NOTES ON THE THEORY OF LARGE DISPLACEMENT WITH SMALL STRAIN

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Summary

In this paper the kinematics of large displacement with small strain is analysed. It has been proved that the partial linearization in strain-displacement relations and equations of motion (or equilibrium) is not correct, because the neglected non-linear terms have the same magnitude as the ones left. It was shown that the "small rotation tensor ω_L" does not rotate but describes the vector-product. The real rotation tensor with both small and large rotation is given with the aid of polar disintegration. The rotation tensor with quadratic approximation is given, too. The boundary-value problem of large displacement with small strains is given.

It has been shown that the state of a body is characterized locally by the kinematical and dynamical relations of continua. But the small strain with large displacement is the result of the "smallness" of one of the global measurement of the analyzed body. Hence, the "smallness" global can be taken into consideration at the numerical approximation of the equation of state: The basis functions depend linearly or quadratically on the coordinate pointing in the direction of "thinness".

The typical problems which can be solved with the aid of the theory of large displacement and small strain are enumerated here.

Symbols

$\mathcal{A}$	Affin strain tensor in the neighbourhood of a point
$\mathcal{A}_L$	Linear approximation of the affin strain tensor; small affin strain tensor
$\mathbf{G}$	Metric tensor in deformed state
$\mathbf{g}$	Metric tensor in non-deformed state
$\mathbf{I}$	Identity tensor
$\mathbf{p}_n$	Prescribed surface traction
$\mathbf{Q}$	Orthogonal tensor
$\mathbf{R}$	Position vector from the origin 0 in deformed state
$\mathbf{r}$	Position vector from the origin 0 in non-deformed state
$\mathbf{u}$, u_i	Displacement vector; its components
$\mathbf{v}$, v_i	Prescribed displacement vector; its components
$\boldsymbol{\gamma}$	Measurement tensor of strain
δ_{ij}	Kronecker's delta
$\boldsymbol{\epsilon}$, ε_{ij}	Small strain tensor; its components
$\boldsymbol{\epsilon}_L$	Linear approximation of the small strain tensor, linear strain tensor

Θ	Angle of rotation described by rotation tensor
ϑ	Angle between differencials of position vectors **dr** and **dR**
λ, μ	Lamé moduli
ϱ	Length of radius of curvature
ρ	Mass density
$\boldsymbol{\sigma}, \sigma_{ij}$	Stress tensor; its components
$\boldsymbol{\Omega}$	Rotation tensor in the neighbourhood of a point
$\boldsymbol{\Omega}_L$	Linear approximation of the rotation tensor; small rotation tensor
Ω	Three-dimensional bounded and open domain
ω_L	Traceless part of the small rotation tensor $\boldsymbol{\Omega}_L$
$\boldsymbol{\omega}_L$	Vector collinear with the axis of rotation described by small rotation tensor $\boldsymbol{\Omega}_L$
$\nabla \mathbf{p}$	Gradient tensor of vector **p**
$\nabla \mathbf{R}^*$	Mapping tensor of strain
$\cos(\mathbf{n}, \mathbf{q})$	Components of outward unit normal vector **n** on $\partial\Omega$
mes Ω	Diameter of domain Ω
$\partial\Omega$	Boundary of domain Ω
$\|\| \ \|\|$	Norm
$\langle , \rangle$	Scalar multiplication
$\overline{\otimes}$	Direct multiplication
$\in$	Belong to
*	Notation of transposition

Italic subscripts can be 1, 2, 3. Einstein-convention of the summation is used over the repeated subscripts.

Introduction

The mechanical behaviour of thin-walled bodies has two characteristics. On the one hand, the kinematics of the thin-walled bodies is characterized by large displacement but small strain, on the other hand, the distribution of stresses can be described by linear functions in the direction of the "thinness". Because of the latter, the theory of the thin-walled bodies is regarded as a special numerical solution to three-dimensional problems [4, 6]. In this way, the theory of the thin-walled bodies with large displacement and small strain is regarded as a special numerical solution to the three-dimensional non-linear theory, too.

