GENERALIZED CONDITIONAL JOINTS AS SUBDIFFERENTIAL CONSTITUTIVE MODELS

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Summary

The element of a body whose stresses or strains or their combinations are governed by prescribed conditions are termed conditional joints. During a loading process new contacts develop (locking of gaps) or existing connections become ineffective (plastification) causing physical nonlinearity of the solid. This ideal elastic-plastic-locking behaviour of materials can be described by subdifferential constitutive law and referring continuously non-differentiable strain and complementary energy functionals.

Using the new terminology of subdifferentiation there are possibilities to discuss more generally the constitutive laws of non-differentiable but convex energy functionals of bodies consisting of elastic-plastic, hardening, contacting-locking elements.

Introduction

The ever widening range of materials and structural forms increasingly requires the development of complex mechanical models more exactly, describing the real behaviour of materials and structures, to improve the economy of design and construction of these structures.

The first classic material model assumed the material to behave elastically. The search for economy induced to take plastic material properties into consideration, continuously developed both theoretically and practically since the beginning of the century represented by [1] to [13].

By about the mid-century, first of all in machine construction, but later in building mechanics, analysis of the contact properties of structures has come to the foreground represented by [14] to [22]. By the late 'fifties, [23] has suggested to take the contact character as a material law into consideration by respecting the so-called "locking" behaviour of materials. It has induced the research on the so-called conditional joints by the late 'sixties ([24] to [28]), pointing out that singular points of solids or structures, behaving under either plasticity or contact condition, may be handled as conditional joints, thus, also the contact character may be considered as a material property.

Research on constitutive laws expanded simultaneously with that on the theory of plasticity, feeding on its roots. Pioneering works [29] to [32] have started a surge of investigations ever better founded mathematically ([33] to [45]). By development of computer facilities the numerical treatment of plasticity and contact problems has prospered simultaneously. Among the great many research teams, the Italian school's fundamental works in mathematical programming applications are remarkable ([49] to [51]).

Development of mathematics emitted the clearing of mathematical fundamentals more generally. By the late 'seventies, mathematical formulations of elegance, after French patterns mainly, have led to the possibility of combined handling of elasto-plastic contact (locking) behaviour of materials ([52] to [67]), theoretical and practical confluence of plasticity and contact problems.

This paper is an attempt for the sake of confluence by coordinating conditional joints resulted by mechanical respect, and so-called subdifferential connections due to mathematical approach [67].

The theoretical examination of subdifferential connections and material law relies essentially on fundamental work [60].

The generalized conditional joint

Structural elements or solid points behaving under predefined conditions are called conditional joints. Referring these conditions to forces or stresses, strength-type (static-type) conditional joints, and to displacements or strains geometric-type (kinematic-type) conditional joints can be distinguished. If these phenomena occur at the same connection element or point consecutively then generalized conditional joint is spoken of [25]. For example a behaviour controlled by strength-type condition is attributed to plastification of certain regions of solids; but the contacting-detaching connections, opening-closing cracks or gaps are conceived as conditional joints of geometry-type. As a typical generalized conditional joint the closing crack of a solid, following by plastification can be treated.

Thus, stress or strain discontinuities assigned to the point, in a certain mutual precedence, can be considered as generalized conditional joint.

Behaviour of the generalized conditional joint depends on the loading process, during which the stress/strain relation at the point is governed by the joint's conditions. Considering all the points of the solid as a generalized conditional joint it seems self-intended that the behaviour of the material may be described by the connection conditions.

Let the examined solid be a subspace V of the three-dimensional Euclidean space, with boundary surface S. Let us assume any point of the solid as a generalized conditional joint. Mechanical state of the solid is described by stress and strain fields

$$\sigma_{ij}(x_i) \in R^6 \qquad x_i \in V$$

$$\varepsilon_{ij}(x_i) \in R^6 \qquad x_i \in V$$

of the six-dimensional vector space interpreted in geometry space V. At the generalized conditional connection point, stresses and strains are limited by generalized activization condition [26],

$$g(x_i) = \{F_k(x_i),\ f_l(x_i),\ k = 1, 2, \ldots, m;\ l = 1,\ 2, \ldots, n\}\ x_i \in V$$

where m and n are the number of strength- and geometry-type conditions specified for the same point x_i

$$F_k(x_i) = F(\sigma_{ij}(x_i),\ \alpha^k_{ij}(x_i)) \leq 0, \qquad \sigma_{ij} \in R^6,$$

and

$$f_l(x_i) = f(\varepsilon_{ij}(x_i),\ \beta^l_{ij}(x_i)) \leq 0, \qquad \varepsilon_{ij} \in R^6$$

respectively.

