\pi v}{L} t \right) + \right. \\
& \left. + \frac{2cm_R\pi^2v^3}{3RL^2} \cos^2 \left(\frac{\pi v}{L} t \right) \right] \mathbf{i} + \left[\left(\frac{\pi v^3}{2RL} - \frac{cm_R\pi^3v^3}{3L^3} \right) \sin \frac{\pi v}{L} t + \right. \\
& \left. + \frac{a\pi^2v^3}{2RL^2} \cos \frac{\pi v}{L} t \right] \mathbf{j} - \left[\left(\frac{m_R\pi^3v^3}{4L^3} + \frac{bm_R\pi^3v^3}{3L^3} \right) \sin \frac{\pi v}{L} t + \right. \\
& \left. + \frac{am_R\pi^4v^3}{4L^4} \cos \frac{\pi v}{L} t \right] \mathbf{k} \tag{4.38}
\end{aligned}$$

By replacement of the numerical data one obtains the value of the \mathbf{h} vector at point $L/2 = vt$:

$$\begin{aligned}
|\mathbf{h}_T| = & \left[\left(\frac{cm_Rv^3\pi^3}{12RL^2} - \frac{2cm_R\pi^2v^3}{3RL^2} \right)^2 + \left(\frac{\pi v^3}{2RL} - \frac{cm_R\pi^3v^3}{3L^3} \right)^2 + \right. \\
& \left. + \left(\frac{m_R\pi^3v^3}{4L^3} + \frac{bm_R\pi^3v^3}{3L^3} \right)^2 \right]^{1/2} = 0.3704 \text{ m/s}^3
\end{aligned}$$

The \mathbf{h} -vector can be calculated at point A by making use of the formula:

$$\mathbf{h}_A = \frac{d\mathbf{a}_A}{dt} = 0 \mathbf{i} + \left(\frac{\pi v^3}{2RL} \sin \frac{\pi v}{L} t \right) \mathbf{j} + \left(\frac{m_R\pi^3v^3}{4L^3} \sin \frac{\pi v}{L} t \right) \mathbf{k} \tag{4.39}$$

The numerical value of the \mathbf{h} vector at point A ($L/2 = vt$) is follows:

$$|\mathbf{h}_A| = \left| \frac{d^3\mathbf{r}_A}{dt^3} \right| = \left[\left(\frac{v^3}{2RL} \right)^2 + \frac{m_R\pi^3v^3}{4L^3} \right]^{1/2} = 0.3816 \text{ m/s}^3$$

The difference of the values of the \mathbf{h} vectors is

$$\Delta h = 0.0112 \text{ m/s}^3 \quad (\text{i.e. } 2.9 \text{ per cent})$$

In the aforesaid, by taking for basis the motion of the rigid body (i.e. the system of points) the critical values of the vectors characterizing the motion induced were determined and the diversions in relation with the assumption of the point motion, for the most fundamental cases occurring in planning railway lines were examined. From the results of the examination it is to be seen that in case of the motion taking place on the railway line the assumption of the point motion in the movement-geometric examinations would meet, in general, the practical requirements and the necessity of a kinematic examination of the motion of a rigid body may occur in specific circumstances only.

5. Uniform cosine-transition geometries

In the present paper a particular attention is given to the transition curves having much more favourable characteristic features as continuous, breakless transitions of curvatures, by putting particular emphasis from among them on the

cosine geometry

which, according to the results of the movement-geometric investigations possesses unequivocally the most favourable kinematic, i.e. geometric characteristics.

The *clotoid* transition curve of the linear curvature-transition is significantly more unfavourable from the point of the movement-geometry than the continuous, breakless curvature transitions, since the joints of the curvature function at the beginning and end of the transition are not breakless.

In investigating the different geometries the transition curves applied

- between tangents and curvilinear sections,
- in connection with compound curves and
- in case of reverse curve designs,

in all of the above cases the *transition curve is considered as a special geometry*.

In determining the curvature-transitional geometries the following symbols are used:

R	— radius of circular arc (Fig. 12);
$G = \frac{1}{R}$	— curvature of circular arc, m^{-1} ;
l	— distance of the point examined from the beginning of transition (the joining point of lesser curvature), m ;
G_l	— curvature at the point examined, m^{-1} ;
x, y	— orthogonal coordinates of the point examined in the local system of coordinates, m ;
X, Y	— orthogonal coordinates of end point of the transition curve, m ;
L	— length of curvature transition (i.e. transition curve), m ;
$R_0 = \frac{R_1 R_2}{R_1 - R_2}$	— radius to be calculated in the knowledge of radii of compound curves (Fig. 13), m ;
$R_{or} = \frac{R_1 R_2}{R_1 + R_2}$	— radius to be calculated when knowing the radii of reverse curves (Fig. 14) m ;

From the above symbols it follows that

$$R_1 > R_2 \quad \text{and} \quad G_1 < G_2.$$

5.1 Uniform geometry of the cosine transition curve

In case of transitions between tangent and curve as well as in case of compound and reverse curves, and further for cant transitions (see Chapter 5.2) the *cosine basic geometry* depicted in Fig. 11 is applied.

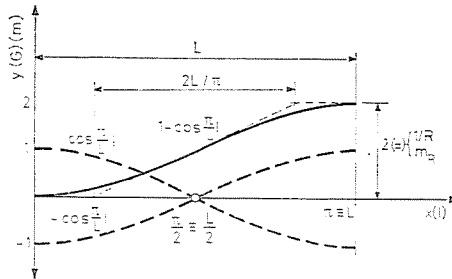


Fig. 11.

By reflecting the segment $0 \leq x \leq \pi$ of the cosine function (depicted in Fig. 11 by short dashed line) in the x -direction (long dashed line) and lengthening the ordinates by the unit length yields the function of the basic cosine geometry (full heavy line in the figure):

$$y = c \left(1 - \cos \frac{\pi}{L} l \right)$$

In a general case (i.e. of compound and reverse curves) at point $x = 0$ $y \neq 0$, wherefore, the *general function of the cosine basic geometry* reads:

$$y = c_0 + c \left(1 - \cos \frac{\pi}{L} l \right) \tag{5.1}$$

where c_0 and c are constants to be calculated in compliance with the problem given.