First, the kinematics of large displacement with small strain is analysed then the boundary-value problem of the theory is written down. In the end, he possible applications are given.

In this paper only isotropic, homogeneous and linearly elastic bodies are analysed.

Survey of literature

The non-linear theory of elasticity was discussed by Novozhilov [9]. Deduction of the classical theory of rods and shells from the three-dimensional problems was made by A. Love [7]. Two- and single-variable problems are regarded as a special approximation method [6], and an exact derivation of the two- and single-variable problems from the three-variable ones was given in a previous paper [4]. The derivation of the boundary-value problem of cables and flexible membranes from three-dimensional problems is given in the paper [5].

Kinematics of large displacement with small strain

Let $\mathbf{r}$ and $\mathbf{R}$ denote the position vector in non-deformed and deformed state, respectively, and $\mathbf{u} = \mathbf{R} - \mathbf{r}$ the displacement vector. Let $\mathbf{g}$ and $\mathbf{G}$ denote the metric tensor in non-deformed and deformed state, respectively. The tensor $\gamma = \mathbf{G} - \mathbf{g}$ is called the measurement tensor of strain. The relation between differencials of position vectors in non-deformed and deformed states of one and the same point of the body $d\mathbf{R} = \nabla\mathbf{R}^* \, d\mathbf{r}$ holds [8]; where the tensor $\nabla\mathbf{R}^*$ is called mapping tensor of strain. Using relations

$$\mathbf{R} = \mathbf{r} + \mathbf{u}(\mathbf{r}) \tag{1}$$

hence

$$\nabla\mathbf{R}^* = \mathbf{I} + \nabla\mathbf{u}^* \tag{2}$$

and

$$\gamma = \nabla\mathbf{u} + \nabla\mathbf{u}^* + \nabla\mathbf{u}\,\nabla\mathbf{u}^*. \tag{3}$$

If the measurement tensor of strain is small, the relation

$$||\gamma|| \ll ||\mathbf{g}|| \tag{4}$$

holds and the small strain tensor of the body is defined by

$$\epsilon = \frac{1}{2}\gamma \tag{5}$$

(see [8]).

From the condition (4) regarding the measurement tensor of strain, it does not follow that the tensors ϵ_L and ω_L are small, that is the relation

$$||\epsilon_L|| \ll 1 \tag{6}$$

and

$$||\omega_L|| \ll 1 \tag{7}$$

hold. Here

$$\epsilon_L = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^*) \tag{8}$$

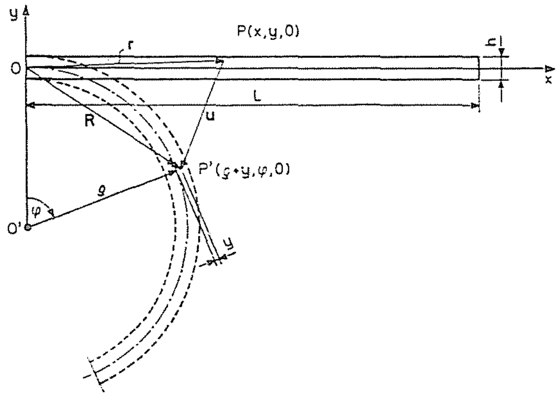

Fig. 1. Large displacement with small strain

is the linear approximation of the small strain tensor, and

$$\omega_L = \frac{1}{2}(\nabla \mathbf{u}^* - \nabla \mathbf{u}) \tag{9}$$

is "the small rotation tensor".