Condition F_k corresponds to the well-known yield condition of the theory of plasticity, thus, F_k is the yield function; condition f_l regulates the locking of connections, thus, advisably, f_l is the so-called locking function [28]. Stress and strain-type constants α_{ij} and β_{ij} in conditions F_k and f_l define the convex sets interpreted in the six-dimensional Euclidean space:

$$K_l = \{\varepsilon_{ij} \mid f_l \leq 0\} \quad \varepsilon_{ij} \in R^6,$$

and

$$K^c_k = \{\sigma_{ij} \mid F_k \leq 0\} \quad \sigma_{ij} \in R^6$$

respectively. Illustrating all the conditions $g = 0$ ($F_k = 0$, $k = 1, 2, \ldots, m$; $f_l = 0$, $l = 1, 2, \ldots, n$) in the six-dimensional coaxial coordinate system σ_{ij}, ε_{ij} leads to a convex hypersurface set of $m + n$ elements corresponding to the number of conditions prescribed for the same point x_i, enveloping convex sets K^c_k and K_l, namely

$$\text{front } K_l = \{\varepsilon_{ij} \mid f_l = 0\}, \quad \varepsilon_{ij} \in R^6,$$

and

$$\text{front } K^c_k = \{\sigma_{ij} \mid F_k = 0\}, \quad \sigma_{ij} \in R^6$$

respectively.

Every element of this hypersurfaces set includes the origin, corresponding to the unloaded state of the conditional joint. Precedence of conditions specified for the same joint namely, the mutual dependence of conditions is illustrated by the relative position of hypersurfaces.

Figure 1 presents a section of six-dimensional hypersurface set $g = 0$ in a simplified form for cases $m = 2$ and $n = 1$, that is, when a geometry condition is surrounded by two strength-type ones. During the loading process, the behaviour of the joint controlled by the consecutive conditions may be observed.

In course of activization of strength meaning ($F_k = 0$); and of geometry meaning ($f_l = 0$) of joints strain and stress increments $d\varepsilon^a_{ij}$, and $d\sigma^a_{ij}$ arise, respectively, in conformity with the normality law

$$d\varepsilon^a_{ij} \in d\Lambda_k \cdot \vartheta F_k(\sigma_{ij})$$

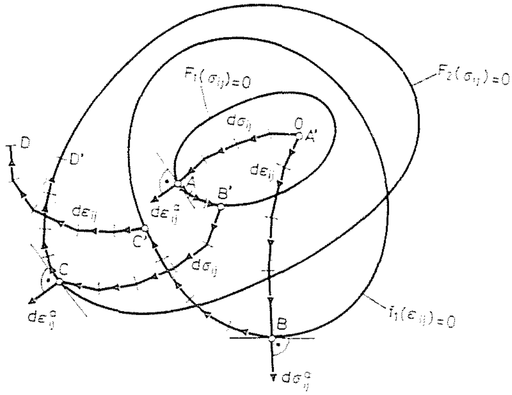

Fig. 1

and

$$d\sigma^a_{ij} \in d\lambda_l \cdot \vartheta f_l(\varepsilon_{ij}), \tag{1}$$

where coefficients $d\Lambda_k \geq 0$ and $d\lambda_l \geq 0$ are the non-negative, multiplier velocities of activization state characteristic increments $d\varepsilon^a_{ij}$ and $d\sigma^a_{ij}$, respectively. These are characterized, in the inactive state of connection

$$\text{for } F_k < 0 \quad \text{by} \quad \Lambda_k = 0, \text{ or,}$$
$$\text{for } f_l < 0 \quad \text{by} \quad \lambda_l = 0;$$

in the active state of the connection

$$\text{for } F_k = 0 \text{ and } dF_k = 0, \text{ by } \Lambda_k \geq 0, \text{ or,}$$
$$\text{for } f_l = 0 \text{ and } df_l = 0, \text{ by } \lambda_l \geq 0;$$

and in the unloading (after active inactive again) state of connection

$$\text{for } F_k = 0 \text{ and } dF_k < 0, \text{ by } \Lambda_k = 0, \text{ or,}$$
$$\text{for } f_l = 0 \text{ and } df_l < 0, \text{ by } \lambda_l = 0.$$