Table 3

Constant values of cosine-transition of curvature

Position of the transition curve	c_0	c
Between tangent and circular curve of radius R	0	$1/2R$
Between compound curves ($R_1 > R_2$) $R_0 = R_1R_2/R_1 - R_2$	$1/R_1$	$1/2R_0$
Between reverse curves $R_{or} = R_1R_2/R_1 + R_2$	$1/R_1$	$1/2R_{or}$

In case of a cosine curvature-transition we have:

$$G_l = c_0 + c \left(1 - \cos \frac{\pi}{L} l \right) \quad m^{-1}. \quad (5.2)$$

The values of the constants c_0 and c are indicated in dependence of the situation of the transition curve in Table 3.

a) *Cosine transition curve between tangent and circular curves*

The geometrical determination of the *cosine transition curve between the tangent and curvilinear section* and the knowledge needed to marking out the transition curve is dealt with in connection with Fig. 12.

The curvature function of the transition curve considering Eq. (5.2), Table 3 and Fig. 11 reads

$$G_l = \frac{1}{2R} \left(1 - \cos \frac{\pi}{L} l \right) \quad m^{-1}, \quad (5.3)$$

namely, $c_0=0$, and according to (5.2) at the middle of the transition curve ($l = 0.5 L$) $c = 1/2R$.

The trigonometrical function of tangent is:

$$\tau_l = \int_0^l G_l dl = \frac{1}{2R} \left(l - \frac{L}{\pi} \sin \frac{\pi}{L} l \right) \quad (5.4)$$

At the end point of the transition curve ($l = L$) the inclination of the tangent is

$$\tau_L = \frac{L}{2R}$$

The system of equations for the arc-length parameter of the orthogonal coordinates, required to a marking out in the system of coordinates x, y is, in accordance with the accuracy of staking out, and by considering the first two terms of the power series given by the equation:

$$\begin{aligned} x = \int_0^l \cos \tau_l dl \approx l \left(1 - \frac{L^2}{16\pi^2 R^2} \right) - \frac{l^3}{24R^2} + \frac{L^3 \sin \frac{\pi}{L} l}{4\pi^3 R^2} + \\ + \frac{L^3 \sin \frac{2\pi}{L} l}{32\pi^3 R^2} - \frac{L^2 l \cos \frac{\pi}{L} l}{4\pi^2 R^2} \quad m \end{aligned} \quad (5.5)$$

$$\begin{aligned}
 y = \int_0^l \sin \tau_l \, dl \approx & -\frac{137 L^4}{1152\pi^4 R^3} - \frac{L^2}{2\pi^2 R} + \frac{l^2}{4R} \left(1 - \frac{L^2}{16\pi^2 R^2}\right) - \\
 & - \frac{l^4}{192R^3} + \frac{L^2 \cos \frac{\pi}{L} l}{2\pi^2 R} \left(1 + \frac{5L^2}{24\pi^2 R^2} - \frac{l^2}{8R^2}\right) + \\
 & + \frac{L^4 \cos \frac{2\pi}{L} l}{128\pi^4 R^3} + \frac{lL^3 \sin \frac{\pi}{L} l}{8\pi^3 R^3} + \frac{lL^3 \sin \frac{2\pi}{L} l}{64\pi^3 R^3} + \frac{L^4 \cos^3 \left(\frac{\pi}{L} l\right)}{144\pi^4 R^3} \quad m \quad (5.6)
 \end{aligned}$$

The coordinates of the end point of the transition curve can be calculated by using formulae (5.5) and (5.6):

$$X = L \left(1 - 0.02267 \frac{L^2}{R^2}\right) \quad m \quad (5.7)$$

$$Y = L^2 \left(\frac{0.14868}{R} - 0.00274 \frac{L^2}{R^3}\right) \quad m \quad (5.8)$$

The further data necessary to mark out the transition curve using the symbols of Fig. 12 are as follows:

$$\tau_L = \frac{L}{2R} \quad (5.9)$$

$$f = Y - (R - R \cos \tau_L) \quad m \quad (5.10)$$

$$x_0 = X - R \sin \tau_L \quad m \quad (5.11)$$

$$t = Y \operatorname{ctg} \tau_L \quad m \quad (5.12)$$

$$t_h = X - t \quad m \quad (5.13)$$

$$t_r = Y \operatorname{cosec} \tau_L \quad m \quad (5.14)$$

The *approximate* layout formulae of the transition curve are, in case of the condition $x \approx l$, as follows:

$$y = \int_0^x \tau_x \, dx = \frac{x^2}{4R} - \frac{L^2}{2\pi^2 R} \left(1 - \cos \frac{\pi}{L} x\right) \quad m \quad (5.15)$$

$$Y = \frac{L^2}{R} \left(\frac{1}{4} - \frac{1}{\pi^2}\right) \approx 0.149 \frac{L^2}{R} \quad m \quad (5.16)$$

$$f = \frac{\pi^2 - 8}{8\pi^2} \frac{L^2}{R} = \frac{L^2}{42.23R} \quad m \quad (5.17)$$

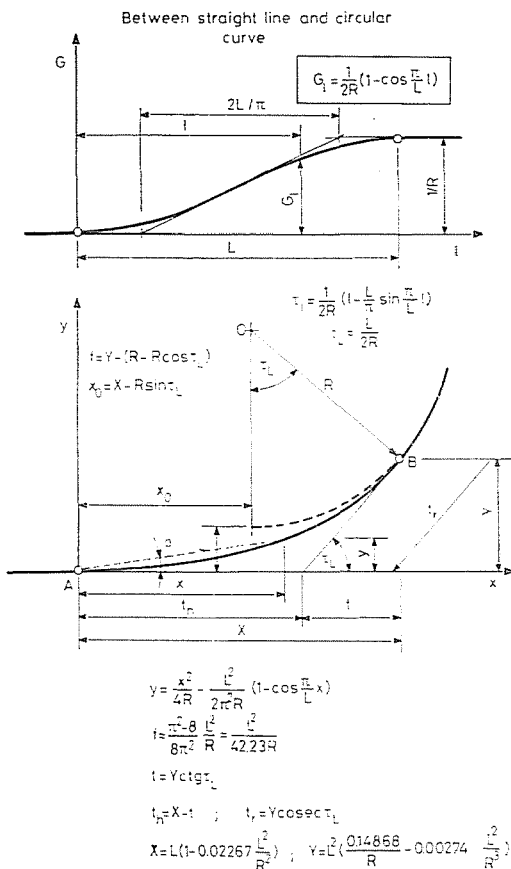


Fig. 12. Cosine transition curve I.

b) *Cosine transition curve between circular curves of identical direction*

Between lines curving in the same direction (i.e. in case of compound curves) the symbols used for the geometric determination and the marking out of the cosine transition curve are indicated in Fig. 13.

