SPECIAL MACRO-ELEMENTS FOR SOLVING PLANE STRESS PROBLEMS

N. N. SHAPOSHNIKOV, I. CSERHALMI and T. NAGY

Department of Civil Engineering Mechanics,
Technical University, H-1521 Budapest

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Summary

A simple method has been presented for producing macro-elements for solving plane stress problems, as well as a method for existence testing, emerging in the finite element method, after (1).

Introduction

Generalization of the application of the finite element method has the strive to develop possibly simple but efficient element types as concomitant. Simple is understood as simplicity of relationships for the elements, possible to be produced in closed form, while efficiency, as adequate accuracy in case of division into rather few elements. Simultaneous fulfilment of both requirements usually fails. Element complexity is fudamentally determined by the complexity of the assumed displacement function (in case of a polynomial, by its number of degrees), and the relation between degrees of freedom of the displacement function and the element (sub-, iso- and superparametric elements). Plane elasticity problems are mostly solved by means of triangular elements, having — besides of known advantages (such as simple relationships for the element, and close approximation of an arbitrary domain) — the disadvantage that, assigning a different triangle network to a given node system of a domain, a different final result is obtained.

Now, a rectangular macro-element composed of triangles will be presented, likely of eliminating these inconvenients, and to meet in limiting case the basic relationships of elasticity.

For the sake of lucidity, the emerging relationships will be presented for rectangular triangle elements of a homogeneous isotropic material, but they can be extended to more general cases.

Basic relationships

Fundamental equations of plane elasticity in matrix form are:
equations of equilibrium:

$$\mathbf{B}\sigma + p = 0 \tag{1}$$

strain-displacement equations:

$$\varepsilon = \mathbf{B}^* z \tag{2}$$

stress/strain equations

$$\sigma = \mathbf{D}\varepsilon \tag{3}$$

where:

$$\mathbf{B} = \begin{bmatrix} \frac{\partial}{\partial x} & & \frac{\partial}{\partial y} \\ & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

$$p = \begin{bmatrix} p_x \\ p_y \end{bmatrix}; \qquad \sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau \end{bmatrix} (\tau_{xy} = \tau_{yx} = \tau)$$

$$\varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma \end{bmatrix} (\gamma_{xy} = \gamma_{yx} = \gamma); \qquad z = \begin{bmatrix} u \\ v \end{bmatrix}$$

in case of a linear elastic material and plane stress problem:

$$D = \frac{1}{E} \begin{bmatrix} 1 & \mu & \\ \mu & 1 & \\ & & 2(1+\mu) \end{bmatrix}.$$

Elimination of stresses and strains leads to the Lamé equation of elasticity. In matrix form:

$$\mathbf{B}\mathbf{D}^{-1}\mathbf{B}^* z + p = 0$$

or, in particular:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{1-\mu}{2}\frac{\partial^2 u}{\partial y^2} + \frac{1+\mu}{2}\frac{\partial^2 v}{\partial x\,\partial y} + \frac{1-\mu^2}{E} p_x = 0 \\ \frac{\partial^2 v}{\partial y^2} + \frac{1-\mu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1+\mu}{2}\frac{\partial^2 u}{\partial x\,\partial y} + \frac{1-\mu^2}{E} p_y = 0. \end{aligned} \tag{5}$$

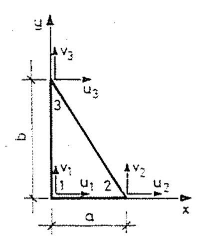

Fig. 1

Stiffness matrix of the rectangular triangle element in Fig. 1 for a linear displacement function becomes (see e.g. p. 185 in (1)):

$$\mathbf{K} = \frac{Eh}{12(1-\mu^2)} \begin{bmatrix} 6\beta+3(1-\mu)\beta^{-1} & 3(1+\mu) & -6\beta & -3(1-\mu) & -3(1-\mu)\beta^{-1} & -6\mu \\ 3(1+\mu) & 6\beta^{-1}+3(1-\mu)\beta & -6\mu & -3(1-\mu)\beta & -3(1-\mu) & -6\beta^{-1} \\ -6\beta & -6\mu & 6\beta & 0 & 0 & 6\mu \\ -3(1-\mu) & -3(1-\mu)\beta & 0 & 3(1-\mu)\beta & 3(1-\mu) & 0 \\ -3(1-\mu)\beta^{-1} & -3(1-\mu) & 0 & 3(1-\mu) & 3(1-\mu)\beta^{-1} & 0 \\ -6\mu & -6\beta^{-1} & 6\mu & 0 & 0 & 6\beta^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & & & & & \\ \vdots & & & & & \\ k_{61} & \dots & \dots & \dots & \dots & k_{66} \end{bmatrix} \qquad (6)$$

where $\beta = \frac{b}{a}$ and h is the thickness of the plate.