For the verification of this statement let us consider two examples. First, the rigid body rotation will be analysed. Let $\mathbf{Q}$ denote the tensor of rotation. The position vector in a deformed state is

$$\mathbf{R} = \mathbf{Q}\mathbf{r}, \tag{10}$$

the displacement vector is

$$\mathbf{u} = (\mathbf{Q} - \mathbf{I})\mathbf{r}, \tag{11}$$

and the strain tensor is

$$\epsilon = \frac{1}{2}[(\mathbf{Q} - \mathbf{I}) + (\mathbf{Q}^* - \mathbf{I}) + (\mathbf{Q} - \mathbf{I})(\mathbf{Q}^* - \mathbf{I})]. \tag{12}$$

Obviously, the strain tensor is identically equal to zero because $\mathbf{Q}\mathbf{Q}^* = \mathbf{I}$ ($\mathbf{Q}$ is orthogonal). At the same time neither ϵ_L, nor ω_L is zero, and they are not small either. (This example is mentioned by Gol'demblat and Lur'e, too [1, 8].)

The rigid body rotation refers to the whole body, so a thin, long beam bent into a circular arc will be considered here (Fig. 1.). On the basis of geometric considerations

$$\varphi = \frac{x}{\varrho}. \tag{13}$$

The problem seems to be a two-variable one, in this way, the position vectors in non-deformed and deformed state are:

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \varrho - (\varrho + y) \sin \frac{x}{\varrho} \\ -\varrho + (\varrho + y) \cos \frac{x}{\varrho} \end{bmatrix}, \tag{14}$$

the gradient tensor of displacement is

$$\nabla \mathbf{u} = \begin{bmatrix} \frac{\varrho + y}{\varrho} \cos \frac{x}{\varrho} - 1 & \sin \frac{x}{\varrho} \\ -\frac{\varrho + y}{\varrho} \sin \frac{x}{\varrho} & \cos \frac{x}{\varrho} - 1 \end{bmatrix}, \tag{15}$$

and the measurement tensor of strain is

$$\gamma = \left(\frac{2y}{\varrho} + \left(\frac{y}{\varrho} \right)^2 \right) \begin{bmatrix} \cos^2 \frac{x}{\varrho} & -\sin \frac{x}{\varrho} \cos \frac{x}{\varrho} \\ -\sin \frac{x}{\varrho} \cos \frac{x}{\varrho} & \sin^2 \frac{x}{\varrho} \end{bmatrix}. \tag{16}$$

The measurement tensor of strain will be small in case the relation

$$\left| \frac{2y}{\varrho} + \left(\frac{y}{\varrho^2} \right)^2 \right| \ll 1 \qquad y \in \left[-\frac{h}{2} ; \frac{h}{2} \right] \tag{17}$$

holds, e.g. the relation

$$\frac{h}{\varrho} \ll 1 \tag{18}$$

holds, too. In this case, the gradient tensor of displacement $\nabla \mathbf{u}$ and tensors ϵ_L and ω_L will not be small and they will not be negligible as compared with 1. For this reason, the argument of functions cosine and sine should be small, e.g. the relation

$$\frac{x}{\varrho} \ll 1 \qquad x \in [0, \mathrm{L}] \tag{19}$$

should apply which refers to the prevailing of

$$\nabla \mathbf{u} \ll 1. \tag{20}$$

It is obvious from the above mentioned that in the case of large displacement with small strain, the strain tensor is non-linear, and the non-linear parts of it are "needed", so that the strain tensor is small indeed. So, in the identical mathematical transformation

$$\epsilon = \epsilon_L + \frac{1}{2} (\epsilon_L - \omega_L)(\epsilon_L + \omega_L) \tag{21}$$

no partial linearization on the basis of the magnitude of ϵ_L and ω_L is possible in the case of large displacement. The reason for it is that the whole sum is needed for the relation to hold (4). So, in literature the partial linearization on the basis of magnitudes ϵ_L and ω_L is not correct in the case of large displacement with small strain [8, 9]. Naturally, the neglection of ϵ_L in the identical mathematical transformation

$$\nabla \mathbf{u} = \epsilon_L - \omega_L \tag{22}$$

in the equation of motion (or equilibrium) is not correct either [9]. In the case of large displacement with small strain none of the non-linear terms can be neglected, the partial linearization cannot be executed with the aid of transformations (21) and (22).