The calculation of the geometry of the transition curve should be performed similarly to that recognized in case of the transition curve between the tangent and circular curve, *using the uniform cosine basic geometry*. The *function of curvature* determining the geometry of the transition curve by taking into account Fig. 13, i.e. Table reads, as follows:

$$G_l = \frac{1}{R_1} + \frac{1}{2R_0} \left(1 - \cos \frac{\pi}{L} l \right) \quad m^{-1} \quad (5.18)$$

wherein

$$R_0 = R_1 R_2 / (R_1 - R_2) \quad m, \quad (5.19)$$

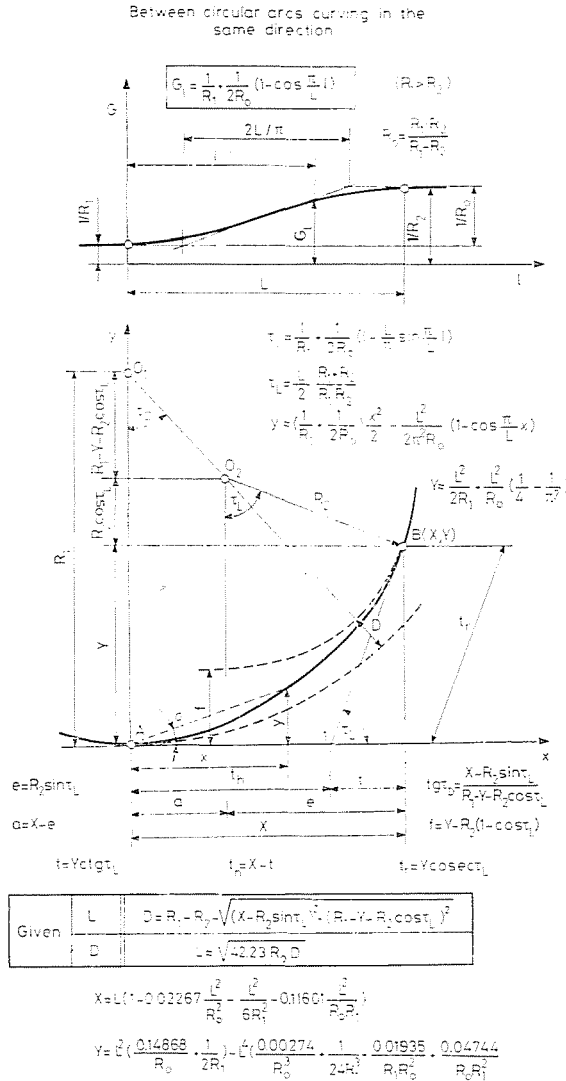


Fig. 13. Cosine transition curve II.

and R_1 and R_2 are the radii of the circular curves of the compound curve ($R_1 > R_2$) m.

The tangent trigonometric function of the cosine transition curve between the circular arcs curving in the same direction is:

$$\tau_l = \int_0^l G dl = \frac{l}{R_1} + \frac{1}{2R_0} \left(l - \frac{L}{\pi} \sin \frac{\pi}{L} l \right) \quad (5.20)$$

The inclination of the tangent at the end of the transition curve is given by the formula ($l = L$)

$$\tau_L = \frac{L}{R_1} + \frac{L}{2R_0} = \frac{L}{2} \frac{R_1 + R_2}{R_1 \cdot R_2}$$

For marking out the transition curve the orthogonal system of coordinates x, y is used, where the geometric beginning of the transition curve is the origin of the local system of coordinates. Consequently, the *cosine transition curve between circular arcs curving in the same direction* is considered, as opposed to the earlier practice, as an *individual geometry* and not an intermediate section of a basic geometry.

The system of equations of arc-length parameters of the *orthogonal coordinates needed to marking out*, by taking into account the first two terms of the power series, reads as follows:

$$\begin{aligned} x = \int_0^l \cos \tau_l dl \approx & l \left(1 - \frac{L^2}{16\pi^2 R_0^2} \right) - l^3 \left(\frac{1}{6R_1^2} + \frac{1}{6R_1 R_0} + \frac{1}{24R_0^2} \right) + \\ & + \left(\frac{L^3}{2\pi^3 R_1 R_0} + \frac{L^3}{4\pi^3 R_0^2} \right) \sin \frac{\pi}{L} l - \left(\frac{L^2}{2\pi^2 R_1 R_0} + \frac{L^2}{4\pi^2 R_0^2} \right) l \cos \frac{\pi}{L} l + \\ & + \frac{L^3}{32\pi^3 R_0^2} \sin \frac{2\pi}{L} l \end{aligned} \quad m \quad (5.21)$$

$$\begin{aligned} y = \int_0^l \sin \pi_l dl \approx & -\frac{L^2}{2\pi^2 R_0} - \frac{L^4}{2\pi^4 R_1^2 R_0} - \frac{33L^4}{64\pi^4 R_1 R_0^2} - \frac{137L^4}{1152\pi^4 R_0^3} + \\ & + l^2 \left(\frac{1}{2R_1} + \frac{1}{4R_0} - \frac{L^2}{32\pi^2 R_1 R_0^2} - \frac{L^2}{64\pi^2 R_0^3} \right) - l^4 \left(\frac{1}{24R_1^3} + \right. \\ & + \frac{1}{16R_1^2 R_0} + \frac{1}{32R_1 R_0^2} + \frac{1}{192R_0^3} \left. \right) + \left(\frac{L^3}{2\pi^3 R_1^2 R_0} + \frac{L^3}{2\pi^3 R_1 R_0^2} + \right. \\ & + \left. \frac{L^3}{8\pi^3 R_0^3} \right) l \sin \frac{\pi}{L} l + \left(\frac{L^2}{2\pi^2 R_0} + \frac{L^4}{2\pi^4 R_1^2 R_0} + \frac{L^4}{2\pi^4 R_1 R_0^2} + \right. \\ & + \left. \frac{5L^4}{48\pi^4 R_0^3} - \frac{l^2 L^2}{4\pi^2 R_1^2 R_0} - \frac{l^2 L^2}{4\pi^2 R_1 R_0^2} - \frac{l^2 L^2}{16\pi^2 R_0^3} \right) \cos \frac{\pi}{L} l + \\ & + \left(\frac{L^3}{32\pi^3 R_1 R_0^2} + \frac{L^3}{64\pi^3 R_0^3} \right) l \sin \frac{2\pi}{L} l + \left(\frac{L^4}{64\pi^4 R_1 R_0^2} + \right. \\ & + \left. \frac{L^4}{128\pi^4 R_0^3} \right) \cos \frac{2\pi}{L} l + \frac{L^4}{144\pi^4 R_0^3} \cos^3 \left(\frac{\pi}{L} l \right) \end{aligned} \quad m \quad (5.22)$$

The coordinates of the end point of the transition curve can be calculated by making use of formulae (5.21) and (5.22):

$$X = L - L^3 \left(\frac{1}{6R_1^2} + \frac{0.02267}{R_0^2} + \frac{0.11601}{R_0 R_1} \right) \quad m \quad (5.23)$$

$$Y = L^2 \left(\frac{0.14868}{R_0} + \frac{1}{2R_1} \right) - L^4 \left(\frac{1}{24R_1^3} + \frac{0.00274}{R_0^3} + \frac{0.01935}{R_1 R_0^2} + \frac{0.04744}{R_1^2 R_0} \right) \quad m \quad (5.24)$$

When marking out in practice the transition curve between two circular arcs curving in the same direction two instances may occur.

