VISCOUS AND HYSTERETIC DAMPING IN VIBRATION OF STRUCTURES

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Summary

The study has introduced the solution of a homogenous differential equation system on the basis of eigenvectors for the case of external damping proportional with speed that makes the simple description of approximate solution possible in case of a certain number of eigenvectors. A method was shown to compose the equivalent damping matrix of structures consisting of different damping characteristic elements and for the differential matrix equation system describing vibration. The description of the equivalent differential equation in the given form makes the superposition of the matrix of external and internal damping possible. The equivalent damping matrix was written with eigenvectors obtained from the solution of an original size real eigenvalue task as a sum of matrices belonging to the individual eigenvectors. The knowledge of these component matrices makes possible the test of damped vibration with a certain number of eigenvectors, enabling to determine damping characteristics at these vibration forms.

Damping caused by internal friction is always to be taken into consideration with vibration of structures, at the same time external vibration dampers can be used, too. To consider damping effects, when calculating structures with the method of finite elements, a model is to be worked out efficient even in case of great degree of freedom. The study introduces a linearized model that makes possible to obtain damping effects with suitable accuracy in case of determining a certain number of vibration forms and frequency, that is the partial solution of eigenvalue tasks.

External damping proportional with speed

The differential matrix equation describing vibration is:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{q}(t) \tag{1}$$

To solve the homogeneous differential equation system introducing an unknown

$$\mathbf{y}(t) = \begin{bmatrix} \dot{\mathbf{u}}(t) \\ \mathbf{u}(t) \end{bmatrix} \tag{2}$$

the following differential matrix equation can be written

$$\mathbf{A}\dot{\mathbf{y}}(t) + \mathbf{B}\mathbf{y}(t) = 0 \tag{3}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{O} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -\mathbf{M} & \\ & \mathbf{K} \end{bmatrix}.$$

Looking for the solution with formula

$$\mathbf{y}(t) = e^{\tilde{\lambda} t}\, \tilde{\mathbf{w}} \tag{5}$$

(mark $\sim$ means complex quantities)
the following generalized eigenvalue task is obtained

$$\tilde{\lambda}\mathbf{A}\tilde{\mathbf{w}} = -\mathbf{B}\tilde{\mathbf{w}} \tag{6}$$

where — as matrix $\mathbf{B}$ is not a positive definite — eigenvalues can be real negatives (great damping), or complex conjugated with a negative real part (small damping). Depending on the structure of the damping matrix, if damping does not belong to certain vibration forms the real part of complex eigenvalue pairs can be zero.

It can be proved that eigenvectors $\tilde{\mathbf{w}}$ are orthogonal to matrixes $\mathbf{A}$ and $\mathbf{B}$, enabling to write the solution of the differential equation system (3) on the basis of eigenvectors after their normalization to matrixes $\mathbf{A}$ and $\mathbf{B}$.

$$\mathbf{y}(t) = \sum_{r=1}^{2n} \tilde{\mathbf{w}}_r\, \tilde{\mathbf{w}}_r^* \mathbf{A} \mathbf{y}_0\, e^{\tilde{\lambda}_r t}. \tag{7}$$

Here $\mathbf{y}_0$ contains the initial conditions

$$\mathbf{y}_0 = \begin{bmatrix} \dot{\mathbf{u}}_0 \\ \mathbf{u}_0 \end{bmatrix}.$$

Separating the real solutions belonging to real and complex eigenvalues (n_r is the complex eigenvalue pair number) and considering that

$$\tilde{\mathbf{w}}_r = \begin{bmatrix} \tilde{\lambda}\, \tilde{\mathbf{v}}_r \\ \tilde{\mathbf{v}}_r \end{bmatrix}$$

$$\begin{aligned} \mathbf{u}(t) = \sum_{j=1}^{n_r} 2e^{-\varrho_j t} [\{\mathbf{A}_j \mathbf{M}\dot{\mathbf{u}}_0 + (\mathbf{A}_j \mathbf{C} - \varrho_j \mathbf{A}_j \mathbf{M} - \omega_{jc} \mathbf{B}_j \mathbf{M})\, \mathbf{u}_0\} \cos \omega_{jc} t - \\ - \{\mathbf{B}_j \mathbf{M}\dot{\mathbf{u}}_0 + (\mathbf{B}_j \mathbf{C} - \varrho_j \mathbf{B}_j \mathbf{M} + \omega_{jc} \mathbf{A}_j \mathbf{M})\, \mathbf{u}_0\} \sin \omega_{jc} t] + \\ + \sum_{l=2n_r+1}^{2n} \mathbf{v}_l \mathbf{v}_l^* \{\mathbf{M}\dot{\mathbf{u}}_0 + (-\varrho_l \mathbf{M} + \mathbf{C})\, \mathbf{u}_0\}\, e^{-\varrho_l t}. \end{aligned} \tag{8}$$

