AN ENDOCHRONIC MATERIAL MODEL FOR THE NUMERICAL ANALYSIS OF CONCRETE STRENGTH

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Received June 21, 1984

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Summary

Fundamental equations of the endochronic material model for cohesive granular materials have been presented, together with relationships needed for numerical models. Application possibilities of original and of significantly modified varieties have been illustrated on numerical problems.

Introduction

In civil engineering problems, knowledge of physical characteristics and internal regularities of the material is often of decisive importance. Theory and numerical application results of a material law with several deviations from fundamentals of the classic theory of plasticity will be presented, found to be superior in accuracy to earlier theories for materials of intricate behaviour (e.g. concrete, special rocks, etc.) little accessible to other models for tracing the exact development of internal stresses and deformations.

Essentials of the theory will be recapitulated under 2. Equations adapted to numerical analysis are presented under 3, followed by essential properties of computation programs. A detailed train of computation for linear and plane stress states is found under 4, referring to results obtained by gradually correcting the model.

Essentials of the endochronic material theory

Essential in the endochronic material model is to describe inelastic deformations by means of a special variable, the so-called "internal time". Internal time is an incremental positive scalar variable, its increase uniquely depends on the increase of deformations. Opposite to classic viscoplastic models, here no explicit yield condition is needed.

Fundamentals of the theory are due to K. C. Valanis [1], with primary concern of metallic material properties. Z. P. Bažant, C. L. Hsieh and others extended the sphere of validity to cohesive, granular materials (concrete, rocks, etc.) [2, 3]. The most important difference between the two material types is sensitivity to hydrostatic pressure. This effect is insignificant for metals but for granular materials it is advisably taken into consideration. Neither the effects of inelastic expansion due to large deviator deformations, as well as of the deformation-plastification are negligible for the type of material examined.

Let us recapitulate equations of the endochronic material model, omitting anything but essential steps. For detailed analyses in several fields see e.g. [4].

Endochronic material equations

The examined material is considered to be homogeneous and isotropic, omitting dynamic effects.

Let us start from Schapery's fundamental equation of classic visco-plasticity:

$$\dot{\varepsilon}_{ij} = D^{el}_{ijkl}\,\sigma_{kl} + \frac{\partial F}{\partial \sigma_{ij}} \Phi_1(\sigma_{ij}, \varepsilon_{ij})\,\Phi_2(\dot{\varepsilon}_{ij}) \tag{1}$$

where ε_{ij} — strain tensor velocity; σ_{ij} — stress tensor; F — load function. In the following, subscript i, j refers to coordinates of an orthonormed basis: $x_i(i = 1, 2, 3)$ etc. The second term of the equation is measure of inelastic deformations, it also depends on the strain tensor velocity. Function $\Phi_2(\dot{\varepsilon}_{ij})$ may be considered as continuous and smooth if inelastic material deformations develop gradually. Thereby Φ_2 can be expanded to tensorial Taylor series:

$$\Phi_2(\dot{\varepsilon}_{ij}) = (p_0 + p_{ij}\,\dot{\varepsilon}_{ij} + p_{ijkm}\,\dot{\varepsilon}_{ij}\,\dot{\varepsilon}_{km} + p_{ijkmrs}\,\dot{\varepsilon}_{ij}\,\dot{\varepsilon}_{km}\,\dot{\varepsilon}_{rs} + \ldots)^m \tag{2}$$

where

$$\Phi_2\big|_{|\dot{\varepsilon}_{ij}|\neq 0} > \Phi_2\big|_{|\dot{\varepsilon}_{ij}|=0}\,. \tag{3}$$

Again, quotient of the norm of inelastic deformation increment by the norm of the entire deformation tensor is greater than zero and less than infinity, if norm of the strain velocity tends to infinity:

$$\text{if} \lim \|\dot{\varepsilon}_{ij}\| \to \infty \quad \text{then} \quad 0 < \frac{\|d\,\varepsilon''_{ij}\|}{\|d\,\varepsilon_{ij}\|} < \infty\,. \tag{4}$$

Let us consider the Taylor series taking both assumptions into consideration. If linear term $p_{ij}\dot{\varepsilon}_{ij}$ is kept there, then the first condition cannot be met since $\dot{\varepsilon}_{ij}$ may be either positive or negative. Therefore p_{ij} has to be taken as zero. From a similar consideration, the cubic term has to vanish, to have $p_{ijkmvs} = 0$. Neglecting terms of order four or higher, it is finally:

$$\Phi_2(\dot{\varepsilon}_{ij}) = (p_0 + p_{ijkl}\,\dot{\varepsilon}_{ij}\,\dot{\varepsilon}_{kl})^m\,. \tag{5}$$