Comments

1. The partial linearization on the basis of magnitudes ϵ_L and ω_L is founded on the descriptive idea of the "smallness" of "strain tensor ϵ_L" and "rotation tensor ω_L". The first error is that the magnitudes are characterized by the words "small" and "large" and the expression "if enough small, then can be neglected". The second one is that ϵ_L is not a strain tensor in the case of large displacement, and ω_L is not a rotation tensor at all. This will be dealt with later.

Now, let us investigate the partial linearization. Previously it was shown that from

$$||\epsilon|| \ll 1 \tag{23}$$

relations (6) and (7) cannot be originated, i.e. the squares and products of multiplying ϵ_L and ω_L cannot be neglected. On the other hand, if one of the non-linear terms is small and negligible then, it should be negligible as compared with ϵ not with 1. In this case, as will be shown, the sum of non-linear terms left is negligible as compared with ϵ, too.

Let the signs $\lesssim$ denote that a term is smaller in magnitude, and let the sign $\cong$ denote that it is equal in magnitude.

Now, the case will be examined when ϵ_L is "small", i.e. $\epsilon_L \epsilon_L$ is negligible as compared to ϵ, i.e. the following series of relations hold:

$$||\epsilon_L \epsilon_L|| \lesssim ||\epsilon|| \lesssim 1. \tag{24}$$

Since the magnitudes are smaller than 1, so the magnitudes of the products of raising to power and multiplication of the same magnitudes will be lessened by the same magnitudes. So, from (24) it follows

$$||\epsilon_L|| \cong ||\epsilon||, \tag{25}$$

i.e. the well-known condition for "smallness" of ϵ_L holds

$$||\epsilon_L|| \ll 1. \tag{26}$$

Leaving the term $\epsilon_L \epsilon_L$ from ϵ because of (25)

$$||\epsilon_L|| \cong ||\epsilon_L + \frac{1}{2}\epsilon_L \omega_L - \frac{1}{2}\omega_L \epsilon_L - \frac{1}{2}\omega_L \omega_L||. \tag{27}$$

The consequence of (27) is either

$$||\epsilon_L \omega_L - \omega_L \epsilon_L - \omega_L \omega_L|| \lesssim ||\epsilon_L|| \tag{28}$$

and in this case the proof is ready, because the sum of the left non-linear terms is negligible as compared with ϵ_L and ϵ at the same time because of (25), or

$$||\epsilon_L \omega_L - \omega_L \epsilon_L - \omega_L \omega_L|| \cong ||\epsilon_L|| \lesssim 1. \tag{29}$$

In this case, it should be noted that only one of the four non-linear terms is negligible, so each left term should have a magnitude greater than $\epsilon_L \epsilon_L$:

$$||\epsilon_L \epsilon_L|| \lesssim ||\epsilon_L \omega_L||; \ ||\epsilon_L \epsilon_L|| \lesssim ||\omega_L \epsilon_L||; \ ||\epsilon_L \epsilon_L|| \lesssim ||\omega_L \omega_L||. \tag{30a, b, c}$$

As a consequence of (30a, b) the magnitude of ω_L is 1. Indeed if its magnitude is less than 1, then the following series of relations hold:

$$||\epsilon_L \epsilon_L|| \cong ||\omega_L \epsilon_L|| \cong ||\epsilon \omega_L|| \cong ||\omega_L \omega_L|| \tag{31}$$

and so does relation (28). However, if the magnitude of ω_L is 1, then the magnitude of $\omega_L \omega_L$ is also equal to 1, and

$$||\epsilon_L \omega_L - \omega_L \epsilon_L - \omega_L \omega_L|| \cong ||-\omega_L \omega_L|| \cong 1. \tag{32}$$

Relation (32) contradicts relation (29), so the magnitude cannot be equal to 1, or greater than 1. So its magnitude should be less than 1, in this way relations (30), as well as (28) prevail. It is proved that if ϵ_L is small, and its square is negligible, than the sum of the left non-linear terms is negligible, too.