Case A): the position of two circular curves is given (in Fig. 13 by the coordinates of the origins O_1 and O_2) and we look for the data of the transition curve L corresponding to this situation. A close approximate value of the length of the intermediate cosine transition curve to be obtained can be determined with the aid of the tangent-angle method as follows:

$$L \approx \sqrt{42.23 R_0 D} \quad m \quad (5.25)$$

with $D = R_1 - R_2 - O_1 O_2$ m.

In the practical solution of the above problem the length of the transition curve cannot, as a matter of course, be shorter than the minimum length obtained on the basis of the h vector.

Case B): the transition curve is given L , and we are to find the position of the circular curve of radius R_2 related to that of radius R_1 , i.e. the distance D (Fig. 13):

$$D = R_1 - R_2 - \sqrt{(X - R_2 \sin \tau_L)^2 + (R_1 - Y - R_2 \cos \tau_L)^2} \quad m \quad (5.26)$$

The actual spacing $\overline{O_1 O_2}$ issued in this case is

$$\overline{O_1 O_2} = R_1 - R_2 - D \quad m \quad (5.27)$$

The position of the system of coordinates x, y to be used to mark out is definite for both cases by knowing the direction $\overline{O_1 O_2}$ and the angle

$$\tau_D = \text{arc tg} \frac{X - R_2 \sin \tau_L}{R_1 - Y - R_2 \cos \tau_L} \quad (5.28)$$

The further data needed to mark out the transition curve are (see Fig. 13) as follows:

$$\tau_L = \frac{L}{2} \frac{R_1 + R_2}{R_1 \cdot R_2} \quad (5.29)$$

$$f = Y - (R_2 - R_2 \cos \tau_L) \quad m \quad (5.30)$$

$$e = R_2 \sin \tau_L \quad m \quad (5.31)$$

$$a = X - e \quad m \quad (5.32)$$

$$t = Y \operatorname{ctg} \tau_L \quad m \quad (5.33)$$

$$t_h = X - t \quad m \quad (5.34)$$

$$t_r = Y \operatorname{cosec} \tau_L \quad m \quad (5.35)$$

The *approximate* formulae for laying out the cosine transition curve between compound curves are, for the condition $x \approx l$, as follows (Fig. 13):

$$y = \frac{x^2}{2R_1} + \frac{x^2}{4R_0} - \frac{L^2}{2\pi^2 R_0} \left(1 - \cos \frac{\pi}{L} x \right) \quad m \quad (5.36)$$

$$Y = \frac{L^2}{2R_1} + \frac{L^2}{R_0} \left(\frac{1}{4} - \frac{1}{\pi^2} \right) \approx 0.5 \frac{L^2}{R_1} + 0.149 \frac{L^2}{R_0} \quad m \quad (5.37)$$

$$e = R_2 \operatorname{arc} \tau_L = L \frac{R_1 + R_2}{2R_1} \quad m,$$

$$a = X - e = L \left(1 - \frac{R_1 + R_2}{2R_1} \right) \quad m,$$

$$f = Y - \frac{e^2}{2R_2} \approx 0.5 \frac{L^2}{R_1} + 0.149 \frac{L^2}{R_0} - \frac{L^2(R_1 + R_2)^2}{8R_2 R_1^2} \quad m$$

c) Cosine transition curve between reverse curves

The *geometrical determination, i.e. the layout of the cosine transition curve between reverse curves* is demonstrated in connection with Fig. 14.

The calculation of the geometry of the transition curve is performed in conformity with the principles recognized in connection with the transition curves to be applied between tangent and circular curves, i.e. between circular curves curving in the same direction.

The curvature function of the cosine transition established on basis of the data in Fig. 14 and Table 3 reads as follows:

$$G_l = \frac{1}{R_1} - \frac{1}{2R_{0r}} \left(1 - \cos \frac{\pi}{L} l \right) \quad m^{-1}, \quad (5.38)$$

where

$$R_{0r} = \frac{R_1 \cdot R_2}{R_1 + R_2} \quad m \quad (5.39)$$

In marking out the cosine transition curve between reverse curves the system of coordinates x, y is used, the origin of which is the geometric beginning of the transition curve. Similarly to the compound curves, the *transition curve* is considered also in this case as an *individual geometry*.

The system of equations of arc-length parameters of the orthogonal coordinates needed to mark out the curve is established by making use of the first two terms of the power series, as follows:

$$x = \int_0^l \cos \tau_l dl \approx l \left(1 - \frac{L^2}{16\pi^2 R_{0r}^2} \right) - l^3 \left(\frac{1}{6R_1^2} - \frac{1}{6R_1 R_{0r}} + \frac{1}{24R_{0r}^2} \right) +$$

$$+ \left(\frac{L^3}{4\pi^3 R_{0r}^2} - \frac{L^3}{2\pi^3 R_1 R_{0r}} \right) \sin \frac{\pi}{L} l - \left(\frac{L^2}{4\pi^2 R_{0r}^2} - \frac{L^2}{2\pi^2 R_1 R_{0r}} \right) l \cos \frac{\pi}{L} l + \frac{L^3}{32\pi^3 R_{0r}^2} \sin \frac{2\pi}{L} l \quad m \quad (5.41)$$

$$y = \int_0^l \sin \tau_l dl \approx \frac{L^2}{2\pi^2 R_{0r}} + \frac{L^4}{2\pi^4 R_1^2 R_{0r}} - \frac{33L^4}{64\pi^4 R_1 R_{0r}^2} + \frac{137L^4}{1152\pi^4 R_{0r}^3} +$$