To eliminate time, let (1) be multiplied by dt, then divide (after rearrangement and reduction) the tensor of inelastic increments by the norm of deformation increment tensor

$$||d\,\varepsilon_{ij}|| = \sqrt{d\,\varepsilon_{ij} d\,\varepsilon_{ij}} \tag{6}$$

to obtain:

$$\frac{||d\,\varepsilon''_{ij}||}{||d\,\varepsilon_{ij}||} = \frac{\partial F}{\partial \sigma_{ij}} \Phi_1(\sigma_{ij}, \varepsilon_{ij}) \left[\frac{p_0}{||\varepsilon_{ij}||^{\frac{1}{m}}} + \frac{p_{ijkl}\,d\,\varepsilon_{ij}\,d\,\varepsilon_{kl}}{d\,\varepsilon_{vs}\,d\,\varepsilon_{vs}}\,||\varepsilon_{ij}||^{2-\frac{1}{m}} \right]^m. \tag{7}$$

Taking the double requirement of the second condition into consideration, it is easy to show that it is only met for $m = 1/2$. Taking it into consideration results in the endochronic material model in concise form:

$$d\,\varepsilon_{ij} = D^{el}_{ijkl}\,d\,\sigma_{kl} + d\,\varepsilon''_{ij} \tag{8}$$

where inelastic deformation

$$d\varepsilon''_{ij} = \frac{\partial F}{\partial \sigma_{ij}} dz \tag{9}$$

dz being the fundamental difference compared to any previous material model:

$$dz = \left\{ \left(\frac{d\zeta}{z_1}\right)^2 + \left(\frac{dt}{\tau_1}\right)^2 \right\}^{\frac{1}{2}} \tag{10}$$

where

$$d\zeta = (p_{ijkl}\,d\varepsilon_{ij}\,d\varepsilon_{kl})^{\frac{1}{2}} \tag{11}$$

z denotes the internal time scale complemented with real time in the endochronic model (real time being the second term, for creep effects). Of course, the first term, function of the rate of deformations, is the decisive one. The other parameters are: z_1, the relaxation strain, and τ_1, the inelastic deformation rate.

A more common, somewhat simplified form of (8) is that decomposed to deviator and hydrostatic parts:

$$de_{ij} = \frac{dS_{ij}}{2G} + de''_{ij} \tag{12a}$$

$$de''_{ij} = \frac{S_{ij}}{2G} dz \tag{12b}$$

$$d\varepsilon = \frac{d\sigma}{3K} + d\varepsilon'' \tag{12c}$$

$$d\varepsilon'' = d\varepsilon^0 + d\lambda + \frac{\sigma}{3K} dz' + d\lambda' \tag{12d}$$

where e_{ij} — deviator part of strain tensor; — mean strain $\left(\varepsilon = \frac{1}{3}\varepsilon_{kk}\right)$, s_{ij} — stress deviator tensor; — mean normal stress $\left(\sigma = \frac{1}{3}\sigma_{kk}\right)$; ε° — stress-independent inelastic strain (e.g. thermal expansion or shrinkage); K and G are moduli of elasticity of volumetric deformation and of shear, resp. (both are variables); λ — inelastic expansion; λ' — shear compaction; z and z' — internal times assigned to distorsion and to hydrostatic effects, respectively.

Effective application of the basic equation depends on how the internal time is assumed. This problem will be examined next.

Assumption of the internal time

Decomposition of Eq. (8) has bisection of dz as concomitant. Thereby (10) becomes:

$$dz = \left\{\left(\frac{d\zeta}{z_1}\right)^2 + \left(\frac{dt}{\tau_1}\right)^2\right\}^{\frac{1}{2}} \tag{13a}$$

$$dz' = \left\{\left(\frac{d\zeta'}{z_2}\right)^2 + \left(\frac{dt}{\tau_2}\right)^2\right\}^{\frac{1}{2}}. \tag{13b}$$

In subsequent examinations, time effect will be omitted (in other words, only "rapid" effects with $\frac{d\zeta}{dt} \gg 1$ and $\frac{d\zeta'}{dt} \ll 1$ will be considered, permitting to neglect the "real" time-dependent second term in the right hand side of (13). Internal times become, in modified form:

$$dz = \frac{d\zeta}{z_1} \quad \text{and} \quad dz' = \frac{d\zeta'}{z_2}. \tag{14}$$