Now, the case is examined when, because of the relation between ϵ_L and ω_L, the expression $\epsilon_L \epsilon_L + \epsilon_L \omega_L - \omega_L \epsilon_L$ is negligible as compared with ϵ:

$$||\epsilon_L \epsilon_L + \epsilon_L \omega_L - \omega_L \epsilon_L|| \lesssim ||\epsilon|| \cong ||\epsilon_L - \frac{1}{2}\omega_L \omega_L|| \lesssim 1. \tag{33}$$

If ϵ_L has such a small value that $\epsilon_L \epsilon_L$ can be neglected as compared with ϵ than due to the above said, the magnitude of ω_L should be smaller than 1, so $\omega_L \omega_L$ is negligible as compared with ϵ_L, i.e. with ϵ. At the same time, the components in the main diagonal of $\omega_L \omega_L$ are square-sums, so they are always positive, but ϵ_L can be both negative and positive. Therefore both ϵ_L and $\omega_L \omega_L$, taken separately, should have a magnitude less than 1. Due to the above said, if the magnitudes of ϵ_L and ω_L are less than 1, then $\omega_L \omega_L$ is negligible as compared with ϵ_L, i.e. with ϵ. It is proved, that if the expression $\epsilon_L \epsilon_L + \epsilon_L \omega_L - \omega_L \epsilon_L$ is negligible, then $\omega_L \omega_L$ is negligible, too.

So, it has been proved that the partial linearization used in literature [8, 9] is not correct because the neglected and left non-linear terms have the same magnitude.

It can be shown, that if the optional part of a non-linear term is regarded as negligible, the sum of the left non-linear terms is negligible, too. The reason for it is that the definition of "smallness" does not refer to ϵ_L or some kind of sum of non-linear terms in ϵ, but to the whole of the measurement tensor of strain γ given by expression (3). So each non-linear term is "needed" for "smallness" to be valid. This reflects the mathematical fact that a curvilinear or surface curvature is indicated in the examined body regarded as inextensible. In the neighbourhood of the one- or two-dimensional domain, relation (4) can apply because the examined neighbourhood is near the inextensible domain, i.e. a relation similar to (18) is in force.

2. The strain of the neighbourhood of a point in the body cannot be described by ϵ_L because the strain tensor — as defined — is the tensor ϵ itself [1, 7, 8, 9]. So the tensor ϵ_L can only be the "whole" strain tensor at the moment, when $\nabla \mathbf{u}$ is small, i.e. the quadratic terms of it are negligible as compared with γ.

3. The tensor ω_L does not describe the rotation of the neighbourhood of a point in the body but it defines the vector product of vector $\boldsymbol{\omega}_L$ $\left(\omega_i = -\frac{1}{2}\varepsilon_{ijk}\omega_{jk}\right)$ where ε_{ijk} is the three-dimensional alternator [10]. So, the vector $\mathbf{q} = \omega_L \mathbf{p}$ is orthogonal to vector $\boldsymbol{\omega}_L$ in the plane and at the same time it is orthogonal to $\mathbf{p}$, too. The sum of tensor ω_L and the identity tensor $\mathbf{I}$ gives the rotation tensor if ω_L is small, and its square is negligible as compared with ω_L, i.e. the tensor

$$\boldsymbol{\Omega}_L = \mathbf{I} + \omega_L \quad (||\omega_L|| \ll 1) \tag{34}$$

is the small rotation tensor [2, 10]. In this case, both (6) and (7) hold i.e. the gradient tensor of displacement is small. Hence the mapping tensors of strain

$$\nabla \mathbf{R}^* = \mathbf{I} + \nabla \mathbf{u}^* \tag{35}$$

form the (commutative) groups under tensor multiplication. This group is called the small mapping group. It can be proved that this group is a Lee-group.