Here ϱ_j and ω_{jc} can be obtained from a complex eigenvalue pair

$$\tilde{\lambda}_j = -\varrho_j + i\,\omega_{jc},$$

while ϱ_l can be obtained from real eigenvalues

$$\lambda_l = -\varrho_l$$

Matrixes $\mathbf{A}_j$ and $\mathbf{B}_j$ can be calculated with the help of real and imaginary parts of complex eigenvectors $\tilde{\mathbf{v}}_j$

$$\tilde{\mathbf{v}}_j = \mathbf{v}_j^v + i\,\mathbf{v}_j^i$$
$$\mathbf{A}_j = \mathbf{v}_j^v \mathbf{v}^{v*} - \mathbf{v}_j^i \mathbf{v}_j^{i*}$$
$$\mathbf{B}_j = \mathbf{v}_j^i \mathbf{v}_j^{v*} + \mathbf{v}\,\mathbf{v}_j^{i*}$$

The solutions belonging to inhomogenous parts can be similarly written on the basis of eigenvectors.

Using the method of finite elements, the degree of freedom of the system can be very high at the same time the different eigenvectors play different roles in the solution, and depending on the initial condition, the solution can be obtained with suitable accuracy in the knowledge of some eigenvectors belonging to the smallest vibration number. In case of solving the solution in a closed form (8) there is no obstacle to perform the calculations, there is no need to calculate the multiplication factors depending on initial conditions of solutions belonging to different eigenvectors.

Calculation of internal, frequency independent friction in case of different damping characteristic elements

Internal frequency independent friction can be written with the aid of complex stiffness and in case of structures consisting of different damping characteristic elements, the differential matrix equation belonging to the vibration is:

$$\mathbf{M}\ddot{\tilde{\mathbf{u}}}(t) + \tilde{\mathbf{K}}\tilde{\mathbf{u}}(t) = \mathbf{q}(t) \tag{9}$$

Looking for the solution of the homogeneous equation in form of:

$$\tilde{\mathbf{u}} = \tilde{\mathbf{v}}\, e^{i\tilde{\delta}t}$$

the generalized complex eigenvalue task is obtained:

$$\tilde{\mathbf{K}}\,\tilde{\mathbf{v}} = \tilde{\lambda}\mathbf{M}$$

($\tilde{\lambda} = \tilde{\delta}^2$).As shown in [1] there is no possibility to write the solution in a closed form similar to (8), the constants belonging to the suitable solution of certain eigenvectors are to be calculated from the initial conditions. The real part of the solution according to [2]:

$$\mathbf{u}(t) = [\mathbf{V}^v\,\mathbf{D}_1(t) + \mathbf{V}^i\mathbf{D}_2(t)]\,\mathbf{a} + [\mathbf{V}^v\,\mathbf{D}_2(t) - \mathbf{V}^i\mathbf{D}_1(t)]\,\mathbf{b} \tag{10}$$

Here $\mathbf{V}^v$ and $\mathbf{V}^i$ are matrixes containing the real and imaginary parts of eigenvectors $\tilde{\mathbf{V}}$

$$\tilde{\mathbf{V}} = \mathbf{V}^v + i\,\mathbf{V}^i$$
$$\mathbf{D}_1(t) = \langle e^{-\nu_r/2}\,\omega_{rc}\,t\cos\omega_{rc}\,t\rangle$$
$$\mathbf{D}_2(t) = \langle e^{-\nu_r/2}\,\omega_{rc}\,t\sin\omega_{rc}\,t\rangle$$

where ν_r and ω_{rc} can be obtained from $\bar{\delta}_r$

$$\tilde{\delta}_r = \left(1 + i\frac{\nu_r}{2}\right)\omega_{rc}.$$

The constants in vectors **a** and **b** can be calculated from the initial conditions. If only some of the eigenvectors are calculately the pertaining multiplication factors cannot be determined accurated as the number of constants is lower than the number of the prescribed initial conditions. In this case the method of minimum squares is used to calculate the elements of **a** and **b** or the initial conditions are prescribed at places of decreased number.

The above-mentioned facts justify the necessity to find an equivalent **C** matrix, with which the solution can be obtained in a closed form, in accordance with (8).