For the examined materials, consideration of hardening and of plastification is rather important, possible by transforming internal deviator and volumetric time variables, properly assuming $d\zeta$ and $d\zeta'$. Different attempts have been made for it, see e.g. [1] or [3]. In the actual case:

$$d\zeta = \frac{d\eta}{f(\eta, \varepsilon_{ij}, \sigma_{ij})} \tag{15}$$

where

$$d\eta = F(\varepsilon_{ij}, \sigma_{ij})\, d\xi$$

and

$$d\zeta' = \frac{d\eta'}{h(\eta')} \tag{16}$$

where

$$d\eta' = H(\sigma_{ij})\, d\xi'$$

where $F(\varepsilon_{ij}, \sigma_{ij})$ and $f(\eta, \varepsilon_{ij}, \sigma_{ij})$ are functions of distorsional plastification and hardening, resp.; $H(\sigma_{ij})$ and $h(\eta')$ are functions of compactional plastification, and of hardening upon hydrostatic effect, resp.; ξ and ξ' are incremental scalar variables to indicate the rate of distorsion or compaction. Increments of ξ and ξ' have been defined as:

$$d\xi = \sqrt{J_2(d\varepsilon_{ij})} = \sqrt{\frac{1}{2} de_{ij} de_{ij}} \tag{17a}$$

$$d\xi' = \sqrt{(I_1(d\varepsilon_{ij}))^2} = |d\varepsilon_{11} + d\varepsilon_{22} + d\varepsilon_{33}| \tag{17b}$$

where J_2 is second invariant of the deviator tensor of deformation, and I_1 is first invariant of the complete deformation tensor.

Inelastic expansion and compaction due to shear are expressed by functional relationships:

$$d\lambda = l(\lambda) \cdot L(\lambda, \varepsilon_{ij}, \sigma_{ij})\, d\xi \tag{18a}$$

$$d\lambda' = l'(\lambda) \cdot L'(\lambda', \varepsilon_{ij}, \sigma_{ij})\, d\xi' \tag{18b}$$

where l and L, and l' and L' are expansional, and shear compactional hardening, and plastification effects, respectively.

Use of the endochronic material model in the finite element method

Equations (12) have to be transformed for numerical analyses. Let stresses be expressed as:

$$dS_{ij} = 2G(de_{ij} - de''_{ij}) \tag{19a}$$

$$d\sigma = 3K(d\varepsilon - d\varepsilon'') \tag{19b}$$

or added:

$$dS_{ij} + d\sigma\delta_{ij} = 2Gde_{ij} + 3Kd\varepsilon\delta_{ij} - 2Gde''_{ij} - 3d\varepsilon''\,\delta_{ij}. \tag{20}$$

First two right-hand side terms describe the elastic effect, so they can be written in another form (immediately substituting corresponding components of the inelastic part):

$$d\sigma_{ij} = C_{ijkm}\, d\varepsilon_{km} - 2G\frac{S_{ij}}{2G} dz - 3K \cdot \delta_{ij}\left\{d\varepsilon^{\circ} + d\lambda + d\lambda' + \frac{\sigma}{3K} dz'\right\} \tag{21}$$

Further substitution, and zeroing $d\varepsilon^{\circ}$, yields the form suiting the finite element program:

$$d\sigma_{ij} = C_{ijkm}\, d\varepsilon_{km} - \left\{S_{ij}\frac{F(\varepsilon_{ij}, \sigma_{ij})}{z_1 f(\eta, \varepsilon_{ij}, \sigma_{ij})}\, d\xi + \sigma\frac{H(\sigma_{ij})}{z_2\, h(\eta')}\, d\,\xi\delta_{ij} + 3K\delta_{ij}(d\lambda + d\lambda')\right\}. \tag{22}$$

The finite element method, or nearer, examination of physically nonlinear systems, will not be discussed here. The presented test program has originally been intended for problems of plates of a linear-elastic material, actually extended to the concerned problem. A simple triangle has been applied, with one node of two degrees of freedom at each corner. The stress field is constant inside an element. The scheme of the structogram for the program:

Input of basic data (topology, initial stiffnesses, etc.)

