OPTIMAL DESIGN OF ELASTIC BAR STRUCTURES SUBJECT TO DISPLACEMENT CONSTRAINTS AND PRESCRIBED INTERNAL AND REACTION FORCES

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Summary

The paper presents a method for the minimum weight design of elastic bar structures (trusses and frames) subjected to displacement constraints and prescribed internal and/or reaction forces. The structural optimization leads to a nonlinear mathematical programming problem which can be solved by repeating the elastic analysis and optimization of the structure. The illustrative examples show that the procedure converges rapidly and after a few steps sufficiently accurate results can be obtained.

1. Introduction

This paper presents a method for optimal design of statically indeterminate bar structures (trusses and frames) with piece-wise constant cross-sections and elastic material. The purpose of the optimization is to minimize the weight of the structure subject to the specified constraints. These constraints may include limitations on the displacements and requirements concerning the values of the internal and/or reaction forces at various points of the structure.

A great number of books and papers deal with the optimal design of various structures subjected to displacement and stress constraints (e.g. [1]—[5]). This paper discusses the problem where a certain distribution of the internal and reaction forces, considered advantageous from a practical point of view is given and those cross-sectional areas of a statically indetermined structure are searched, to be determined which yield the prescribed internal forces and the minimum weight of the structure, as well. Since the proposed method is based on the optimal design with displacement constraints published in [6] and [7], therefore the solution of this problem will be briefly described first.

2. Optimal design subject to displacement constraints

2.1. *Description of problem*

Let us consider homogeneous linear elastic bar structures (beams or frames) consisting of constant cross-sectional elements under a given loading condition. Let us denote δ_j the maximal allowable values of displacements

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d_j at point $x = \xi_j$ $(j = 1, 2, \ldots, m)$. Then, the displacement constraints are as follows

$$|d_j| = \left| \sum_{i=1}^{n} \int_{L_i} \frac{M_0(x)\, m_j(x)}{EJ_i} \, dx \right| \leq \delta_j, \quad j = 1, 2, \ldots, m, \tag{1}$$

where x denotes the coordinate measured along the axis of the structure, while $M_0(x)$ and $m_j(x)$ are the moment-distributions due to the external load and the unit virtual load acting in the direction of the jth displacement, respectively. E is the Young's modulus of the material, and J_i is the moment of inertia of the ith element with length L_i. Let us introduce the flexibility coefficient (e_{ij})

$$e_{ij} = \int_{L_i} \frac{M_0(x)\, m_j(x)}{E} \, dx \tag{2}$$

and a function f expressing the relation between the cross-sectional area A_i and the moment of intertia J_i:

$$J_i = f(A_i) \tag{3}$$

Eqs (1) can then, be written as:

$$\left| \sum_{i=1}^{n} \frac{e_{ij}}{f(A_i)} \right| - \delta_j \leq 0, \quad j = 1, 2, \ldots, m \tag{4}$$

In case of homogeneous structures the weight is proportional to the volume, so the optimality criterion is given by

$$V(A_i) = \sum_{i=1}^{n} A_i L_i = \min ! \tag{5}$$

Eqs (4) and (5) represent a nonlinear mathematical programming (MP) problem. There are two main possibilities to solve it: using the optimality criteria (indirect) method or the gradient (direct) methods. The indirect method guarantees the optimal solution by solving the optimality criteria equations. This method was successfully applied to the optimal design of trusses (see [6] and [7]). In case of structural optimization the objective function is generally linear and the constraints are nonlinear. In a number of cases the MP problem can be transformed by using reciprocal design variables in so that the constraints are linear and the objective function is nonlinear. This problem can be solved efficiently by the gradient methods [8], [9].

2.2. *Optimality criteria method*

Using Eqs (4) and (5) the Lagrange function $W(A_i, \lambda_j)$ can be written as

$$W(A_i, \lambda_j) = \sum_{i=1}^{n} A_i L_i + \sum_{j=1}^{m} \lambda_j \left(\left| \sum_{i=1}^{n} \frac{e_{ij}}{f(A_i)} \right| - \delta_j \right), \tag{6}$$

where λ_js are the Lagrange multipliers. The necessary conditions for the optimum are obtained by differentiating Eq. (6) with respect to the design variables A_i. This gives

$$\sum_{j=1}^{m} \lambda_j \frac{e_{ij} f'(A_i)}{L_i f^2(A_i)} = 1. \tag{7}$$

The optimal solution has to satisfy Eqs. (4) and (7). These are $m + n$ nonlinear inequalities and equations and the unknowns are n cross-sectional areas and m Lagrange multipliers. A recurrence relation for the design variables can be written by multiplying both sides of Eq. (7) by A_i^p and taking the pth root. This gives

$$A_i^{(\gamma+1)} = A_i^\gamma \left(\sum_{j=1}^{n} \lambda_j \frac{e_{ij} f'(A_i)}{L_i f^2(A_i)} \right)^{1/p} \tag{8}$$

where $\gamma + 1$ and γ refer to the iteration number and the parameter p determinees the step size. This recurrence relation with $p = 2$ was used in [5, 6] and a detailed discussion of this method can be found in [7].