$$+ l^2 \left(\frac{1}{2R_1} - \frac{1}{4R_{0r}} - \frac{L^2}{32\pi^2 R_1 R_{0r}^2} + \frac{L^2}{64\pi^2 R_{0r}^3} \right) - l^4 \left(\frac{1}{24R_1^3} - \frac{1}{16R_1^2 R_{0r}} + \frac{1}{32R_1 R_{0r}^2} - \frac{1}{192R_{0r}^3} \right) + \left(\frac{L^3}{2\pi^3 R_1 R_{0r}^2} - \frac{L^3}{2\pi^3 R_1^2 R_{0r}} - \frac{L^3}{8\pi^3 R_{0r}^3} \right) l \sin \frac{\pi}{L} l + \left(\frac{L^4}{2\pi^4 R_1 R_{0r}^2} - \frac{L^2}{2\pi^2 R_{0r}} - \frac{L^4}{2\pi^4 R_1^2 R_{0r}} - \frac{5L^4}{48\pi^4 R_{0r}^3} + \frac{l^2 L^2}{4\pi^2 R_1^2 R_{0r}} - \frac{l^2 L^2}{4\pi^2 R_1 R_{0r}^2} + \frac{l^2 L^2}{16\pi^2 R_{0r}^3} \right) \cos \frac{\pi}{L} l +$$

$$+ \left(\frac{L^3}{32\pi^3 R_1 R_{0r}^2} - \frac{L^3}{64\pi^3 R_{0r}^3} \right) l \sin \frac{2\pi}{L} l + \left(\frac{L^4}{64\pi^4 R_1 R_{0r}^2} - \frac{L^4}{128\pi^4 R_{0r}^3} \right) \cos \frac{2\pi}{L} l - \frac{L^4}{144\pi^4 R_{0r}^3} \cos^3 \left(\frac{\pi}{L} l \right) \quad m \quad (5.42)$$

The coordinates of the end point of the transition curve are to be determined by using (5.41) and (5.42):

$$X = L - L^3 \left(\frac{1}{6R_1^2} + \frac{0.02267}{R_{0r}^2} - \frac{0.11601}{R_{0r} R_1} \right) \quad m \quad (5.43)$$

$$Y = L^2 \left(-\frac{0.14868}{R_{0r}} + \frac{1}{2R_1} \right) - L^4 \left(\frac{1}{24R_1^3} - \frac{0.00274}{R} + \frac{0.01935}{R_1 R_{0r}^2} - \frac{0.04744}{R_1^2 R_{0r}} \right) \quad m \quad (5.44)$$

When actually laying out the transition curve between reverse curves, similarly to what has been said in connection with the compound curves, two instances might occur:

Case A): the position of the two reverse curves (in Fig. 14 the coordinates of the points O_1 and O_2) is given. The *close approximate length* of L cosine transition curve can be given by the formula obtained with the aid of the tangent angle method:

$$L \approx \sqrt{42.23 R_{or} D} \quad m \quad (5.45)$$

with $D = \overline{O_1 O_2} - R_1 - R_2$ m .

Also to solve this problem, the condition is valid that the length of the transition curve should not be larger than that defined as a minimum with the aid of the h vector.

Case B): the length of the transition curve, find the relative position of the circular arcs of the reverse curve is given. On the basis of Fig. 14 we have:

$$D = -R_1 - R_2 + \sqrt{(X - R_2 \sin \tau_L)^2 + (R_1 - Y + R_2 \cos \tau_L)^2} \quad m \quad (5.46)$$

and

$$\overline{O_1 O_2} = R_1 + D + R_2 \quad m \quad (5.47)$$

The system of coordinates x, y of marking out, is unequivocally determined by the direction $O_1 O_2$ and the angle known for both of the above cases.

$$\tau_D = \text{arc tg} \frac{X - R_2 \sin \tau_L}{R_1 - Y + R_2 \cos \tau_L} \quad (5.48)$$

Further data needed to mark out the transition curve are, by taking Fig. 14 into account, are the following:

$$\tau_L = -\frac{L}{2} \frac{R_1 - R_2}{R_1 \cdot R_2} \quad (5.49)$$

$$L_1 = \frac{L}{\pi} \text{arc cos} \frac{R_1 - R_2}{R_1 + R_2} \quad m \quad (5.50)$$

$$\tau_0 = \frac{L_1}{R_1} - \frac{1}{2R_{or}} \left(L_1 - \frac{L}{\pi} \sin \frac{\pi}{L} L_1 \right) \quad (5.51)$$

as well as

$$\overline{O1} = y_0 \text{cosec } \tau_0; \quad \overline{A1} = x_0 - y_0 \text{ctg } \tau_0$$

5.2 Uniform cosine cant transition

The calculation of the cosine cant transitions by using the uniform basic geometry is presented for the cases: between tangent and circular curve, between circular arcs curved in the same direction and between reverse curves, to be seen in Fig. 15.

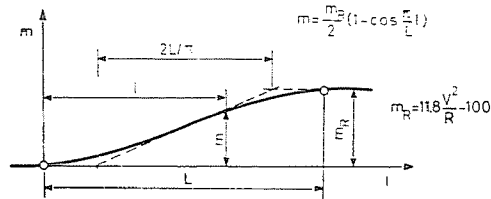
The length of the cant transition agrees in all cases with that of the transition curve.

In case of the cosine cant transition *between tangent and circular curve* the function of cant reads (Fig. 15) as follows

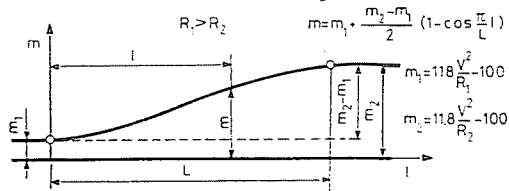
$$m = \frac{m_R}{2} \left(1 - \cos \frac{\pi}{L} l \right) \quad \text{mm} \quad (5.52)$$

wherein $m_R = 11.8 \frac{V^2}{R} - 100$, the amount of cant in a circular curve (mm).

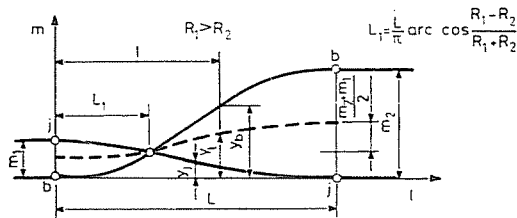
I. Between straight lone and circular curve



II. Between circular arcs curving in the same direction



III. Between reverse curve



$$y_1 = \frac{m_1}{2} + \frac{m_2 - m_1}{4} (1 - \cos \frac{\pi}{L} l)$$

$$y_b = \frac{m_2}{2} (1 - \cos \frac{\pi}{L} l) ; \quad m_2 = 11.8 \frac{V^2}{R_2} - 100 = a_2 \frac{V^2}{R_2}$$

$$y_j = m_1 \frac{m_2}{2} (1 - \cos \frac{\pi}{L} l) ; \quad m_1 = m_2 \frac{G_2}{G_1} = a_2 \frac{V^2}{R_1} = 11.8 \frac{V^2}{R_1} - 100 \frac{R_2}{R_1}$$

Fig. 15. Cosine cant transitions

In case of a cant transition *between circular arcs curving in the same direction* (Fig. 15):

$$m = m_1 + \frac{m_2 - m_1}{2} \left(1 - \cos \frac{\pi}{L} l \right) \quad \text{mm} \quad (5.53)$$

with $m_1 = 11.8 \frac{V^2}{R_1} - 100$ and $m_2 = 11.8 \frac{V^2}{R_2} - 100$ mm

amount of cants in circular curves (mm).