$I = 1$, N LOADS
cycle controls application of load increments. Load values may differ between steps, cycle can direct also unloading and reloading

Iteration cycle to each equilibrium

Establishment of global stiffness matrix in dependence of instantaneous material characteristics

Computation of nodal displacements

Check displacements for convergence
max $|u_i| <$ eps

yes

no

$j = 1$, N ELEMENTS, cycle for all elements of the structure

computation of deformations and of "elastic" and "endochronic" stresses

computation of nodal force excesses from unbalanced element stresses

Printing out displacements, deformations and stresses for the equilibrium condition at the end of load increment, corresponding to the given error limit

Functioning of the program presented in the structogram — essentially, a procedure corresponding to the modified Newton—Raphson method — is illustrated in Fig. 1.

Next, some typical diagrams of results obtained in the analyses will be presented, enhancing the significance of modifications by correcting the program (and the theoretical bases).

Numerical results, and model correction

The first problem involved a compressive test in linear stress state. Control tests were those by Hognestad, Hansen and McHenry [6], and by Popovics [7]. Figure 2a shows scheme of the tested "structure", and 2b the $\sigma - \varepsilon$ diagram at ultimate condition.

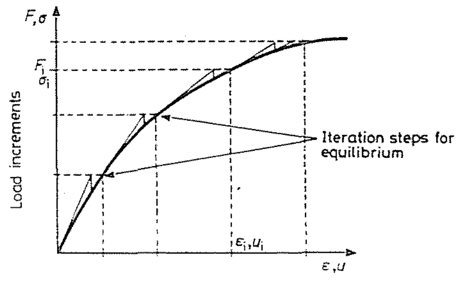

Fig. 1

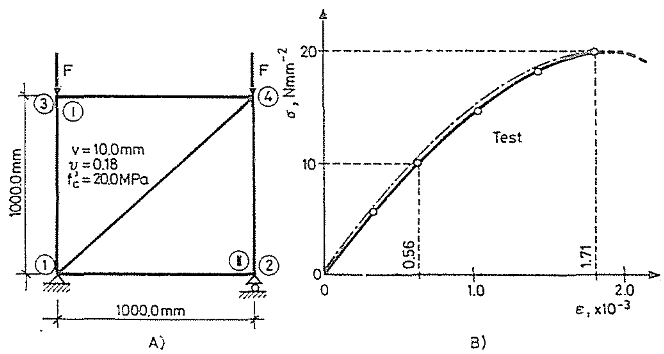

Fig. 2

$\sigma - \varepsilon$ functions for cyclic load, and for several unloadings and reloadings are seen in Figs 3a and b, respectively.

After the first analyses, details to be refined became apparent. Rising limbs of hysteresis loops were too "low", making the material to look like more plastic than in reality was. On the other hand, in reloading after unloading in low stress state, the material appeared too stiff.

To eliminate these errors, the following sophistications have been applied: To inelastic deformations that contained before only the "plastic" term, another component, the term for crumbling upon microcracking effects has been added.

Hence, (12) is replaced by:

$$de_{ij} = \frac{dS_{ij}}{2G} + de''_{ij} \tag{23a}$$

$$de''_{ij} = de^{pl}_{ij} + de^{t}_{ij} \tag{23b}$$

$$de^{pl}_{ij} = \frac{S^*_{ij}}{2G} d\zeta \quad \text{or} \quad de^{t}_{ij} = \frac{\partial \Phi}{\partial e_{ij}} d\varkappa \; . \tag{23c}$$

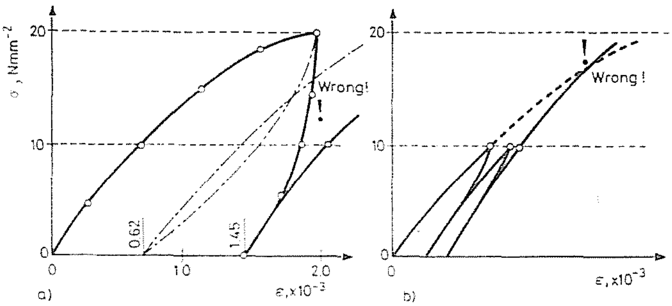

Fig. 3

Other terms remain inaltered. In plastic deformation, modification consists in a transformation by shifting:

$$S_{ij}^{*} = S_{ij} - \alpha_{ij} \tag{24}$$

where α_{ij} is centre of the deviator load surface, needed in cyclic loading.

Of course, the spherical part of stresses has also to be modified: $\sigma^{*} = \sigma - \alpha'$.