Simple macro-elements and stiffness matrices

Selecting the element types

Four possible triangular divisions in the neighbourhood of plate point i are seen in Fig. 2. In the following, cases in Figs 2c and 2d will be ignored since they a priori fail symmetry conditions hence are meaningless for us.

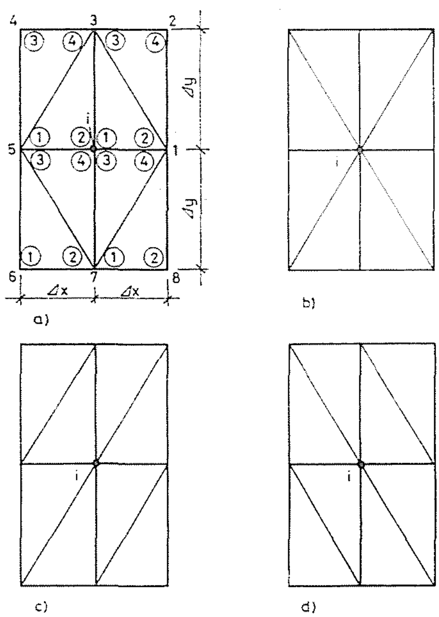

Fig. 2

Discussions will be restricted to divisions in Figs 2a and 2b. They are composed of macro-elements I and II in Fig. 3 with no special transformation, hence stiffness matrices of these two macro-elements have to be established, to be used for writing equilibrium equations at point i. By the time, stiffness matrix (6) will only be applied in its symbolic notation. Stiffness matrix of uncoupled elements in hypermatrix form:

$$\mathbb{K} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} \tag{7}$$

or, in particular:

$$\mathbb{K} = \left[\begin{array}{cccccc:cccccc} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & & & & & & \\ k_{21} & & & & & k_{26} & & & & & & \\ & & & & & & & & & & & \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} & & & & & & \\ \hdashline & & & & & & k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ & & & & & & & & & & & \\ & & & & & & k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{array}\right]. \tag{8}$$

Stiffness matrix of macro-element I

Using Eq. (8), stiffness matrix of element I is obtained as:

$$\mathbf{K}_{\mathrm{I}} = \mathbf{L}_{\mathrm{I}}^{*}\,\mathbb{K}\mathbf{L}_{\mathrm{I}} \tag{9}$$

where — taking "global" and "local" element numbering, and the need to transform displacements of the triangle given by nodes ②, ③, ④ into consideration —

$$\mathbf{L}_{\mathrm{I}} = \left[\begin{array}{cccccccc} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ \hline & & & & & & -1 & \\ & & & & & & & -1 \\ & & & & -1 & & & \\ & & & & & -1 & & \\ & & -1 & & & & & \\ & & & -1 & & & & \end{array}\right] \tag{10}$$

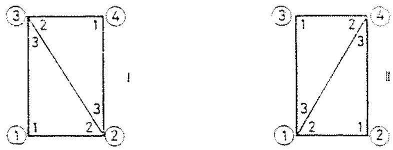

Fig. 3

$$\mathbb{K}_{I} = \left[\begin{array}{cc|c|c|c|c|c|c}
k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & 0 & 0 \\
k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & 0 & 0 \\
\hline
k_{31} & k_{32} & k_{33}+k_{55} & k_{34}+k_{56} & k_{35}+k_{53} & k_{36}+k_{54} & k_{51} & k_{52} \\
k_{41} & k_{42} & k_{43}+k_{65} & k_{44}+k_{66} & k_{45}+k_{63} & k_{46}+k_{54} & k_{61} & k_{62} \\
\hline
k_{51} & k_{52} & k_{53}+k_{35} & k_{54}+k_{36} & k_{55}+k_{33} & k_{56}+k_{34} & k_{31} & k_{32} \\
k_{-1} & k_{62} & k_{63}+k_{45} & k_{64}+k_{46} & k_{65}+k_{43} & k_{66}+k_{44} & k_{41} & k_{42} \\
\hline
0 & 0 & k_{15} & k_{16} & k_{13} & k_{14} & k_{11} & k_{12} \\
0 & 0 & k_{25} & k_{26} & k_{23} & k_{24} & k_{21} & k_{22}
\end{array}\right] \qquad (11)$$

