MARKOV'S INEQUALITY IN THE CASE OF RANDOM VARIABLE OF CONCAVE DISTRIBUTION

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Abstract

By elemental methods it is proved in this paper that the following inequalities are existing.

If probability variable ζ is non-negative, has continuous distribution and an expected value M and its distribution function on section $(0, +\infty)$ is concave, then

$$\frac{M}{2\varepsilon} \geqq P(\zeta \geqq \varepsilon), \quad \varepsilon > 0.$$

In the case if probability variable ζ has possible values $x_i > 0$ $(i = 0, 1, 2, \ldots, n)$, its expected value is M, its probability distribution and possible values satisfy conditions

$$P(\zeta = x_{i-1}) \geqq P(\zeta = x_i), \quad x_{i+1} - x_i \geqq x_i - x_{i-1} \ (i = 1, 2, \ldots, n)$$

then

$$\frac{M}{2\, x_{i-1}} \geqq P(\zeta \geqq x_i), \quad (i = 1, 2, \ldots, n).$$

1. Below, one of the different special cases will be discussed when the distribution function $F(x)$ of the non-negative random variable ξ of expected value M presents itself concave over the section $(0, +\infty)$.

The expected value of the non-negative continuously distributed random variable ξ of expected value M is as follows:

$$M = \int_0^\infty x\, f(x)\, dx = \int_0^\varepsilon x\, f(x)\, dx + \int_\varepsilon^\infty (x - \varepsilon)\, f(x)\, dx + \varepsilon \int_\varepsilon^\infty f(x)\, dx$$

$$M = \varepsilon P(\xi \geqq \varepsilon) + \int_0^\varepsilon x\, f(x)\, dx + \int_\varepsilon^\infty (x - \varepsilon)\, f(x)\, dx, \quad \varepsilon > 0, \tag{1}$$

where $f(x)$ is the density function of the random variable ζ.

Markov's inequality is yielded from inequality (1) by the omission of the following function:

$$G(\varepsilon) = \int_0^\varepsilon x\, f(x)\, dx + \int_\varepsilon^\infty (x - \varepsilon)\, f(x)\, dx = M - \varepsilon(1 - P(\varepsilon)) \quad \varepsilon > 0 \tag{2}$$

From the trivial equality

$$M = \varepsilon P(\xi \geqq \varepsilon) + M - \varepsilon(1 - F(\varepsilon)) \tag{3}$$

the following equality is yielded on the basis of concavity of $F(x)$:

$$G(\varepsilon) \geqq M - \frac{M}{F(M)} \frac{F(M)}{M} \varepsilon \left(1 - \frac{F(M)}{M} \varepsilon\right) \geqq M - \frac{M}{4F(M)} \quad 0 < \varepsilon < M$$

and from inequality (3) the following inequality is yielded using the above expression:

$$\frac{M}{4F(M)\,\varepsilon} \geqq P(\xi \geqq \varepsilon), \qquad 0 < \varepsilon < M \tag{4}$$

For further investigations the following equality is used as a starting basis:

$$\int_0^M (M - x) f(x) \; dx = \int_M^\infty (x - M) f(x) \; dx \tag{5}$$

From equation (5) the following is obtained:

$$[(M - x)F(x)]_0^M + \int_0^M F(x) \; dx = \int_M^{2\lambda M} (x - M) f(x) \; dx +$$

$$+ \int_{2\lambda M}^\infty (x - M) f(x) \; dx, \quad \lambda \geqq 1,$$

$$\int_0^M F(x)dx \geqq [(x - M)F(x)]_M^{2\lambda M} \int_M^{2\lambda M} - F(x)\, dx + (2\lambda - 1)\, M(1 - F(2\lambda M)),$$

$$\int_0^{2\lambda M} F(x) \; dx \geqq (2\lambda - 1) M F(2\lambda M) + (2\lambda - 1) M (1 - F(2\lambda M)),$$

$$\int_0^{2\lambda M} F(x)\, dx \geqq (2\lambda - 1)\; M, \qquad \lambda \geqq 1. \tag{6}$$

On the other hand, with the use of concavity the following inequality is yielded:

$$2\lambda M F\;(\lambda M) \geqq \int_0^{2\lambda M} F(x) \; dx \tag{7}$$

This can also be read off from Fig. 1.

From inequalities (6) and (7) it follows that

$$2\lambda M F(\lambda M) \geqq \int_0^{2\lambda M} F(x) \; dx \geqq (2\lambda - 1)\; M,$$

$$2\lambda M F(\lambda M) \geqq (2\lambda - 1) M, \tag{8}$$

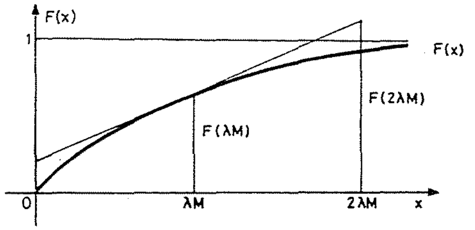

Fig. 1

and hence

$$F(\lambda M) \geqq 1 - \frac{1}{2\lambda}, \tag{9}$$

$$\frac{1}{2\lambda} \geqq 1 - F(\lambda M)$$

$$\frac{M}{2\lambda M} \geqq P(\xi \geqq \lambda M), \qquad \lambda \geqq 1. \tag{10}$$

And by means of $\varepsilon = \lambda M$ the following is yielded:

$$\frac{M}{2\varepsilon} \geqq P(\xi \geqq \varepsilon), \qquad \varepsilon \geqq M. \tag{11}$$

From inequality (9), in the case of $\lambda = 1$, the inequality $F(M) \geqq \frac{1}{2}$ is yielded, and further on, with the use of inequality (4) the following inequality is got:

$$\frac{M}{2\ \varepsilon} \geqq P(\xi \geqq \varepsilon), \qquad 0 < \varepsilon \leqq M \tag{12}$$

Thereupon, it can be said: if ξ is a non-negative random variable of concave distribution function and of expected value M, then in the case of any positive ε, the inequality

$$\frac{M}{2\ \varepsilon} \geqq P(\xi \geqq \varepsilon) \tag{13}$$

applies.