Φ in (23b) stands for the crumbling function, with properties accessible to detailed definition (see e.g. in [8]). It is rather similar in form to the Prager—Drucker plasticity condition. In conformity with Dougill's [8] suggestion:

$$\Phi = \frac{1}{2} e_{km}^{*} e_{km}^{*} + h_2(\varepsilon) - H_2 . \tag{25}$$

Substituting it into (23c), the deviator increment of deformation at failure:

$$de_{ij}^{t} = e_{ij}^{*} d\varkappa \text{ where } e_{ij}^{*} = e_{ij} - \alpha'_{ij} \tag{26}$$

is a transformed variable again. $d\varkappa$ stands for a new kind of internal time:

$$d\varkappa = c \cdot F_t \cdot d\xi \tag{27}$$

where c — multiplier parameter of cyclic load (see later); and F_t — function for the plasticizing effect of microcracks. Of course, also $d\xi$ and $d\xi'$ have to be modified by c and c', respectively. Under monotonous load $c = c' = 1$ (see in [5]).

Another significant modification is kinematic hardening introduced in cyclic loading. Accordingly, centers of the actual load surfaces (α_{ij}, α'_{ij} and α') jump to the maximum, and minimum, stress and strain points at the beginning of unloading, and of reloading, respectively.

Loading and unloading may be influenced also by load direction-dependent c and c' parameter values. In analyzing the strain energy dissipated in

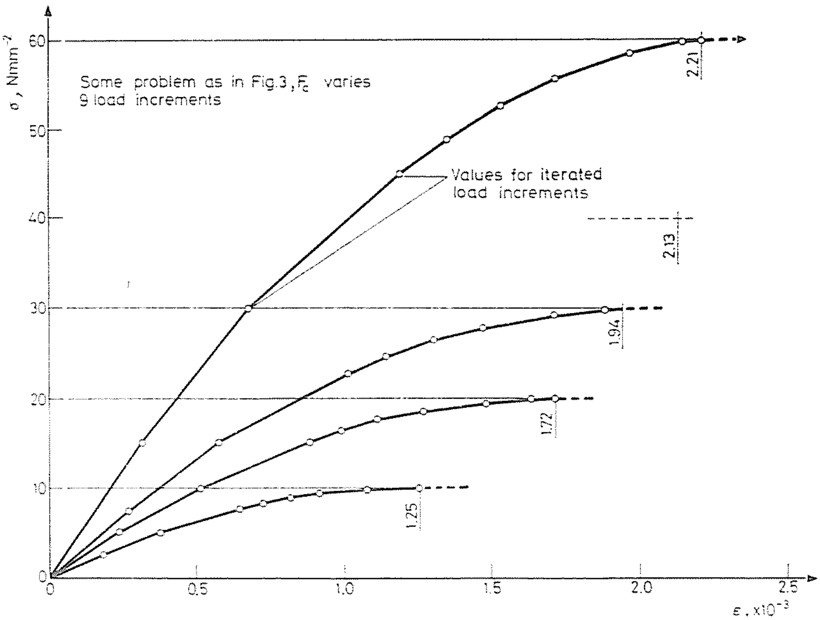

Fig. 4

the load cycle, it may be deduced that two different values have to be assumed for c and c', separately for loading and for unloading (see in [5]). Let subscripts n and ν refer to unloading and reloading, resp., then the two correction parameters are related as:

$$C_n < C_\nu < 2C_n. \tag{28}$$

The loading — unloading — reloading criterion as a function of deviator and volumetric work ($dW = s_{km}\, de_{km}$ and $dW' = 3\sigma\, d\varepsilon$) decides when to replace the original parameter $c = c' = 1$ by c_n or c_v, namely:

$dw \geq 0$ and $w = w_0 \rightarrow c = 1$ *original* load (29a)
$dw' \geq 0$ and $w' = w_0 \rightarrow c' = 1$

$dw < 0 \rightarrow c = c_n\ dw' < 0 \rightarrow c' = c_n$ *unloading* (29b)

$dw \geq 0$ and $w < w_0 \rightarrow c = c_v$ *reloading* (29c)
$dw' \geq 0$ and $w' < w_0 \rightarrow c' = c_v$

where w_0 and w_0' are instantaneous maxima of w and w'.