Calculation of the cosine transition *between reverse circular curves* on basis of Fig. 15:

$$y_b = \frac{m_2}{2} \left(1 - \cos \frac{\pi}{L} l \right) \quad \text{mm} \quad (5.54)$$

$$y_j = m_1 - \frac{m_1}{2} \left(1 - \cos \frac{\pi}{L} l \right) \quad \text{mm} \quad (5.55)$$

where

$$m_2 = 11.8 \frac{V^2}{R_2} - 100 \quad \text{and} \quad m_1 = 11.8 \frac{V^2}{R_1} - 100 \frac{R_2}{R_1} \quad \text{mm.}$$

(The latter equation results from the condition of the zero point of the curvature.)

6. Actual problems in the field of the geometry of motion

In order to practically represent the motion-geometric examinations dealt with in the foregoing the approximate solutions to some simple fundamental problems are presented.

The examinations have in general, a double objective:

- determination of the unknown functions $f(V)$ and
- $f(R)$ required on the basis of the acceleration vector and the \mathbf{h} vector, thereafter, starting from the critical state, calculation of the value desired.

A) *The examination according to the acceleration vector* (“acceleration-aspect”) is based on the inward (i.e. its opposed outward) radial acceleration applied on a vehicle moving along a curved track, the formula of which reads:

$$|\mathbf{a}_n| = G v^2 = \frac{v^2}{\rho} \quad \text{m/s}^2 \quad (6.1)$$

which in case of the state of motion $v = \text{const}$ (m/s), ($|\mathbf{a}_t| = 0$) means at the same time the value of the acceleration ($|\mathbf{a}_n| = |\mathbf{a}_n|$).

B) *In case of an examination on the basis of the h-vector* (“ \mathbf{h} vector aspect”) starting from the assumption $v = \text{const}$. of the acceleration-aspect, by neglect-

ing the torsion of the track, the magnitude of the \mathbf{h} vector for the motion along a curved track is expressed according to (3.27) by the formula

$$|\mathbf{h}| = \sqrt{(v^3 G^2)^2 + \left(v^2 \frac{dG}{dt}\right)^2} \quad \text{m/s}^3$$

An introduction of practical considerations of the trajectory function $G = f(l)$ yields

$$\frac{dG}{dt} = \frac{dG}{dl} \frac{dl}{dt} = \frac{dG}{dl} v,$$

the magnitude of the \mathbf{h} vector being as follows

$$|\mathbf{h}| = v^3 \sqrt{G^4 + \left(\frac{dG}{dl}\right)^2} \approx \frac{V^3}{3.6^3} \frac{dG}{dl} \quad \text{m/s}^3 \quad (6.2)$$

since $G^4 \rightarrow 0$.

In formula (6.2) the velocity V , given in km/h and dG/dl , is the derivative of the function of curvature of the curved path with respect to the length of curve.

B.1 The approximate magnitude of the \mathbf{h} vector on a track of a circular arc with transition curve.

The maximum of the functions dG/dl of the transition curves which might be taken into consideration in designing railway tracks is the following:

$$\left(\frac{dG}{dl}\right)_{\max} = \frac{\alpha}{RL} \quad \text{m}^{-2} \quad (6.3)$$

wherein: R — radius of circular curve joining to transition curve (m),
 L — length of transition curve (m).

The value of α is, in case of continuous, breakless curvature functions for cosine transition curve $\pi/2 = 1,57$

for a 4th degree parabola and for a sinusoidal transition curve 2

(in case a of clotoid transition curve at the beginning and end points of the transition the function of curvature has break points, therefore here, the derivative dG/dl is not defined).

Simultaneous consideration of (6.2) and (6.3) yields the approximate magnitude of the \mathbf{h} vector at a critical point of the transition curve, by the formula:

$$|\mathbf{h}| = \frac{\alpha V^3}{3.6^3 RL} \quad \text{m/s}^3 \quad (6.4)$$

B.2 Approximate magnitude of the \mathbf{h} vector in case of a direct joint of tangent and circular curve; on the basis of (6.2):

$$|\mathbf{h}| = \frac{V^3}{3.6^3} \frac{dG}{dl} \approx \frac{V^3}{3.6^3} \frac{\Delta G}{\Delta l} = \frac{V^3}{3.6^3} \frac{1}{R \cdot d} \quad \text{m/s}^3 \quad (6.5)$$

wherein: V — speed, km/h;
 R — radius of circular curve, m;
 d — length percieving change in curvature (in case of four-axle vehicles bogie-base, in case of two-axle vehicles wheel-base; in the calculations of the present paper $d = 17$ m).

6.1 Examination of the case of curve without transition

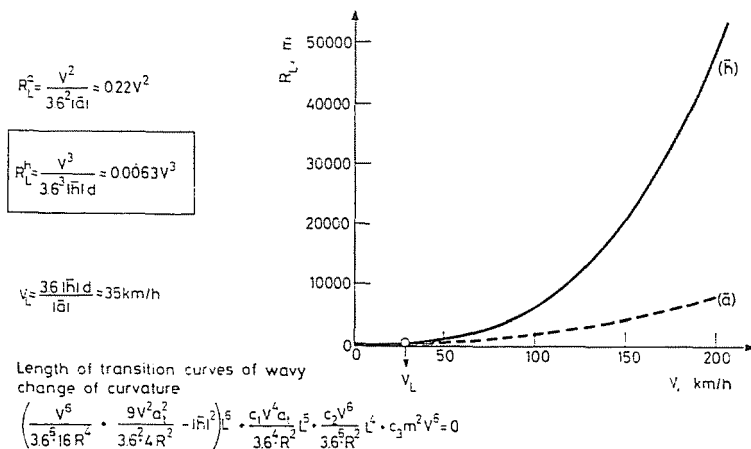
Direct joining of straight and circular curve sections without transition is permitted only in case where at the tangent point the magnitude of neither the acceleration nor the \mathbf{h} vector surpasses the permissible threshold value.

The value of the limiting radius determined on the basis of the acceleration (6.1), above which the straight line and the curved section might be joined without transition is obtained by making use of formula:

$$R_L^a = \frac{V^2}{3.6^2 |a|} \approx 0.22 V^2 \quad \text{m} \quad (6.6)$$

in case where according to Table 1 $a = 0.35 \text{ m/s}^2$.