The tensor

$$\mathcal{A}_L = \mathbf{I} + \epsilon_L \tag{36}$$

is the small affin tensor, and the multiplication of $\mathcal{A}_L$ and $\boldsymbol{\Omega}_L$ satisfies the following series of relations

$$\nabla\mathbf{R}^* = \mathcal{A}_L \circ \boldsymbol{\Omega}_L = \boldsymbol{\Omega}_L \circ \mathcal{A}_L = \mathbb{I} + \epsilon_L + \omega_L. \tag{37}$$

The small strain tensors $\mathcal{A}$ and the small rotation tensors $\boldsymbol{\Omega}_L$ form the subgroups of the small mapping group.

4. In case of large displacement, the relation (20) does not hold, hence the mapping tensor of strain $\nabla\mathbf{R}^*$ cannot be regarded as a multiplication of the two tensors linear in $\nabla\mathbf{u}$. But each tensor can be regarded as a product of multiplication of a symmetrical and orthogonal tensor [2, 10]:

$$\nabla\mathbf{R}^* = \boldsymbol{\Omega} \circ \mathcal{A}. \tag{38}$$

Here $\mathcal{A}$ is the symmetrical tensor describing the strain of the neighbourhood of a point in the body and generating the same metric tensor as $\nabla\mathbf{R}^*$. Here $\boldsymbol{\Omega}$ is the orthogonal tensor, describing the rotation of the neighbourhood of the point in the body. Both $\mathcal{A}$ and $\boldsymbol{\Omega}$ can be determined unambiguously [2, 10]. (Of course, $\mathcal{A}$ and $\boldsymbol{\Omega}$ are not commutative.) The rotation of the neighbourhood takes place around the single real principal direction of $\boldsymbol{\Omega}$ with angle Θ, which is determined by

$$\cos\Theta = \frac{1}{2}(\mathbb{T}\mathrm{r}\,\boldsymbol{\Omega} - 1) \tag{39}$$

[2, 10].

The tensors $\mathcal{A}$ and $\boldsymbol{\Omega}$ neglecting the third and higher power of tensor $\nabla\mathbf{u}$ are

$$\mathcal{A} = \mathbb{I} + \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^*) - \frac{1}{8}(\nabla\mathbf{u}\,\nabla\mathbf{u} + \nabla\mathbf{u}^*\,\nabla\mathbf{u}^*) + \frac{3}{8}\nabla\mathbf{u}\,\nabla\mathbf{u}^* - \frac{1}{8}\nabla\mathbf{u}^*\,\nabla\mathbf{u}, \tag{40}$$

$$\boldsymbol{\Omega} = \mathbb{I} + \frac{1}{2}(\nabla\mathbf{u}^* - \nabla\mathbf{u}) + \frac{3}{8}\nabla\mathbf{u}\,\nabla\mathbf{u} - \frac{1}{8}\nabla\mathbf{u}^*\,\nabla\mathbf{u}^* - \frac{1}{8}\nabla\mathbf{u}\,\nabla\mathbf{u}^* - \frac{1}{8}\nabla\mathbf{u}^*\,\nabla\mathbf{u}. \tag{41}$$

As follows from (38) there are two kinds of rotations. One of them is the rotation of the neighbourhood of a point and the other one is the rotation of direction. The first kind of rotation is determined by rotation tensor $\boldsymbol{\Omega}$. The rotation of direction is determined by the mapping tensor of strain $\nabla\mathbf{R}^*$ by means of relation $\mathbf{dR} = \nabla\mathbf{R}^*\,\mathbf{d}r$. The rotation of direction $\mathbf{dr}$ takes place around the vector $\mathbf{p} = \mathbf{dr} \times \mathbf{dR}$ with angle ϑ which can be determined from

$$\cos\vartheta = \frac{\langle \mathbf{dr}, \mathbf{dR} \rangle}{||\mathbf{dr}||\;||\mathbf{dR}||} \tag{42}$$

[2, 10]. The effects of the two rotations i.e. the tensors $\boldsymbol{\Omega}$ and $\nabla\mathbf{R}^*$ are not equal to each other (see Fig. 2). The mapping tensor of strain $\nabla\mathbf{R}^*$ can be

regarded not *only* as a rotation tensor, because in this case it should be orthogonal on the whole examined domain. Hence, $\nabla \mathbf{R}^*$ is constant and it describes *only* the rigid body rotation.