Stiffness matrix of macro-element II

Using Eq. (8), stiffness matrix of element II is obtained as:

$$\mathbb{K}_{II} = \mathbb{L}^{*}_{II}\,\mathbb{K}\mathbb{L}_{II}. \qquad (12)$$

Here, taking "local" and "global" numbering according to Fig. 3b, as well as element displacements relative to Fig. 1 into consideration, matrices $\mathbb{L}_{II}$ and $\mathbb{K}_{II}$ become:

$$\mathbb{L}_{II} = \left[\begin{array}{cccccccc}
 & & -1 & & & & & \\
 & & & 1 & & & & \\
-1 & & & & & & & \\
 & 1 & & & & & & \\
 & & & & & & -1 & \\
 & & & & & & & 1 \\
\hline
 & & & & 1 & & & \\
 & & & & & -1 & & \\
 & & & & & & 1 & \\
 & & & & & & & -1 \\
1 & & & & & & & \\
 & -1 & & & & & &
\end{array}\right] \qquad (13)$$

$$\mathbb{K}_{II} = \left[\begin{array}{c|c|c|c|c|c|c|c}
k_{33}+k_{55} & -k_{34}-k_{56} & k_{31} & -k_{32} & k_{51} & -k_{52} & k_{35}+k_{53} & -k_{36}-k_{54} \\
-k_{43}-k_{65} & k_{44}+k_{66} & -k_{41} & k_{42} & -k_{61} & k_{62} & -k_{45}-k_{63} & k_{46}+k_{64} \\
\hline
k_{13} & -k_{14} & k_{11} & -k_{12} & 0 & 0 & k_{15} & -k_{16} \\
-k_{23} & k_{24} & -k_{21} & k_{22} & 0 & 0 & -k_{25} & k_{26} \\
\hline
k_{15} & -k_{16} & 0 & 0 & k_{11} & -k_{12} & k_{13} & -k_{14} \\
-k_{25} & k_{26} & 0 & 0 & -k_{21} & k_{22} & -k_{23} & k_{24} \\
\hline
k_{53}+k_{35} & -k_{54}-k_{36} & k_{51} & -k_{52} & k_{31} & -k_{32} & k_{55}+k_{33} & -k_{56}-k_{34} \\
-k_{63}-k_{45} & k_{64}+k_{46} & -k_{61} & k_{62} & -k_{41} & k_{42} & -k_{65}-k_{43} & k_{66}+k_{44}
\end{array}\right] \qquad (14)$$

Equilibrium equations at plate point i

Examination of the arrangement in Fig. 2a

In knowledge of stiffness matrices $\mathbf{K}_{\mathrm{I}}$ and $\mathbf{K}_{\mathrm{II}}$, equilibrium equations for point i can be written. Displacement vectors at $i = 1, 2, \ldots, 8$ are denoted as $z_i, z_1, \ldots, z_8$. Their coefficients are composed of the proper 2×2 blocks of matrices $\mathbf{K}_{\mathrm{I}}$ and $\mathbf{K}_{\mathrm{II}}$ in conformity with Fig. 2a, presented in Table 1 like for difference operators.

Table 1

$\mathbf{K}_{\mathrm{II}}^{23} = \mathbf{0}$	$\mathbf{K}_{\mathrm{I}}^{24} + \mathbf{K}_{\mathrm{II}}^{13}$	$\mathbf{K}_{\mathrm{I}}^{14} = \mathbf{0}$
$\mathbf{K}_{\mathrm{I}}^{43} + \mathbf{K}_{\mathrm{II}}^{21}$	$\mathbf{K}_{\mathrm{I}}^{11} + \mathbf{K}_{\mathrm{I}}^{44} + \mathbf{K}_{\mathrm{II}}^{22} + \mathbf{K}_{\mathrm{II}}^{33}$	$\mathbf{K}_{\mathrm{I}}^{12} + \mathbf{K}_{\mathrm{II}}^{34}$
$\mathbf{K}_{\mathrm{I}}^{41} = \mathbf{0}$	$\mathbf{K}_{\mathrm{I}}^{42} + \mathbf{K}_{\mathrm{II}}^{31}$	$\mathbf{K}_{\mathrm{II}}^{32} = \mathbf{0}$