For a constant speed, motion the magnitude of the \mathbf{h} vector calculated according to (6.5) and not taking the transition into consideration, with the



value of $|\mathbf{h}| = 0.2 \text{ m/s}^3$ (Table 2) we have:

$$R_L^h \approx \frac{V^3}{3.6^3 |\mathbf{h}| d} = 0.0063 V^3 \quad \text{m} \quad (6.7)$$

The limiting functions R_L^a and R_L^h are depicted in Fig. 16. Consequently, the speed above which, from the viewpoint of the examination the \mathbf{h} vector is critical, issues from the equality $R_L^a = R_L^h$ and reads

$$V_L = \frac{3.6 |\mathbf{h}| d}{|\mathbf{a}|} \approx 35 \text{ km/h} \quad (6.8)$$

In case of a motion of constant acceleration ($a_t = \text{const.}$) considering (3.27) and not taking into account the tangent-directed component one obtains

$$|\mathbf{h}| \approx 3va_t G - v^3 \frac{G}{d}$$

from which considering $|\mathbf{h}| = 0.2 \text{ m/s}^3$ (Table 2) and $|\mathbf{a}_t| = 0.35 \text{ m/s}^2$ (Table 1) results the limiting radius in omitting the transition curve as follows

$$R_L^h = 1.46 V + 0.0063 V^3 \quad \text{m.} \quad (6.9)$$

(It is to be noted that in calculating the above the approximate value of the \mathbf{h} vector the omission of the tangent-directed component $V^3 G^2$ in case of values of several thousands of metres does not cause but an error of a few cms.)

6.2 Determination of the minimum length of the transition curve

On a line with transition curves, due to the permanent change of the curvature the length of the transition *cannot be determined according to the acceleration-aspect* and so many railways have no other choice but to establish the length of the transition curve on an empirical basis ($L = 10 V \cdot m$, where V is the speed in km/h and m is the cant in m).

The length of transition, on the basis of the third order movement-characteristic is to be calculated from formula (3.27) of the \mathbf{h} vector keeping in mind that the magnitude of the \mathbf{h} -vector should not surpass the permitted threshold value, not even at the critical point of transition (see Table 2).

In case of transitions with a wavy change of curvature (cosine-, fourth-order-parabole- and sinusoidal transition curves) the critical point is the mid-point of the transition. From the magnitude of the \mathbf{h} vector (3.27), after simplification, one obtain the following sixth order equation for the length

of transitions with a wavy change of curvature

$$\left(\frac{v^6}{16R^4} + \frac{9v^2|a_t|^2}{4R^2} - |h|^2 \right) L^6 + c_1 \frac{v^4|a_t|}{R^2} L^5 + c_2 \frac{v^6}{R^2} L^4 + c_3 m^2 v^6 = 0 \quad (6.10)$$

the single positive root of which is the solution to the equation.

6.3 Determination of the maximum permissible speed in a circular curve without cant and transition curve

In practice, this instance occurs in circular curves of a great radius and in turnout curves without a cant.

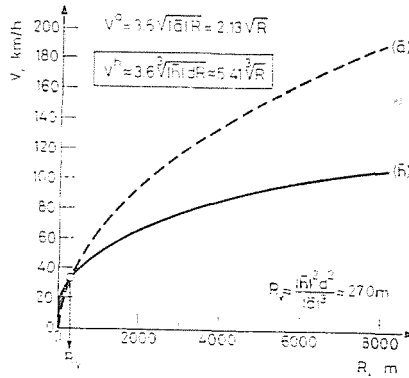


Fig. 17. Maximum permissible speed along circular curves without transition, non-superelevated

The maximum permissible speed on the basis of acceleration (6.1) is the following

$$V^a = 3.6 \sqrt{|a|R} \approx 2.13 \sqrt{R} \quad \text{km/h} \quad (6.11)$$

if according to Table 1: $a = 0.35 \text{ m/s}^2$. Taking the h vector into account (Table 2) $|h| = 0.2 \text{ m/s}^3$; according to (6—5) we have

$$V^h \approx 3.6 \sqrt[3]{|h|dR} = 5.41 \sqrt[3]{R} \quad \text{km/h} \quad (6.12)$$

Figure 17 shows the limiting functions V^a and V^h . By taking them into consideration, the limiting radius above which, in calculating the maximum permissible speed the h vector is critical, by using equality $V^a = V^h$ the following can be determined:

$$R_V = \frac{|h|^2 d^2}{|a|^3} \approx 270 \text{ m} \quad (6.13)$$

Examination of the minimum permissible radius of circular curve with transition curves

The value of the minimum permissible radius of circular curves calculated from the formula of cant on the *basis of acceleration is*

$$R_{\min}^a = \frac{0.0118V^2}{m + 0.153|a|} \quad \text{m (6.14)}$$

(where m is the cant, in m).

From the formula (6.4) of the 3rd order motion characteristic

$$R_{\min}^h = \frac{\alpha V^3}{3.6^3|h|L} \quad \text{m (6.15)}$$

From the comparison of the formulae R_{\min}^a and R_{\min}^h , i.e. (6.14) and (6.15) it is to be seen that while the minimum radius calculated on basis of the acceleration for a given speed depends only on the characteristics of the circular curve (m , a) and is not affected by the existence of the transition; the radius to be calculated from the magnitude of the h vector *calls the attention to the significance of the transition* (L , α). Further, from relationship (6.15) R_{\min}^h one can see that by increasing the length of L the value of R_{\min} may be decreased.

Thus, the circular curve of constant cant should always be examined as a *uniform geometry* since it arises from the geometric continuity of the railway line, a circular curve of constant cant cannot occur but together with a transition curve. The separate examination of the circular curve with a motion of constant speed would be misleading, as in that case only radial acceleration is indicated, the value of the lateral h vector is equal to zero. This, however, would result in narrowing down the examination and neglecting the critical state.