2.3. *Wolfe's reduced gradient method*

Completing the MP problem defined by Eqs (4) and (5) with the minimum size constraints

$$A_i \geq A_i^{\min} \qquad (i = 1, 2, \ldots, n) \tag{9}$$

it can be reduced to an other one, where a nonlinear objective function has to be minimized subjected to linear constraints.

Since $J_i = f(A_i)$ is a monotonic function, A_i can be expressed with the inverse function

$$A_i = f^{-1} \left(\frac{1}{X_i} \right), \tag{10}$$

where the new reciprocal design variable X_i is given by

$$X_i = \frac{1}{f(A_i)}. \tag{11}$$

Then the mathematical programming formulation of the problem is

$$\sum_{i=1}^{n} L_i f^{-1} \left(\frac{1}{X_i} \right) = \min ! \tag{12}$$

subject to the *linear* constraints

$$\sum_{i=1}^{n} e_{ij} X_i \leq \delta_j, \quad (j = 1, 2, \ldots, m) \tag{13}$$

$$0 \leq X_i \leq \frac{1}{f(A_i^{\min})} \,. \tag{14}$$

Eqs (12), (13) and (14) can be solved by Wolfe's reduced gradient method [8, 9] and the optimal values of the original design variables A_i $(i = 1, 2, \ldots, n)$ can be obtained by using Eq. (10).

The flexibility coefficients e_{ij} are constant for statically determinate structures, so the solution of Eqs (12), (13) and (14) gives the optimal solution of the problem in one step. For statically indeterminate structures, however, the flexibility coefficients also become a function of the design variables. In this case the structural optimization algorithm consists of two main steps. The first step is to analyze the structure and to compute the flexibility coefficients e_{ij} $(i = 1, 2, \ldots, n, j = 1, 2, \ldots, m)$. The second step is to redistribute the material that is to solve the MP problem by the optimality criteria method or by Wolfe's gradient method using the constant values e_{ij} obtained in the first step. These two main steps have to be repeated until the differences between the results of two subsequent steps are sufficiently small.

In case of trusses $f(A_i) = A_i$, which simplifies the solution of the problem.

3. Optimal design subject to prescribed internal and reaction forces

In case of statically indeterminate structures the distribution of the internal forces depends on the design variables A_i $(i = 1, 2, \ldots, n)$. When changing these design variables, different distributions of the internal forces can be obtained. Thus, by a suitable choice of the design parameters a desired distribution of the internal forces can be ensured.

In the following a method will be presented for calculating the design variables A_i which minimize the volume of the structure and ensure that the internal and/or reaction forces at given points of the structure take their desired values.

Consider a homogeneous, linear elastic, statically indeterminate bar structure (frame or truss) with piece-wise constant cross-sections under a given loading condition. Let us denote by C_j $(j = 1, 2, \ldots, n)$, the desired values of the internal and/or reaction forces at the points $x = \xi_j$. To solve our problem we must reduce the structure by cutting the corresponding constraints at $x = \xi_j$ and replace them by the prescribed values of the internal and reaction forces C_j. At the jth cut of the actual structure the relative displacement caused by the external loads and the prescribed $C_1, C_2, \ldots, C_m$ internal and reaction forces must be equal to zero. This is expressed as follows:

$$\sum_{i=1}^{n} \int_{L_i} \frac{M(x) \, m_j(x)}{EJ_i} \, dx = 0 \qquad j = 1, 2, \ldots, m \,. \tag{15}$$

Here $M(x)$ is the moment distribution calculated from the external loads and the prescribed internal and reaction forces and $m_j(x)$ is the moment distribution due to the virtual unit load $C_j = 1$ acting at $x = \xi_j$. Both $M(x)$ and $m_j(x)$ have to be computed on the reduced structure. Then the optimal design problem in question can be formulated as follows:

$$V(A_i) = \sum_{i=1}^{n} L_i A_i = \min ! \tag{16}$$

subject to Eqs (15) and to the constraints $A_i \geq A_i^{\min}$. This is a special case of the MP problem described in Section 2. The only difference is that now (1) are equations rather than inequalities and $\delta_j = 0$. The solution of this problem is similar to that described in Section 2.

It is to be noted that the number of prescribed internal and reaction forces and their places and magnitudes cannot be optional. They must be prescribed so that after cutting the corresponding constraints, the reduced structure remain stable and the number of elements with constant cross-section be large enough. The investigation of this problem is outside the scope of this paper.

4. Optimal design subject to displacement constraints and prescribed internal forces

The optimal design methods described above can be generalized for cases, where both displacement constraints and internal or reaction forces are prescribed. In this general case the volume of the structure (12) must be minimized subject to the inequalities (13), (14) and the equations (15). The details of this problem will be illustrated by the solution of an example.