The symbols $\vartheta F_k(\sigma_{ij})$ and $\vartheta f_l(\varepsilon_{ij})$ in (1) are the sets of so-called gradient tensors, that is: where F^k_{ij} and f^l_{ij} are elements of a normal cone constituted by outer normal vectors at points $\sigma_{ij} \in$ front K^c_k and $\varepsilon_{ij} \in$ front K_l of six-dimensional convex hypersurfaces $F_k = 0$ and $f_l = 0$, respectively. If functionals F_k and f_l are differentiable at points σ_{ij} and ε_{ij}, resp. then the normal cones contain a single element F^k_{ij} and f^l_{ij}, resp.; if they cannot be differentiated but subdifferentiated, then the normal cones consists of sets of several elements. For normal cones containing more than a single element, the extension of the Koiter's generalized yield law [31, 32] for the case of generalized activization law is spoken of. More exactly: a vector $d\varepsilon^a_{ij}$ or $d\sigma^a_{ij}$ belonging to a singular point $\sigma_{ij} \in$ front K_k^c or $\varepsilon_{ij} \in$ front K_l lies among, or is coincident with the normal vectors belonging to the regular points near the concerned point.

There upon the activization law can be formulated, namely: activization state characteristic increments can only arise where the activization function has a value of zero, that is, the activization function is potential function of activization state characteristic increments. Furthermore, functionals F_k and f_l are called the superpotentials of the connection, and the generalized conditional joints are called subdifferential connections [60]. The subdifferential connection will be detailed in the next chapter.

In the case of generalized conditional joint, the orthogonality law prevails, namely:

$$d\varepsilon_{ij}^a \cdot d\sigma_{ij} = 0, \quad \sigma_{ij} \in K_k^c, \tag{4}$$

or

$$d\sigma_{ij}^a \cdot d\varepsilon_{ij} = 0, \quad \varepsilon_{ij} \in K_l,$$

that is, if e.g. $\sigma_{ij} \in$ front K_k^c, namely in the active state of the connection, $F_k(\sigma_{ij}) = 0$ and $dF_k(\sigma_{ij}) = 0$, then vector $d\varepsilon_{ij}^a \geq 0$ is element of the normal cone, but $d\sigma_{ij}$ of the tangential cone, hence $d\varepsilon_{ij}^a \cdot d\sigma_{ij} = 0$. With unloading of the connection, if $F_k(\sigma_{ij}) = 0$ and $dF_k(\sigma_{ij}) < 0$ then $d\varepsilon_{ij}^a = 0$; and in the inactive state of the connection, of $F_k(\sigma_{ij}) < 0$ and $dF_k(\sigma_{ij}) > 0$, then $d\varepsilon_{ij}^a = 0$, as well. Thus, relationships (4) are equally valid in the inactive, active, and unloading state of the connection.

The subdifferential connection

Let U denote the six-dimensional linear space constituted by generalized displacement vectors of a mechanical system (the solid) interpreted in a three-dimensional Euclidean space, and F its six-dimensional linear space constituted by generalized force vectors. Be U and F dual spaces, $u \in U$ and $f \in F$ a dual element pair.

Transformations $A : X \to F \quad X \subset U$ or $B : Y \to U \quad Y \subset F$ are termed connective operators of the mechanical system [60], where

$$f \in A(u) \subset F \qquad \forall u \in X \subset U,$$

or

$$u \in B(f) \subset U \qquad \forall f \in Y \subset F.$$

Sets

$$D_A = \{f \mid f \in Y, \quad B(f) \neq \theta\}$$

and

$$D_B = \{u \mid u \in X, \quad A(u) \neq \theta\}$$

are termed domains of connective operators A and B, while sets

$$\{A(u)\} \quad \text{and} \quad \{B(f)\}$$

are the ranges of the connective operators. If for $\forall u \in X$ or $\forall f \in Y$, sets $A(u)$ or $B(f)$ consist at most of a single element each, then the connection is called unique.

Connective operators mean a transformation between spaces of forces and of displacements, so that they lead to the material law, considering the solid points as connection points.

A connection under the validity of

$$A(u) = \vartheta\Phi(u), \qquad \forall u \in U$$

or

$$B(f) = \vartheta\Phi^c(f), \qquad \forall f \in F$$

where Φ and Φ^c denote convex functionals, interpreted in space U, and F, respectively, is called a subdifferential connection. Namely then $f \in A(u)$ and $u \in B(f)$ are elements of the set of subgradients of functionals $\Phi(u)$, and $\Phi^c(f)$, respectively

$$f \in \vartheta\Phi(u) \quad \text{and} \quad u \in \vartheta\Phi^c(f)$$

where dual functionals Φ and Φ^c $(\Phi^c)^c = \Phi)$ are termed superpotential, and conjugated superpotential of the connection, as generalized potentials. Introduction of the concept of superpotential is due to Moreau [54], further generalized by Panagiotopoulos [61] relying on maximal monotonous operators.