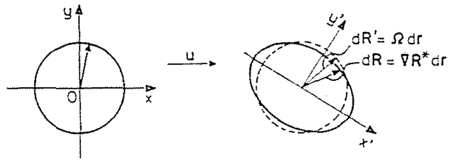

Fig. 2. Rotation of neighbourhood of a point and rotation of a direction

The equations of the mechanical state with large displacement and small strain

In this section indicial notation and Einstein-convention of summation will be used over the repeated subscripts. The Italic subscripts can be 1, 2, 3.

These forms of relation are based on [9] using relation (4).

Strain-displacement relations:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} + \frac{\partial u_s}{\partial x^i}\frac{\partial u_s}{\partial x^j}\right). \tag{43}$$

Equations of motion are

$$\frac{\partial}{\partial x^r}\left[\left(\delta_{pk} + \frac{\partial u_k}{\partial x^p}\right)\sigma_{pr}\right] = \varrho\,\frac{\partial^2 u_k}{\partial t^2}. \tag{44}$$

Hook's law has form:

$$\sigma_{ij} = \lambda \varepsilon_{ss}\,\delta_{ij} + 2\mu\varepsilon_{ij}. \tag{45}$$

Boundary conditions are

$$u_i = v_i, \qquad x^i \in \partial u\,\Omega \tag{46}$$

$$\left(\delta_{pk} + \frac{\partial u_k}{\partial x^p}\right)\sigma_{pq}\,(\cos \mathbf{n}, \mathbf{q}) = \hat{p}_{nk}, \qquad x^i \in \partial p\,\Omega \tag{47}$$

where $\cos(n, q)$ are the components of outward unit vector $\mathbf{n}$ on the surface $\partial\Omega$ and $\hat{p}_{nk}$ are the components of the prescribed surface traction at the same point in the deformed state. The prescribed surface traction $\hat{\mathbf{p}}_n$ may depend on the position of unit normal vector $\mathbf{n}$. In the case of a non-conservative tracing load

$$\hat{p}_{nk} = \left(\delta_{sk} + \frac{\partial u_k}{\partial x^s}\right) p_{ns} \tag{48}$$

where p_{ns} are the components of the prescribed surface traction at the same point of the surface $\partial\Omega$ in non-deformed state.

Initial conditions should be given at $t = t_0$ for the solution of dynamic problems. The initial conditions in the non-linear and linear state are identical, so they are not given here.

Some typical characters of the mechanical state of a thin-walled body with large displacement and small strain

In Section "Kinematics..." it was shown that in theory only one a priori hypothesis is used, namely the measurement tensor of strain should be small, and written in form

$$||\gamma|| \ll ||g||. \tag{49}$$

Above it was proved that on the basis of (49), no partial linearization can be done and both ϵ_L and ω_L are "large". The reason for this is explained in the following. The relations describing the kinematical and dynamical behaviour of a body is based on the differential geometry and the body is characterized by these relations locally — in the neighbourhood of a point. Hence, all further hypotheses of linearization should be written in terms of local magnitudes. The relation (49) itself is written in a local form. But at the same time the relation (49) expresses global "smallness"; namely, the ratio of the global measurement (h) in the body and curvature radius (ϱ) of the inextensible line or surface should be smaller than 1,

$$\left|\frac{h}{\varrho}\right| \ll 1. \tag{50}$$

Hence the "small strain" and "large rotation" should "be drawn into" the theory using global relations. In the solution it can only be done at one point, at the choice of the basis function.