Thereby equilibrium equation at i:

$$(\mathbf{K}_{\mathrm{I}}^{11} + \mathbf{K}_{\mathrm{I}}^{44} + \mathbf{K}_{\mathrm{II}}^{22} + \mathbf{K}_{\mathrm{II}}^{33}) z_i + (\mathbf{K}_{\mathrm{I}}^{12} + \mathbf{K}_{\mathrm{II}}^{34}) z_1 + \mathbf{0} \cdot z_2 + \\ + (\mathbf{K}_{\mathrm{I}}^{24} + \mathbf{K}_{\mathrm{II}}^{13}) z_3 + \mathbf{0} \cdot z_4 + (\mathbf{K}_{\mathrm{I}}^{43} + \mathbf{K}_{\mathrm{II}}^{21}) z_5 + \mathbf{0} \cdot z_6 + \\ + (\mathbf{K}_{\mathrm{I}}^{42} + \mathbf{K}_{\mathrm{II}}^{31}) z_7 + \mathbf{0} \cdot z_8 + p = \mathbf{0}. \tag{16}$$

Displacements of points $1, \ldots, 8$ are written by expanding function z taken as continuous into Taylor series in the neighbourhood of point i, illustrated in Table 2 like for differential operators (indicating approximations of the

Table 2

$z - z_x \Delta x + z_y \Delta y + z_{xx} \frac{\Delta^2 x}{2} - z_{xy} \Delta x \Delta y + z_{yy} \frac{\Delta^2 y}{2} + O(\Delta^3)$	$z + z_y \Delta y + z_{yy} + 0(\Delta^3)$	$z + z_x \Delta x + z_y \Delta y + z_{xx} \frac{\Delta^2 x}{2} + z_{xy} \Delta x \Delta y + z_{yy} \frac{\Delta^2 y}{2} + O(\Delta^3)$
$z - z_x \Delta x + z_{xx} \frac{\Delta^2 x}{2} + O(\Delta^3)$	z	$z + z_x \Delta x + z_{xx} \frac{\Delta^2 x}{2} + O(\Delta^3)$
$z - z_x \Delta x - z_y \Delta y + z_{xx} \frac{\Delta^2 x}{2} + z_{xy} \Delta x \Delta y + z_{yy} \frac{\Delta^2 y}{2} + O(\Delta^3)$	$z_y - zy\Delta y + z_{yy} \frac{\Delta^2 y}{2} + O(\Delta^3$	$z \times z_x \Delta x - z_y \Delta y + z_{xx} \frac{\Delta^2 x}{2} - z_{xy} \Delta x \Delta y + z_{yy} \frac{\Delta^2 y}{2} + O(\Delta^3)$

function at the proper point in each tetragon). Here subscripts of vector z refer to partial differentiation with respect to the given variable, that is:

$$z=\begin{bmatrix}u\\ v\end{bmatrix}\quad z_x=\begin{bmatrix}\dfrac{\partial u}{\partial x}\\[2mm] \dfrac{\partial v}{\partial x}\end{bmatrix}\quad z_y=\begin{bmatrix}\dfrac{\partial u}{\partial y}\\[2mm] \dfrac{\partial v}{\partial y}\end{bmatrix}\quad z_{xx}=\begin{bmatrix}\dfrac{\partial^2 u}{\partial x^2}\\[2mm] \dfrac{\partial^2 v}{\partial x^2}\end{bmatrix}$$

$$z_{yy}=\begin{bmatrix}\dfrac{\partial^2 v}{\partial y^2}\\[2mm] \dfrac{\partial^2 v}{\partial y^2}\end{bmatrix}\quad z_{xy}=z_{yx}=\begin{bmatrix}\dfrac{\partial^2 u}{\partial x\,\partial y}\\[2mm] \dfrac{\partial^2 v}{\partial x\,\partial y}\end{bmatrix}. \tag{17}$$

Detailed writing of block matrices in Table I using Eqs (11), (14) and (6) yields Table 3 (of elements each to be multiplied by Eh 12 $(1-\mu^2)$. Intermultiplying Tables 2 and 3 (multiplying block counterparts) and taking (17)

Table 3

0	0	$-6(1-\mu)\beta^{-1}$	0	0	0
0	0	0	$-12\beta^{-1}$	0	0
-12β	0	$24\beta+12(1-\mu)\beta^{-1}$		-12β	0
0	$-6(1-\mu)\beta$		$24\beta^{-1}+12(1-\mu)\beta$	0	$-6(1-\mu)\beta$
0	0	$-6(1-\mu)\beta^{-1}$	0	0	0
0	0	0	$-12\beta^{-1}$	0	0

into consideration, all coefficients but those of z_{xx} and z_{yy} will be zero, and (16) is transformed to:

$$\frac{Eh}{12(1-\mu)^2}\left\{\begin{bmatrix}-24\beta & \\ & -12(1-\mu)\beta\end{bmatrix}\begin{bmatrix}\dfrac{\partial^2 u}{\partial x^2}\\[2mm] \dfrac{\partial^2 v}{\partial x^2}\end{bmatrix}\frac{\Delta^2 x}{2}+\right.$$

$$\left.+\begin{bmatrix}-12(1-\mu)\beta^{-1} & \\ & -24\beta^{-1}\end{bmatrix}\begin{bmatrix}\dfrac{\partial^2 u}{\partial y^2}\\[2mm] \dfrac{\partial^2 v}{\partial x^2}\end{bmatrix}\frac{\Delta^2 y}{2}\right\}+ \tag{18}$$

$$+\begin{bmatrix}p_x\\ p_y\end{bmatrix}\Delta x\,\Delta y\cdot h+O(\Delta^3)=0.$$

Here, reckoning with $\beta = \frac{\Delta y}{\Delta x}$, after matrix multiplications, we obtain:

$$\frac{Eh}{(1-\mu)^2}\left[-\frac{\partial^2 u}{\partial x^2}\Delta x\,\Delta y - (1-\mu)\frac{\partial^2 u}{\partial y^2}\frac{\Delta x\,\Delta y}{2}\right] + p_x\,\Delta x\,\Delta y\cdot h + O(\Delta^3) = 0$$

$$\frac{Eh}{(1-\mu^2)}\left[-(1-\mu)\frac{\partial^2 v}{\partial x^2}\frac{\Delta x\,\Delta y}{2} - \frac{\partial^2 v}{\partial y^2}\Delta x\,\Delta y\right] + p_y\,\Delta x\,\Delta y\cdot h + O(\Delta^3) = 0. \tag{19}$$

After simplification in limiting case (where $0(\Delta^3) \to 0$):

$$\frac{\partial^2 u}{\partial x^2} + \frac{1-\mu}{2}\frac{\partial^2 u}{\partial y^2} + \frac{1-\mu^2}{E}p_x = 0$$

$$\frac{1-\mu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{1-\mu^2}{E}p_y = 0. \tag{20}$$

Confrontation with (5) shows the mixed derivative to be absent, and the other terms to agree, that is, for such a network (element division), fundamental equations of elasticity are not recovered, thus, in limiting case, the selected macro-element or the triangle element yields no correct solution.

Examination of the arrangement in Fig. 2b

Similar as in the previous chapter equilibrium in the neighbourhood of point i is expressed by:

$$\frac{\partial^2 u}{\partial x^2} + (1+\mu)\frac{\partial^2 v}{\partial x\,\partial y} + \frac{1-\mu}{2}\frac{\partial^2 u}{\partial y^2} + \frac{1-\mu^2}{E}p_x = 0$$

$$\frac{1-\mu}{2}\frac{\partial^2 v}{\partial x^2} + (1+\mu)\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{1-\mu^2}{E}p_y = 0. \tag{21}$$

The suggested macro-element

The above raise the idea to compose the rectangular element from four trangular elements by forming its stiffness matrix as mean of stiffness matrices of elements I and II:

$$\mathbf{K} = \frac{1}{2}(\mathbf{K}_{\mathrm{I}} + \mathbf{K}_{\mathrm{II}}). \tag{22}$$

In these case, even without detailed analyses, the final result is seen to be average from Eqs (20) and (21) agreeing with Eqs (5). Thereby an element meeting relationships of elasticity in boundary transition has been created.

Numerical analyses

Numerical analyses have been made using the suggested element type to be compared with other results found in publications. Experience shows "fitness" of the stiffness matrix (of any element type) to be highly dependent on the Poisson's ratio. For usual μ values ($\mu = 0.2$—0.3) the suggested element type yields practically satisfactory solution with a quite low number of elements. The effect of varying μ needs further, detailed analyses.

Conclusions

The statements above let cocnlude on

— the superiority of the macro-element composed of simple elements over their components;

— the advisability of exacter analyses than convergence examinations usual for finite elements (independence of deformations, exemptness from deformations in rigid-body displacements, etc.).

Analyses above refer to rectangular networks alone, but a network of general tetragonal elements joining four in a node can be aptly transformed to rectangular. Relevant analyses would have a prohibitive extension for a paper, only it is mentioned that relative numerical analyses have led to favourable conclusions.

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Prof. N. N. Shaposhnikov, Moscow Institute of Railway Engineering (MIIT), 15 ul. Obraztsova, Moscow, USSR

Imre Cserhalmi, Dr. Tamás Nagy — H-1521 Budapest