According to Fig. 18 the point of intersection of functions R_{\min}^a and R_{\min}^h determines the value of speed V_R above which the effect of the h vector

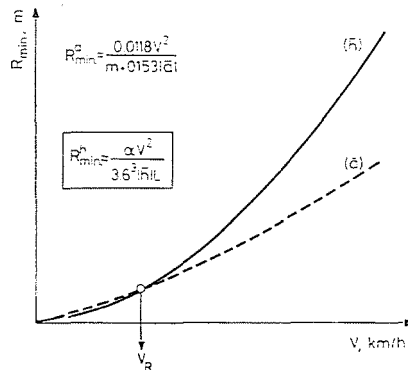


Fig. 18. Minimum permissible radius of a circular curve with transition

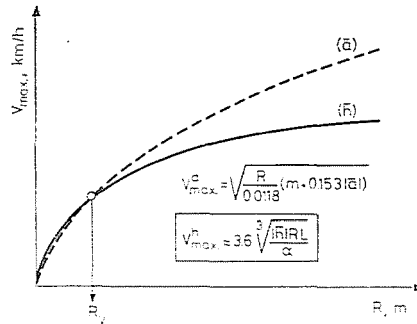


Fig. 19. The maximum permissible speed in case of a circular curve with a transition curve is critical:

$$V_R = \frac{3.6 \cdot |h| \cdot L}{\alpha \left(|a| + \frac{m}{0.153} \right)} \quad \text{km/h (6.16)}$$

6.5 Determination of the maximum permissible speed for the case of a circular curve with transitions

The permissible maximum speed in case of a given cant calculated from the acceleration is obtained by the formula

$$V_{\max}^a = \sqrt{\frac{m + 0.153|a|}{0.0118}} R \quad \text{km/h (6.17)}$$

and from the value of the h vector (1.6—4.)

$$V_{\max}^h = 3.6 \sqrt[3]{\frac{|h|R \cdot L}{\alpha}} \quad \text{km/h (6.18)}$$

Functions V_{\max}^a and V_{\max}^h are represented in Fig. 19. The abscissa of the point of intersection of the functions in comparison to which the radii of the circular curves are greater, and therefore, the h vector is critical, can be calculated on the basis of the equality $V_{\max}^a = V_{\max}^h$ by equation

$$R_v = \frac{|h|^2 L^2}{\alpha^2 \left(|a| + \frac{m}{0.153} \right)^3} \quad \text{m (6.19)}$$

Examination of radii of vertical curves applied in changes of grading

The radius of the vertical curve can be calculated on the basis of the acceleration with the aid of the formula

$$R_f^a = \frac{V^2}{3.6^2 |a_f|} \quad \text{m, (6.20)}$$

wherein a_f designates the acceleration permissible in the vertical plane ($a_f = 0.35 \text{ m/s}^2$).

From the formula of the 3rd order characteristic of motion (6—5)

$$R_f^h = \frac{V^3}{3.6^3 |h_f| d} \approx \frac{V^3}{238} \quad \text{m (6.21)}$$

Taking the motion in the vertical plane into account $|h_f| = 0.3 \text{ m/s}^3$.

The limiting speed can be calculated from equality $R_f^a = R_f^h$ on the basis of (6.20) and (6.21) above which the effect of the h vector is critical (Fig. 20).

$$V_f = 3.6 \frac{|h| d}{|a|} \approx 52.5 \text{ km/h} \quad (6.22)$$

Here, it is to be noted that in practice, several railways, considering the small radius resulting from the probable acceleration, determine the value of the vertical radii from the empirical relation $R_f = V^2$ in lieu of the acceleration aspect. Figure 20 clearly demonstrates that the above proceeding represents a significant "overdimensioning" which is otherwise also to be seen from the comparison with the values calculated from the formula $R_f = V^3/328$ justified by the geometry of motion (Table 4).

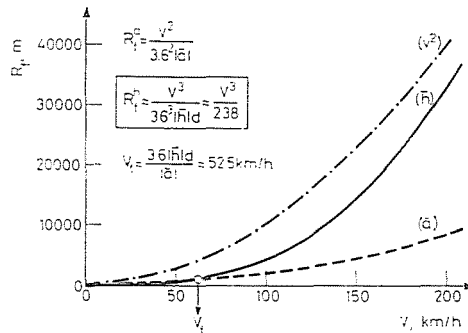


Fig. 20. Examination of the radii of vertical curves applied in changes of grading

Table 4

Magnitudes of radii of vertical curves applied in changes of grading

R_f m	V , km/h							
	60	80	100	120	130	140	150	160
$R_f = V^3/238$	908	2 152	4 202	7 260	9 231	11 529	14 181	17 210
$R_f = V^2$	3 600	6 400	10 000	14 400	16 900	19 600	22 500	25 600

From the instances dealt with in the foregoing, above speed $V = 35 \dots 52.5$ km/h (i.e. the radius $E = 270$ m), and in calculating the length of the transition it is the effect of the h vector which is to be taken as critical.

At the same time it can be stated that in case of the examination performed on the basis of the acceleration the error made will be ever greater with an increase in speed (i.e. the radius of the circular curve) as is to be seen in Figs 16 . . . 20.

Annexe I.

Country	Unbalanced outward radial acceleration	Maximum cent	3rd order characteristic	Velocity of raising	Max. inclina- tion of cant ($1/V$)
	$ a_0 _{\max}$ m/s ²	m_{\max} mm	$ h_0 _{\max}$ m/s ³	$ v_{0m} $ mm/s	$n/V = \text{ctg } \alpha/V$
U.K. (BR)	0.65	150	0.25	(38–57)	10
Austria (ÖBB)	0.65(0.85)	160	—	—	6
Bulgaria (BDŽ)	0.65	—	—	28(–35)	—
Belgium (SNCF)	0.65	160	—	—	6
Czechoslovakia (ČSD)	0.65	150	0.4	28(–35)	10
Denmark (DSB)	0.65	150	—	—	8
Finland (VR)*	0.69	150	—	—	10
France (SNCF)	0.98	160	0.3	50	5.5
Netherlands (NS)	0.78	150	—	—	8
Japan Tokaido	0.5–0.88 0.65–0.39	180	0.14	45	—
Jugoslavia (JŽ)	0.65	150	—	—	10
Poland (PKP)	0.6–0.65	150	0.7 (1.0)	(28–35)	8
Luxemburg (CFL)	0.65	150	—	—	8
Hungary (MÁV)	0.65	150	—	—	10
					(8)
GDR (DR)	0.65 (0.85)	150	0.5 (1.0)	28–35	8
GFR (DB)	$V < 160$ km/h: 0.85 $V > 160$ km/h: 0.65	150	—	28(–35)	10 (8)
Norway (NSB)	0.65 (0.69)	150	—	—	5.7
Italy (FS)	0.85	160	0.3 (0.15–0.2)	(40–44)	7 (5.2)
Rumania (CFR)	0.60	—	—	28–35	—
Sweden (SJ)	0.65	150	—	—	8 (6)
Switzerland (SBB-CFF)	0.65	150	0.24	—	6 (5)
Soviet Union (SzZsD)*	0.7	—	0.6	28–35	—
USA	0.5	152	—	32	—

* Gauge: 1524 mm

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