Let the "global smallness" of the body be explained. Now, the body is regarded as a direct product of a one- and a two-dimensional domain: $\Omega = \Omega_1 \otimes \otimes \Omega_2$. If Ω_1 is a curvature, then Ω_2 is a plane (surface) domain and vica versa. If the relation

$$\max \text{mes } \Omega_2 \ll \min \text{mes } \Omega_1 \tag{51}$$

holds, the body is called thin-walled. In case Ω_1 is a curvature, the body is called a rod, but if Ω_2 is a surface, the body is called a shell [6].

If relation (51) applies, the movement of the body can be characterized by large displacement and small strain, because the neighbourhood of domain Ω_1 is small, and the relation

$$\left|\frac{\max \text{mes } \Omega_2}{\varrho}\right| \ll 1 \tag{52}$$

can be satisfied. The relation (52) is the condition of smallness of strain, if domain Ω_1 moves nearly non-deformed in space.

Due to (52), it is sufficient to restrict oneself to a few basis functions applying constant, linear and quadratic functions in the direction of "thinness" [4]. Two problems arise owing to this approximation. The first one is that the stresses determined by displacement through the strain-displacement relations and Hook's law cannot result in equilibrium. The second one is that the boundary conditions, as such cannot be satisfied.

The solution to the first problem is as follows. The stresses required for equilibrium should be approximated independently from displacement. As a consequence, these stresses cause no strain and Hook's law does not hold to them. These stresses are "statically determinate" ones.

The solution to the second problem is that the prescribed surface traction should be numerically approximated. The moment of identical order of surface traction, compared to the point of domain Ω (i.e. the axis of rod or middle surface of shell) should be added to each equation of equilibrium. The moment of the prescribed surface traction will be interpreted as a body force in the equation of equilibrium of force. The body force is the force per unit length for the rod, and a force per unit area for the shell. A "body moment" analogue with the body force cannot occur in the equation of equilibrium of moment [3]. There is one exception to this in the case of a rod: the torsion moment. It can be shown that the shear force distributed over the surface "formally" giving bending moment is no other as pure shear.

The small strain practically means a movement of domain Ω_1 during which it does not change its measurement, i.e. its metric tensor. A large displacement can be produced both in the case of free and rigid support, respectively. In the latter case, if the investigated body is a rod, the distance between the supports should be smaller than the length of the rod. If the investigated body is a shell, than the area of the minimal surface stretched out over the supports should be smaller than the area of middle surface of the shell. The conditions of inextensibility are transformed into anholonomic constraint relations. The deformation of domain Ω_1 can be expressed by the change in magnitude of the force distributed over domain Ω_1. The magnitude should be divided by the length or area of element of domain Ω_1 in a deformed state.

In the case of large displacement and small strain there are two kinds of boundary-value problems.

1. If the outward load is given, the motion should be determined. The inverse problem, if the motion of each point of domain Ω_1 is given, the outward load should be determined.

2. If the motion of (a part of) boundary of domain Ω_1 is given (with length of rod or area of shell), the outward load as well as the motion of the point of domain Ω_1 should be determined.

In the case of the second type of boundary-value problem the anholonomic constraint relations have great importance because the given kinematical boundary conditions can be satisfied by many domains but the solution can be given by those having the same length or area as the analysed rod or shell, respectively. A great many boundary-value problems of cable and flexible membrane shell are of the second type.

The theory of large displacement and small strain has only one a priori hypothesis, namely, the measurement tensor of strain is small. It means that the theory can be used for analysis of the cable and flexible shell, as well as of the thin rod and shell in case when the curvature radius is large by orders of magnitude than the "thinness" of the thin-walled body. Hence the theory can be used for the analysis of global buckling of rods and shells, for the analysis of the global post-buckling behaviour (in case of a rod it is called elastica) and for the analysis of local buckling of rods and shells. In case of local buckling neither the hypothesis of small strain, nor the hypothesis of elastic behaviour (nor both) is satisfied, so the theory is unsuitable for describing local post-buckling behaviour of a thin-walled body.

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