In (2) Цейтлин suggests describing the differential equation belonging to the internal friction in the following form:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{M}(\mathbf{M}^{-1}\mathbf{K})^{1/2}\mathbf{\Gamma}\,\mathbf{u}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{q}(t). \tag{11}$$

The matrix of dissipation factors $\mathbf{\Gamma}$ is determined from the suggestion that the particular solution of inhomogeneous equation should be equal with the real part of the solution obtained from equation

$$\mathbf{M}\ddot{\tilde{\mathbf{u}}}(t) + (\mathbf{K}_u u + i\,\mathbf{K}_v)\,\tilde{\mathbf{u}}(t) = \mathbf{q}(t) \tag{12}$$

when exciting with a harmonic force. According to [2]

$$\mathbf{\Gamma} = \mathbf{K}_u^{-1}\mathbf{K}_v. \tag{13}$$

If eigenvectors and vibration numbers, orthonormal to $\mathbf{M}$ belonging to an undamped position ($\mathbf{v}_r$ and ω_r) are known

$$\mathbf{C} = (\mathbf{M}\mathbf{V}\langle\omega_r\rangle\,\mathbf{V}^*\mathbf{M})\,\mathbf{K}_u^{-1}\mathbf{K}_v. \tag{14}$$

The disadvantage of the method introduced is that both real and complex stiffness matrixes are to be produced and to determine matrix **C**, matrix $\mathbf{K}_u$ is to be inverted. As the degree of freedom can be quite high the production, storage of the invers matrix (it will not have a tape structure) is a difficult task.

If matrix $\mathbf{K}_u$ is substituted instead of **K** is equation (11) and the deduction is repated, expression (13) is obtained for matrix $\mathbf{\Gamma}$ again. If eigenvectors obtained from eigenvalue tasks

$$\mathbf{K}_u\mathbf{V} = \omega_{ru}^2\,\mathbf{M}\mathbf{V} \tag{15}$$

are normalized to M, considering that

$$\mathbf{V}^*\,\mathbf{M}\mathbf{V} = \mathbf{E}$$

$$\mathbf{V}^*\,\mathbf{K}_u\mathbf{V} = \langle\omega_{ru}^2\rangle$$

the damping matrix is:

$$\mathbf{C} = \mathbf{MV} \left\langle \frac{1}{\omega_{ru}} \right\rangle \mathbf{V}^* \mathbf{K}_v \tag{16}$$

that is matrix $\mathbf{K}_u$ is not to be inverted, the damping matrix can be produced easier with the eigenvectors of eigenvalue task in (15). Thus the damping matrix is produced from components in accordance with different eigenvectors:

$$\mathbf{C} = \sum_{r=1}^{n} \frac{1}{\omega_{ru}} \mathbf{M}\mathbf{v}_r \mathbf{v}_r^* \mathbf{K}_v . \tag{17}$$

Let us consider the case when the structure has identical damping characteristics. Then

$$\tilde{\mathbf{K}} = (u + iv)\, \mathbf{K}$$

and the homogeneous differential equation can be written as

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \left(\sum_{r=1}^{n} \frac{v}{u\, \omega_{ru}} \mathbf{M}\mathbf{v}_r \mathbf{v}_r^* \mathbf{K}_u \right) \dot{\mathbf{u}}(t) + \mathbf{K}_u \mathbf{u}(t) = 0 \tag{18}$$

With the substitution

$$\mathbf{u}(t) = \mathbf{V}\mathbf{x}(t)$$

and after multiplication with matrix $\mathbf{V}$ from the left the differential equation breaks up into n number differential equations with one unknown

$$\ddot{x}_r + c_r \dot{x}_r + \omega_{ru}^2 x = 0 .$$

Here

$$c_r = \mathbf{V}^* \left(\sum_{r=1}^{n} \frac{v}{u\, \omega_{ru}} \mathbf{M}\mathbf{v}_r \mathbf{v}_r^* \mathbf{K}_u \right) \mathbf{V} .$$

If only a certain number of eigenvectors appear in the expression in brackets, the number of values c_r obtained are in accordance with them, viz. the damping matrix in the sum (17) "distributes" the dissipation characteristics of the structure according to certain eigenvectors. This means that the calculations can be performed with a certain number of eigenvectors, for these vibration forms the pertaining damping characteristics can be obtained potentially while for the other ones they cannot be obtained.