It is needless to interpret functionals Φ and Φ^c on the whole space U and F, but it is sufficient to interpret $\Phi = \Phi_0$, and $\Phi^c = \Phi_0^c$ in a convex subsets $X \subset U$, and $Y \subset F$, respectively, and for $u \notin X$ and $f \notin Y$ to stipulat $\Phi = +\infty$, and $\Phi^c = +\infty$. Thereby, by introduction the indicator of convex sets [55], interpretation of Φ and Φ^c can be extended to the whole spaces U and F, respectively:

$$\Phi(u) = \Phi_0(u) + I_X(u) \qquad \forall u \in U,$$

or

$$\Phi^c(f) = \Phi_0^c(f) + I_Y^c(f) \qquad \forall f \in F,$$

where the indicator functionals of the convex sets are

$$I_X(u) = \begin{cases} 0, & \text{for} \quad u \in X, \\ \infty, & \text{for} \quad u \notin X, \end{cases}$$

and

$$I_Y^c(f) = \begin{cases} 0, & \text{for} \quad f \in Y, \\ \infty, & \text{for} \quad f \notin Y, \end{cases}$$

respectively.

Dual functionals Φ and Φ^c are affected by variational inequalities:

$$\Phi(u_1) - \Phi(u) \geq \langle f, u_1 - u \rangle \quad \forall u_1 \in U, \text{ for } u \in U,$$

and

$$\Phi^c(f_1) - \Phi^c(f) \geq \langle u, f_1 - f \rangle \quad \forall f_1 \in F, \text{ for } f \in F.$$

and since f and u are subgradients of Φ and Φ^c at points u and f, equality

$$\Phi(u) + \Phi^c(f) = \langle u, f \rangle$$

is valid in Fenchel transformation [52].

In occurrence of the special cases

$$\Phi_0(u) = 0 \quad \text{or} \quad \Phi_0^c(f) = 0,$$

hence

$$\Phi(u) = I_X(u) \quad \text{or} \quad \Phi^c(f) = I_Y^c(f)$$

for the connection, it is called an ideal unilateral (conditional) connection. Now, f and u are subgradients of the indicators, directly, i.e.:

$$f \in \vartheta I_X(u), \qquad \forall u \in U,$$

and

$$u \in \vartheta I_Y^c(f), \qquad \forall f \in F,$$

where f and u are elements of a normal cone composed of outer normal vectors at points u and f of convex sets X and Y, respectively. For $u \in \text{int}\, X$, and $f \in \text{int}\, Y$, the normal cones contain only the zero element; for $u \in \text{front}\, X$, and $f \in \text{front}\, Y$, normal cones may contain nonzero elements. If functionals Φ and Φ^c are differentiable at points $u \in \text{front}\, X$, and $f \in \text{front}\, Y$ resp., the normal cones contain a single element.

If convex subsets X or Y equal to the whole spaces U or F, i.e., $X \equiv U$, and $Y \equiv F$, that is, $I_X(u) = 0$ and $I_Y^c(f) = 0$ (being meaningless the conditions $u \notin U$ and $f \notin F$) so that

$$\Phi(u) = \Phi_0(u) \text{ and } \Phi^c(f) = \Phi_0^c(f),$$

it is called a bilateral (unconditional) connection where functionals Φ_0 and Φ_0^c are differentiable everywhere.

Subdifferential connection as material model, the subdifferential constitutive law

Generalization of the concept of differentiability of convex functionals, interpretation of subdifferential and of subdifferential connections are seen to permit generalized discussion of conditional connections, hence, of the constitutive law. Namely, also for material models indicated by convex, not everywhere differentiable strain and complementary strain energy density functions $W(\varepsilon_{ij})$ and $W^c(\sigma_{ij})$, resp., relations between stress tensor σ_{ij} and strain tensor ε_{ij}

$$\sigma_{ij} \in \vartheta W(\varepsilon_{ij}) \qquad \varepsilon_{ij} \in R^6$$