5. Numerical examples

Example 1

The problem is to minimize the volume of an elastic two-span continuous beam of given geometry and material properties subjected to a single force (Fig. 1). The design variables are the cross-sections of the four elements of the structure. $(A_i, i = 1, 2, 3, 4.)$ The reaction force at support B is prescribed; its desired value is 6 kN. The prescribed minimum size of the rectangular cross-sections with ratio $b/a = 2$ is $A^{\min} = 30$ cm^2.

Replacing the middle support by a vertical force with magnitude 6 kN, the vertical displacement at this place has to be equal to zero:

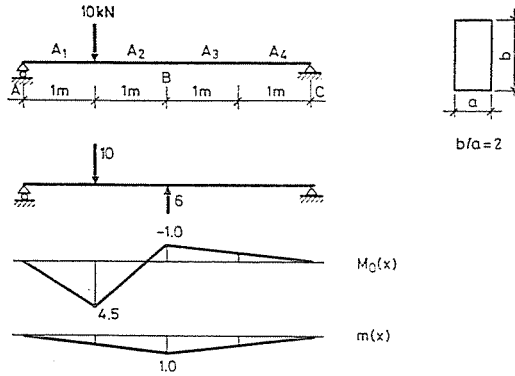$$\sum_{i=1}^{n} \int \frac{M_m}{EJ_i} dx = 0.$$

Fig. 1

In case of the assumed rectangular cross-section $J_i = \dfrac{A_i^2}{6}$, so the MP problem can be formulated as:

$$V = \sum_{i=1}^{n} L_i A_i = \sum_{i=1}^{n} A_i = \min !$$

subject to

$$\frac{0.75}{A_1^2} + \frac{1.083}{A_2^2} - \frac{0.583}{A_3^2} - \frac{0.0833}{A_4^2} = 0$$

and

$$A_i \geq 30 \qquad (i = 1, 2, 3, 4).$$

The solution of this problem is $A_1 = 46.7$ cm², $A_2 = 52.9$ cm², $A_3 = 30$ cm². $A_4 = 30$ cm².

Example 2

An elastic frame subjected to a single force is shown in Fig. 2. The design variables are the cross-sectional areas A_i, $(i = 1, 2, \ldots, 6)$ of the six elements. The cross-sections are rectangular with a ratio of $b/a = 1.5$. (In this case $J_i = A_i^2/8$.) Young's modulus of the material is $E = 2 \cdot 10^7$ kN/m². The problem is to find the values A_i which minimize the volume of the structure and satisfy the following constraints:

a) The desired value of the bending moment at point A is 80 kNm.

b) The desired value of the bending moment at point B is 140 kNm.

c) The maximum allowable value of the horizontal displacement at the upper level is $\delta = 0.1$ m.

d) The minimum value of the cross-sectional areas is 0.05 m².

Then the formulation of the mathematical programming problem is:

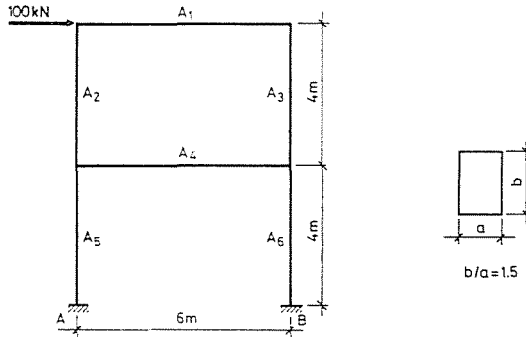$$V = \sum_{i=1}^{6} L_i A_i = \min !$$

Fig. 2

and

$$\sum_{i=1}^{6} \int_{L_i} \frac{M_0 M_1}{EJ_i} \, dx = 0,$$

$$\sum_{i=1}^{6} \int_{L_i} \frac{M_0 M_2}{EJ_i} \, dx = 0,$$

$$\sum_{i=1}^{6} \int_{L_i} \frac{M_0 M_3}{EJ_i} \, dx - \delta \leq 0,$$

$$A \geq A^{\min}, \qquad (i = 1, 2, \ldots, 6)$$

where M_0 is the moment distribution of the reduced structure (with hinges at points A and B) due to the 100 kN horizontal load and the prescribed bending moments acting at the points A and B. M_1, M_2 and M_3 are the moment distributions due to the virtual unit moments acting at A and B and the virtual unit horizontal force acting at the upper level, respectively.

Since the reduced structure is statically indeterminate, the moment distributions depend on the design parameters A_i. Thus, the solutions can be obtained by repeating the two main steps: the structural analysis and the redistribution of the material. The results of the iteration are given in Table I.

Table I

Area of cross-sections [m²]	Steps			
	0	1	2	5
A_1	0.084	0.096	0.059	0.056
A_2	0.084	0.084	0.072	0.074
A_3	0.084	0.057	0.050	0.050
A_4	0.084	0.189	0.131	0.131
A_5	0.084	0.050	0.050	0.050
A_6	0.084	0.068	0.066	0.065
Volume [m³]	2.35	2.75	2.09	2.08

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