Simultaneous effect of external and internal damping

The differential matrix equation belonging to matrixes

$$\mathbf{M}\ddot{\tilde{\mathbf{u}}}(t) + \mathbf{C}\dot{\tilde{\mathbf{u}}}(t) + (\mathbf{K}_u + i\, \mathbf{K}_v)\, \tilde{\mathbf{u}}(t) = \mathbf{q}(t) \tag{18}$$

was suggested among others by Gupta [3], who calculated the frequency and damping characteristics of the damped system on this basis. Let us examine first the special case when the matrix of external damping is proportional with the mass matrix while the characteristics of internal damping are identical at every element of the structure. The homogeneous equation

$$\mathbf{M}\ddot{\tilde{\mathbf{u}}}(t) + \beta\,\mathbf{M}\dot{\tilde{\mathbf{u}}}(t) + (u + iv)\,\mathbf{K}\tilde{\mathbf{u}}(t) = 0 \tag{19}$$

with substitution $\tilde{\mathbf{u}}(t) = \mathbf{V}\tilde{\mathbf{y}}(t)$ and after multiplication with matrix $\mathbf{V}^*$ from the left ($\mathbf{V}$ is the matrix of eigenvectors normalized to $\mathbf{M}$ belonging to the undamped case) the equation system is broken up:

$$\ddot{\tilde{y}}_r(t) + \beta\,\dot{\tilde{y}} + (u + iv)\,\omega_r^2\,\tilde{y}_r(t) = 0\,. \tag{20}$$

Looking for the solution in formula:

$$\tilde{y}_r(t) = \tilde{c}\,e^{i\tilde{\delta}_r t}$$

$$\tilde{\delta}_{r1,2} = i\frac{\beta}{2} \pm \omega_r\sqrt{u - \frac{\beta^2}{4\omega_r^2} + iv} = i\frac{\beta}{2} \pm \omega_r(d_r + if_r) = i\left(\frac{\beta}{2} + \omega_r f_r\right) \pm \omega_r\,d_r.$$

Leaving out the divergent member from the solution in a way shown in [2] the real part of $\tilde{y}_r(t)$

$$y_r(t) = e^{-\nu_r\omega_{rc}t/2}\,[c_1\cos\omega_{rc}t + c_2\sin\omega_{rc}t] \tag{22}$$

Here

$$\omega_{rc} = \omega_r\,d_r$$

$$\nu_r = \frac{\beta + 2\omega_r f_r}{\omega_{rc}}\,.$$

As in (21) the quantity under the root is complex $\tilde{\delta}_{r1,2}$ will be complex too, that is a small damping belongs to every vibration form. If inside friction had been neglected

$$\tilde{\delta}_{r,1} = i\frac{\beta}{2} \pm \omega_r\sqrt{1 - \frac{\beta^2}{4\omega_r^2}}$$

great damping would have occurred at certain vibration forms depending on the value of β (and on that of ω_r). It is evident by a contradiction that internal friction stops this great damping. Its cause is that vibratory motion viz. small damping was supposed when introducing complex stiffness. One method to solve the contradiction is to carry out calculations considering internal friction and not considering it. In the latter case the vibration forms with small damping can be chosen. To obtain the final solution the pertaining characteristics are taken from the model calculated with internal friction while the ones belonging to high damping are taken from the model calculated without internal friction. The damping effects originating in the non-linear elastic character of the material have an effect in this case, too, but their extent can be neglected compared to external damping causing high damping. The suggested method requires the calculations twice and this solution will not be homogeneous either, cannot be written in a closed form (8).

The requirement to describe external and internal damping with one equivalent model proportional with speed is a must. Резников [4] constructs the equivalent matrix with eigenvectors and eigenvalues belonging to equation (18) that makes the use of relation (8) possible for further calculations. From the above mentioned facts it is evident that the method gives satisfactory results if small dampings belong to every vibration form. The method introduced in the following tries to eliminate this deficiency.

(12) is the matrix of dissipation factors only in case of internal friction and can be written in the form:

$$\mathbf{\Gamma} = \mathbf{K}_u^{-1}\mathbf{K}_r$$

and it was produced with equivalence prescribed for excitation

$$\mathbf{q}(t) = \mathbf{q}e^{i\alpha t}$$

looking for the solution in case of excitation in form

$$\tilde{\mathbf{u}}(t) = \tilde{\mathbf{u}}e^{i\alpha t}$$

from equation (12)

$$[(\mathbf{K}_u - \alpha^2 \mathbf{M}) + i\,\mathbf{K}_r]\,\tilde{\mathbf{u}} = \mathbf{q}. \tag{23}$$

If the solution is calculated from equation (18)

$$[(\mathbf{K}_u - \alpha^2 \mathbf{M}) + i(\mathbf{K}_v + \alpha\,\mathbf{C})]\,\tilde{\mathbf{u}} = \mathbf{q}. \tag{24}$$