and

$$\varepsilon_{ij} \in \vartheta W^c(\sigma_{ij}) \qquad \sigma_{ij} \in R^6$$

remain valid, where $W(\varepsilon_{ij})$ is a convex functional interpreted in the six-dimensional R^6 Euclidean space defined by scalar product $\langle\sigma_{ij}, \varepsilon_{ij}\rangle$, with its conjugated $W^c(\sigma_{ij})$:

$$W^c(\sigma_{ij}) = \sup_{\varepsilon_{ij}\in R^6} \{\sigma_{ij}\,\varepsilon_{ij} - W(\varepsilon_{ij})\} \qquad \forall \varepsilon_{ij} \in R^6,$$

where also $W^c(\sigma_{ij})$ is a convex functional. Functionals W and W^c are termed superpotential and conjugated superpotential, resp., of the constitution law. Derivation as subgradients of convex functionals W and W^c is responsible for the monotonous increasing character of connective operators $\sigma_{ij}(\varepsilon_{ij})$ and $\varepsilon_{ij}(\sigma_{ij})$, that is:

$$W(\varepsilon^1_{ij}) - W(\varepsilon_{ij}) \geq \sigma(\varepsilon^1_{ij} - \varepsilon_{ij}) \qquad \forall \varepsilon^1_{ij} \in R^6, \quad \text{for} \quad \varepsilon_{ij} \in R^6,$$

and

$$W^c(\sigma^1_{ij}) - W^c(\sigma_{ij}) \geq \varepsilon_{ij}(\sigma^1_{ij} - \sigma_{ij}) \qquad \forall \sigma^1_{ij} \in R^6, \quad \text{for} \quad \sigma_{ij} \in R^6,$$

corresponding to Drucker's stability postulate [29, 30]. Functionals W and W^c are defined as:

$$W(\varepsilon_{ij}) = \begin{cases} <\infty & \text{for} \quad \varepsilon_{ij} \in K \subset R^6, \\ \infty & \text{for} \quad \varepsilon_{ij} \notin K, \end{cases}$$

and

$$W^c(\sigma_{ij}) = \begin{cases} <\infty & \text{for} \quad \sigma_{ij} \in K^c \subset R^6, \\ \infty & \text{for} \quad \sigma_{ij} \notin K^c, \end{cases}$$

where functionals W and W^c may be subdifferentiated for $\varepsilon_{ij} \in K$ and $\sigma_{ij} \in K^c$, resp., what means that sets $\vartheta W(\varepsilon_{ij})$ and $\vartheta W^c(\sigma_{ij})$ of their subdifferentials are no empty sets for any fixed ε_{ij} or σ_{ij}, while for $\varepsilon_{ij} \notin K$ or $\sigma_{ij} \notin K^c$, that is, if $W(\varepsilon_{ij}) = \infty$ or $W^c(\sigma_{ij}) = \infty$, then $\vartheta W(\varepsilon_{ij}) = \theta$, and $\vartheta W^c(\sigma_{ij}) = \theta$.

Though, convex sets K and K^c are:

$$K = \{\varepsilon_{ij} \mid f(\varepsilon_{ij}) \leq 0\}, \ \text{int}\, K = \{\varepsilon_{ij} \mid f(\varepsilon_{ij}) < 0\},$$
$$\text{front}\, K = \{\varepsilon_{ij} \mid f(\varepsilon_{ij}) = 0\},$$

and

$$K^c = \{\sigma_{ij} \mid F(\sigma_{ij}) \leq 0\}, \ \text{int}\, K^c = \{\sigma_{ij} \mid F(\sigma_{ij}) < 0\},$$
$$\text{front}\, K^c = \{\sigma_{ij} \mid F(\sigma_{ij}) = 0\},$$

having indicators $I_K(\varepsilon_{ij})$ and $I^c_K(\sigma_{ij})$ leading to energy functionals W and W^c as:

$$W(\varepsilon_{ij}) = W_0(\varepsilon_{ij}) + I_K(\varepsilon_{ij}) \qquad \varepsilon_{ij} \in R^6,$$

and

$$W^c(\sigma_{ij}) = W^c_0(\sigma_{ij}) + I^c_K(\sigma_{ij}) \qquad \sigma_{ij} \in R^6,$$

where $I^c_K(\sigma_{ij})$ is conjugated indicator of $I_K(\varepsilon_{ij})$. In particulars:

$$I_K(\varepsilon_{ij}) = \begin{cases} \lambda \cdot f(\varepsilon_{ij}) = 0, & \text{for} \quad \varepsilon_{ij} \in K \subset R^6, \\ \infty, & \text{for} \quad \varepsilon_{ij} \notin K, \end{cases}$$

and

$$I_K^c(\sigma_{ij}) = \begin{cases} \Lambda \cdot F(\sigma_{ij}) = 0, & \text{for} \quad \sigma_{ij} \in K \subset R^6 \\ \infty, & \text{for} \quad \sigma_{ij} \notin K^c. \end{cases}$$