Earlier stiffness increment in the new loading limb may be eliminated by simple means. The error resulted from the excessive increase of ζ in cyclic loading, so that hardening functions yielded excessive stiffness for the material. Let us determine the ζ° value where the loading process turns unloading, and computation has to be continued from the same point ζ° in reloading, a correction producing the proper "behaviour".

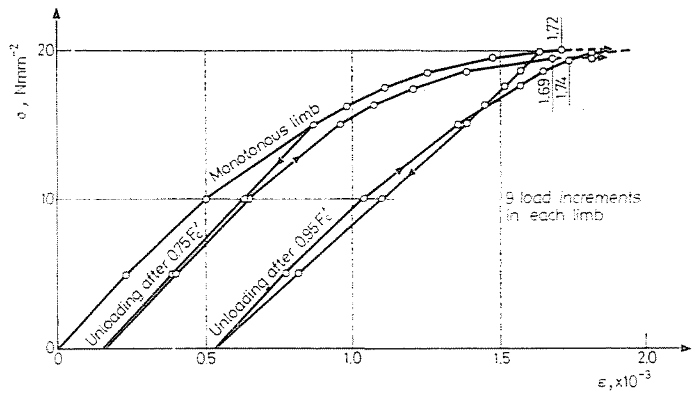

Fig. 5

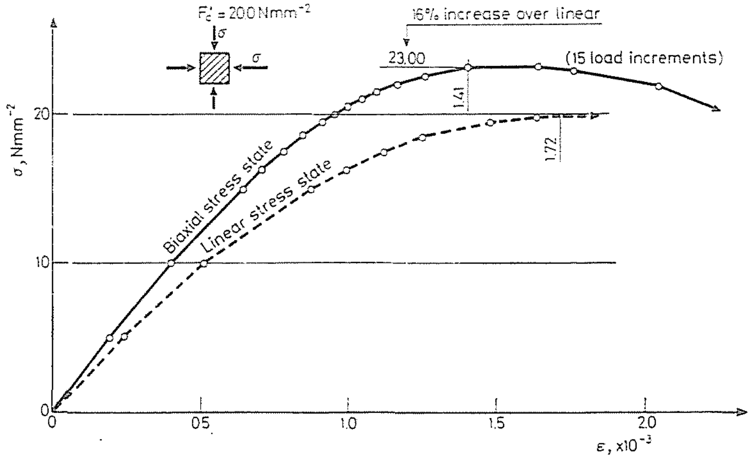

Fig. 6

In concluding the discussion of modifications, let us mention that introduction of the term for crumbling has, at the same time, somewhat modified the computation of the variation of stiffness parameters.

Second term in

$$de_{ij} = \frac{dS_{ij}}{2G} - e_{ij}\frac{dG}{G} \tag{30}$$

being for ultimate deformation,

$$d\varkappa = -\frac{dG}{G} \rightarrow dG = -G \cdot dx \tag{31}$$

also G vs. $d\chi$ may be indicated.

Computation results obtained with the modified material model are illustrated in Figs 4 to 7.

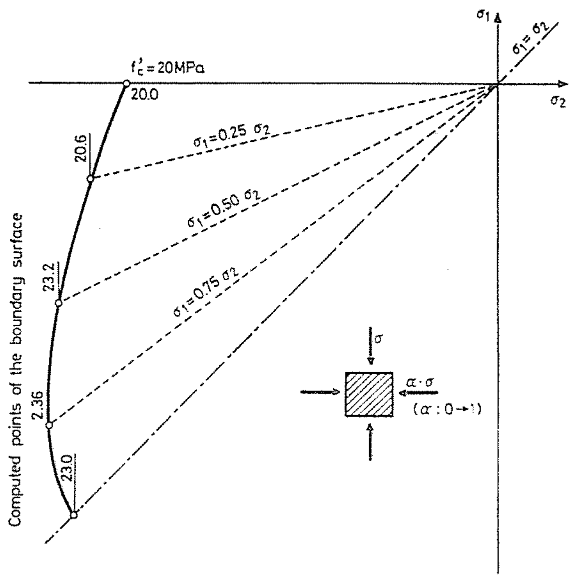

Fig. 7

Further research is going on at the Department of Civil Engineering Mechanics on theoretical and numerical uses of this model in the analysis of complex stress effects (tension, important shear effects, triaxial stress state).

Acknowledgement

The Author is deeply indebted to Prof. Luigi Cedolin (Politecnica di Milano, Dip. di Ingegneria Strutt.) for his kind assistance and for having made him available his experimental measurements for determining the model material constants.

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