Comparing relations (23) and (24) we receive:

$$\mathbf{\Gamma} = \mathbf{K}_u^{-1}\mathbf{K}_v + \alpha\,\mathbf{K}_u^{-1}\mathbf{C} = \mathbf{\Gamma}_1 + \mathbf{\Gamma}_2. \tag{25}$$

Accordingly the damping matrix consists of two parts, too:

$$\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2.$$

Here $\mathbf{C}_1$ is the matrix given by (16) in accordance with the internal friction while $\mathbf{C}_2$ is in accordance with (16)

$$\mathbf{C}_2 = \sum_{r=1}^{n} \mathbf{M}\mathbf{v}_r\mathbf{v}_r^* \,\mathbf{C} = \mathbf{C}.$$

We obtained the trivial solution that the damping matrix can be achieved as a sum of the external damping matrix and the equivalent matrix of internal friction while the vibrations result from the differential equation system using a certain number of eigenvectors:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \left\{\sum_{r=1}^{n}\mathbf{M}\mathbf{v}_r\,\mathbf{v}_r^*\left(\mathbf{C} + \frac{1}{\omega_{ru}}\,\mathbf{K}_v\right)\right\}\dot{\mathbf{u}}(t) + \mathbf{K}_u\,\mathbf{u}(t) = 0 \tag{26}$$

Results of numerical experiments

The figure shows a double-support beam of constant cross-section. The beam is divided into five, identical length parts and the bending vibrations of the system with ten degrees of freedom are tested. The external dampers are operated at points marked in the figure. At the given point vibration

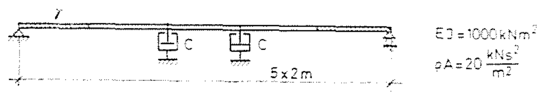

Fig. 1

forms 5 and 10 have nodal points thus at these vibration forms external damping has no effect. To characterize external damping a damping value was taken thus high damping occurred at the frequency point in accordance with the first vibration form.

Table 1 contains angular eigen frequency (ω_{rc}) and damping characteristics (ϱ_r, ν_r) belonging to undamped and damped vibrations.

Table 1

Frequencies and damping characteristics in case of external damping

r	$c = 0$	$c = 30$ kNs/m		
	ω_r [1/s]	ω_{rc} [1/s]	ϱ_r [1/s]	ν_r
1	0.6980	0	6.530 0.0908	—
2	2.796	2.756	1.171	0.8495
3	6.331	5.746	0.5149	0.1792
4	11.42	10.55	3.173	0.6019
5	19.37	19.37	0	0
6	27.95	27.07	2.265	0.1676
7	40.70	40.53	0.9127	0.0450
8	57.81	57.62	0.6817	0.0237
9	77.80	77.69	0.7774	0.0200
10	88.74	88.74	0	0

Also internal friction was calculated. We supposed that an internal friction characterized with $\nu = 0.02$ on the first part ($\nu = v/\pi$ where v is the logarithmic decrement). Table 2 shows that this internal friction has different effects at different vibration forms. Its specific value, distributed to the whole bar, is $\nu_{\text{average}} = 0.02/5 = 0.004$. At eight vibration forms the deviation from the mean is within 20%, at the same time the damping characteristic at the first vibration form is only 25% of the mean. In case of external damping internal friction increases values ϱ_r and ν_r where there was a low damping. In this case, in accordance with high damping, value ϱ_r does not change in a

Table 2

Frequencies and damping characteristics in case of external and internal damping

r	$c = 0$			$c = 30$ kNs/m		
	ω_{rc}	ϱ_r	ν_r	ω_{rc}	ϱ_r	ν_r
1	0.6980	0.00034	0.0010	0	6.530 0.0908	—
2	2.796	0.00429	0.0031	2.752	1.174	0.8531
3	6.331	0.0146	0.0046	5.745	0.5301	0.1845
4	11.42	0.0259	0.0045	10.54	3.197	0.6067
5	19.36	0.0387	0.0040	19.36	0.0386	0.0040
6	27.95	0.0484	0.0035	27.07	2.315	0.1710
7	40.70	0.0690	0.0034	40.53	0.9809	0.0484
8	57.81	0.1427	0.0049	57.62	0.8256	0.0287
9	77.80	0.2735	0.0070	77.69	1.052	0.0279
10	88.73	0.1772	0.0040	88.73	0.1770	0.0040

demonstrable way. Where there was no damping effect from external damping (vibration forms 5 and 10) damping characteristics are in accordance with those originating from internal friction.

The calculations were repeated for the case when only some eigenvectors were written in expression (26). The damping characteristics belonging to these vibration forms were obtained with accuracy in accordance with the eigenvector calculation that justifies the facts mentioned for the solution of vibration calculation by eigenvalue calculation.

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