Namely, if $\varepsilon_{ij} \in \text{int } K$, then $\lambda = 0$, and for $\varepsilon_{ij} \in \text{front } K$, it is $f(\varepsilon_{ij}) = 0$. Similarly, if $\sigma_{ij} \in \text{int } K^c$, then $\Lambda = 0$, and for $\sigma_{ij} \in \text{front } K^c$ it is $F(\sigma_{ij}) = 0$.

Since strength- and geometry-type conditional functions $F(\sigma_{ij}) \leq 0$ and $f(\varepsilon_{ij}) \leq 0$ specified for subdifferential connections are convex, in space R^6 everywhere subdifferentiable functionals the material law becomes:

$$\sigma_{ij} \in \vartheta W_0(\varepsilon_{ij}) + \vartheta I_K(\varepsilon_{ij}),$$

and

$$\varepsilon_{ij} \in \vartheta W_0^c(\sigma_{ij}) + \vartheta I_K^c(\sigma_{ij}).$$

Let us form the set of subgradients of indicator $I_K(\varepsilon_{ij})$ at point ε_{ij}:

$$\vartheta I_K(\varepsilon_{ij}) = \begin{cases} \lambda \cdot \vartheta f(\varepsilon_{ij}) = \lambda \cdot f_{ij}, & \text{for } \varepsilon_{ij} \in K, \\ \theta, & \text{for } \varepsilon_{ij} \notin K, \end{cases}$$

however, for $\varepsilon_{ij} \in \text{int } K$, it is $\vartheta I_K(\varepsilon_{ij}) = \theta$, but for $\varepsilon_{ij} \in \text{front } K$, it is $\vartheta I_K(\varepsilon_{ij}) \neq \theta$.

Similarly

$$\vartheta I_K^c(\sigma_{ij}) = \begin{cases} \Lambda \cdot \vartheta F(\sigma_{ij}) = \Lambda \cdot F_{ij}, & \text{for} \quad \sigma_{ij} \in K^c, \\ \theta, & \text{for} \quad \sigma_{ij} \notin K^c. \end{cases}$$

By geometrial interpretation, sets $\vartheta I_K(\varepsilon_{ij})$ and $\vartheta I_K^c(\sigma_{ij})$ constitute the normal cones of outer normal vectors $\lambda \cdot f_{ij}$ and $\Lambda \cdot F_{ij}$ at points ε_{ij} and σ_{ij} of enveloping surfaces of convex sets K and K^c. For $F(\sigma_{ij}) \leq 0$, and $f(\varepsilon_{ij}) \leq 0$, the cones contain the zero element alone, for $F(\sigma_{ij}) = 0$, and $f(\varepsilon_{ij}) = 0$, in addition to the zero element, also further nonzero elements may be contained. If $I_K(\varepsilon_{ij})$ and $I_K^c(\sigma_{ij})$ are functionals everywhere differentiable above K and K^c, resp., then the normal cone contains a single element.

Thus, the subdifferential constitutive law may be summarized as follows:

$$\sigma_{ij} \in \begin{cases} \vartheta W_0(\varepsilon_{ij}) + \lambda \cdot f_{ij}, & \text{for} \quad \varepsilon_{ij} \in K, \\ \theta, & \text{for} \quad \varepsilon_{ij} \notin K, \end{cases}$$

$$\varepsilon_{ij} \in \begin{cases} \vartheta W_0^c(\sigma_{ij}) + \Lambda \cdot F_{ij}, & \text{for} \quad \sigma_{ij} \in K^c, \\ \theta, & \text{for} \quad \sigma_{ij} \notin K^c. \end{cases}$$

Because of the subdifferentiability of convex functionals, as generalization of the classic Legendre transformation, the Fenchel transformation [52] is valid, namely:

$$W(\varepsilon_{ij}) + W^c(\sigma_{ij}) = \sigma_{ij} \cdot \varepsilon_{ij}, \quad \sigma_{ij}, \varepsilon_{ij} \in R^6,$$

but here W and W^c are functionals not differentiable everywhere!

In the special case of $W_0 = 0$, and $W_0^c = 0$, that is, if:

$$W(\varepsilon_{ij}) = I_K(\varepsilon_{ij}) \quad \text{and} \quad W^c(\sigma_{ij}) = I_K^c(\sigma_{ij}),$$

a perfectly closing, or perfectly plastic material is spoken of.

In this case:

$$\sigma_{ij} \in \vartheta I_K(\varepsilon_{ij}) = \begin{cases} \lambda \cdot f_{ij}, & \text{for} \quad \varepsilon_{ij} \in K, \\ \theta, & \text{for} \quad \varepsilon_{ij} \notin K, \end{cases}$$

and

$$\varepsilon_{ij} \in \vartheta I_K^c(\sigma_{ij}) = \begin{cases} \Lambda \cdot F_{ij}, & \text{for} \quad \sigma_{ij} \in K^c, \\ \theta, & \text{for} \quad \sigma_{ij} \notin K^c. \end{cases}$$

The Fenchel transformation is also then valid, hence, if

$$\varepsilon_{ij} \in K \text{ and } W(\varepsilon_{ij}) = I_K(\varepsilon_{ij}) \quad \text{then}$$

$$W^c(\sigma_{ij}) = \sigma_{ij} \cdot \varepsilon_{ij} - I_K(\varepsilon_{ij}) = \lambda \cdot f_{ij}\ \varepsilon_{ij}, \quad \varepsilon_{ij} \in K.$$

Namely then

$$I_K(\varepsilon_{ij}) = \begin{cases} 0, & \text{for} \quad \varepsilon_{ij} \in K, \\ \infty, & \text{for} \quad \varepsilon_{ij} \notin K, \end{cases}$$

and

$$\sigma_{ij} \in \vartheta I_K(\varepsilon_{ij}) = \begin{cases} \lambda \cdot f_{ij}, & \text{for} \quad \varepsilon_{ij} \in K, \\ \theta, & \text{for} \quad \varepsilon_{ij} \notin K. \end{cases}$$

Similarly, if

$$\sigma_{ij} \in K^c \text{ and } W^c(\sigma_{ij}) = I_K^c(\sigma_{ij}), \quad \text{then}$$

$$W(\varepsilon_{ij}) = \varepsilon_{ij}\,\sigma_{ij} - I_k^c(\sigma_{ij}) = \Lambda \cdot F_{ij} \cdot \sigma_{ij}, \quad \sigma_{ij} \in \mathbb{K}^c,$$

namely then:

$$I_K^c(\sigma_{ij}) = \begin{cases} 0, & \text{for} \quad \sigma_{ij} \in K^c, \\ \infty, & \text{for} \quad \sigma_{ij} \notin K^c, \end{cases}$$

and

$$\varepsilon_{ij} \in \vartheta I_K^c(\sigma_{ij}) = \begin{cases} \Lambda \cdot F_{ij}, & \text{for} \quad \sigma_{ij} \in K^c, \\ \theta, & \text{for} \quad \sigma_{ij} \notin K^c. \end{cases}$$

Thus, in an ideal unilaterial connection:

$$\left.\begin{aligned} W(\varepsilon_{ij}) &= I_K(\varepsilon_{ij}) = \lambda \cdot f(\varepsilon_{ij}) = 0 \\ \sigma_{ij}(\varepsilon_{ij}) &= \lambda \cdot f_{ij} \\ W^c(\sigma_{ij}) &= \lambda \cdot f_{ij} \cdot \varepsilon_{ij} \end{aligned}\right\} \text{if } \varepsilon_{ij} \in K,$$

and

$$\left.\begin{aligned} W^c(\sigma_{ij}) &= I_K^c(\sigma_{ij}) = \Lambda \cdot F(\sigma_{ij}) = 0 \\ \varepsilon_{ij}(\sigma_{ij}) &= \Lambda \cdot F_{ij} \\ W(\varepsilon_{ij}) &= \Lambda \cdot F_{ij} \cdot \sigma_{ij} \end{aligned}\right\} \text{if } \sigma_{ij} \in K^c.$$

The case of simultaneous $\varepsilon_{ij} \in \text{front } K$, and $\sigma_{ij} \in \text{front } K^c_\cdot$ is impossible, since

$$\text{front } K \cap \text{front } K^c = \theta.$$

hence front K and front K^c are disjunct sets. It would mean that the same point of a solid body cannot get in locking and in plastic state at the same time.

For an ideal bilateral connection, if $I_K(\varepsilon_{ij}) = 0$, $\varepsilon_{ij} \in R^6$ and $I^c_K(\sigma_{ij}) = 0$, $\sigma_{ij} \in R^6$, that is, if $K \equiv R_6$, and $K^c \equiv R^6$, then

$$W(\varepsilon_{ij}) = W_0(\varepsilon_{ij}) \text{ and } W^c(\sigma_{ij}) = W^c_0(\sigma_{ij}),$$

and

$$W^c_0(\sigma_{ij}) = \sigma_{ij}\,\varepsilon_{ij} - W_0(\varepsilon_{ij}),$$

an elastic material is spoken of.

The material model interpreted as subdifferential connection was seen to integrate elastic, locking and plastic properties of the material. In course of the loading process, a solid point may get into elastic, plastic or locking state, or even it may be unloaded, thus, it may behave according to different $\sigma_{ij} - \varepsilon_{ij}$ laws controlled as a subdifferential connection.

To have a closer insight into this generalized material law, let it be applied for the simplest case: uniaxial stress/strain state [66], where the subdifferential material law is characterized by a polygontype stress/strain function. Hence the name if "polygonal constitutive law" for the subdifferential material law. Let us consider such a polygonal material law in Fig. 2, as subdifferential curve of convex energy functionals $W(\varepsilon)$ and $W^c(\sigma)$, namely where:

$$\sigma \in \vartheta W(\varepsilon) \quad \text{and} \quad \varepsilon \in \vartheta W^c(\sigma).$$

As seen in the diagram, for $\sigma \in \text{int } K^c_1$, where

$$\text{front } K^c_1 = \{\sigma \mid \alpha'_1 \leq \alpha \leq \alpha_1\},$$

the point behaves as an ideal bilateral connection: perfectly elastic.

However, for $\sigma \in \text{front } K^c_1$, where

$$\text{front } K^c_1 = \{\sigma \mid \sigma = \alpha_1 \text{ or } \sigma = \alpha'_1\},$$

accordingly, $\varepsilon \in K_2$ (but $\varepsilon \notin K_1 \subset K_2$) where

$$K_2 = \{\varepsilon \mid \beta'_2 \leq \varepsilon \leq \beta_2\}$$

the point behaves as an ideal unilateral connection: perfectly plastic. Furthermore, for $\varepsilon \in \text{front } K_2$ where

$$\text{front } K_2 = \{\varepsilon \mid \varepsilon = \beta_2 \text{ or } \varepsilon = \beta'_2\},$$

accordingly $\sigma \in K^c_3$ (but $\sigma \notin K^c_2 \subset K^c_3$) where

$$K^c_3 = \{\sigma \mid \alpha'_3 \leq \sigma \leq \alpha_3\},$$

the point behaves as an ideal unilateral connection again: perfectly closing.

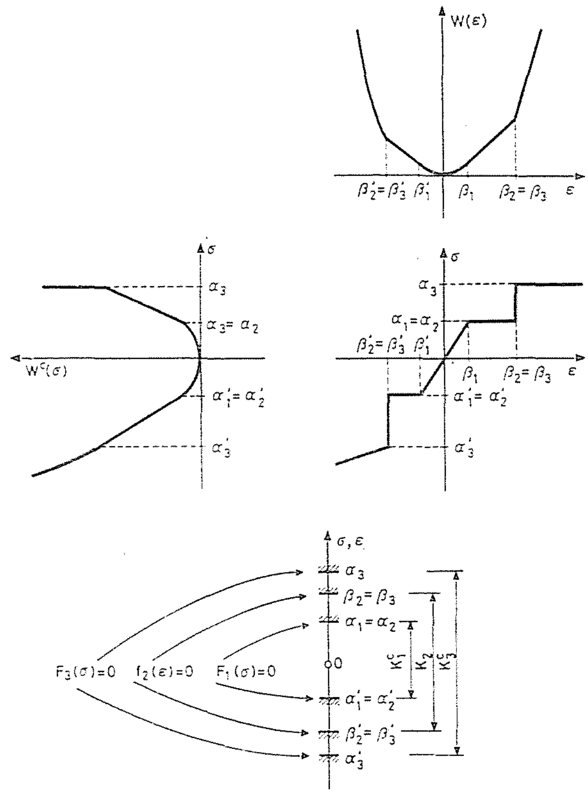

Fig. 2

By way of complementing, hypersurfaces referring to conditional joints envelope surfaces of closed convex sets K_i and K_i^c, have also been represented, but each closed, convex hypersurface only by a single point pair, in conformity with the uniaxial state.

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