Besides, the following should be noted:

a) If ξ is uniformly distributed over the section $(0, 2M)$, then the minimum of the function $G(\varepsilon)$ is $\frac{1}{2} M$, and its exponential distribution, in the case of random variable ξ, will be represented by the following equality:

$$\frac{M}{e\ \varepsilon} \geqq P(\xi \geqq \varepsilon), \qquad \varepsilon > 0 \tag{14}$$

b) Inequality (6) is valid for the case of any arbitrary random variable ξ. In the case of a discrete random variable taking the values of $x_1, x_2, \ldots, x_n$ and $x_1, x_2, \ldots, x_n, \ldots,$ respectively with a probability of $P(\xi = x_i) = P_i$ $(i = 1, 2, \ldots)$ the above mentioned fact can be seen by imposing a lower limit on the area T under section $(0, x_k)$ of the distribution function, in the following way:

$$T = \sum_{i=1}^{k-1} (x_k - x_i)P_i = x_k \sum_{i=1}^{k-1} P_i - \sum_{i=1}^{k-1} x_i P_i,$$

$$T = x_k \sum_{i=1}^{k-1} P_i + x_k \sum_{i=k}^{\infty} P_i - \sum_{i=1}^{k-1} x_i P_i \geqq x_k - M$$

$$T \geqq x_k - M \tag{15}$$

Inequality (12) is valid for all discrete random variables for which it can be pointed out that

$$x_k F\left(\frac{x_k}{2}\right) \geqq T \tag{16}$$

in the case of an arbitrary $x_k \geqq 2M$. Then, from inequalities (15) and (16) it is yielded that

$$x_k F\left(\frac{x_k}{2}\right) \geqq x_k - M, \tag{17}$$

$$F\left(\frac{x_k}{2}\right) \geqq 1 - \frac{M}{x_k},$$

$$\frac{M}{x_k} \geqq P\left(\xi \geqq \frac{x_k}{2}\right). \tag{18}$$

Thereupon, it seems useful to deal with such random variables that take the values of $x_0, x_1, \ldots x_n$ with a probability of $P(\xi = x_i) = P_i$ $(i = 0, 1, \ldots, n)$ and

$$P_{i-1} \geqq P_i, \; x_{i+1} - x_i \geqq x_i - x_{i-1}, \qquad (i = 1, 2, \ldots, n) \tag{19}$$

Then to the random variables ξ the function $\widetilde{F}(x)$ is assigned, whose diagram is a polygon fitting the points $(x_i; P_0 + p_1 + \ldots + p_i)$. In case, the conditions in (19) are satisfied, this polygon (the diagram of $\widetilde{F}(x)$ is concave), and the following expressions apply for each x:

$$\widetilde{F}(x) \geqq F(x), \qquad \widetilde{T}(x) \geqq T(x) \tag{20}$$

On the other hand, due to the concavity in the case of any x_k

$$x_k \widetilde{F}\left(\frac{x_k}{2}\right) \geqq \widetilde{T}(x_k) \geqq T(x_k). \tag{21}$$

From inequalities (17) and (18) it follows that

$$x_k \tilde{F}\left(\frac{x_k}{2}\right) \geqq x_k - M,$$

$$\tilde{F}\left(\frac{x_k}{2}\right) \geqq 1 - \frac{M}{x_k},$$

$$\frac{M}{x_k} \geqq 1 - \tilde{F}\left(\frac{x_k}{2}\right). \tag{22}$$

From the definition of $\tilde{F}(x)$ it follows that in case $x_{i-1} < \frac{x_k}{2} \leqq x_i$ then $\tilde{F}\left(\frac{x_k}{2}\right) < F(x_i)$ and $x_k > 2x_{i-1}$. With the use of this, from inequality (22) the following inequality is yielded:

$$\frac{M}{2 \cdot x_{i-1}} \geqq P(\xi \geqq x_i) \tag{23}$$

It should be noted that the conditions in (19) are satisfied by the random variable taking, e.g., values $(k - np)^2$ with a probability of $\binom{n}{k} p^k(1-p)^{n-k}$ $(k = 0, 1, 2, \ldots, n)$. This can be seen immediately in the case of $p = \frac{1}{2}$. And in the case of $p \neq \frac{1}{2}$, it is yielded that the probabilities

$$P(\xi = k) = \binom{n}{k} p^k(1-p)^{n-k} \quad (k = 0, 1, 2, \ldots, n)$$

are the increasing functions of k over the section $0 \leqq k \leqq np$, and those are decreasing functions over the section $np \leqq k \leqq n$ and the superposition of the two concave plane-figures is also concave. This involves that on the basis of inequality (23), $\frac{n}{2}$ experiments are sufficient for the accuracy and reliability ensured by the Bernoulli's inequality in the case of n measurements.

2. And now, the problem will be dealt with: what can be asserted if the condition of concavity is omitted?

If the degenerated random variable of the distribution function

$$F(x) = \begin{cases} 0, & \text{if } x \geqq a, \\ 1, & \text{if } x > a \end{cases} \tag{1}$$

is considered, then it can be seen immediately that Markov's inequality cannot generally be satisfied. This can be represented by Fig. 2.

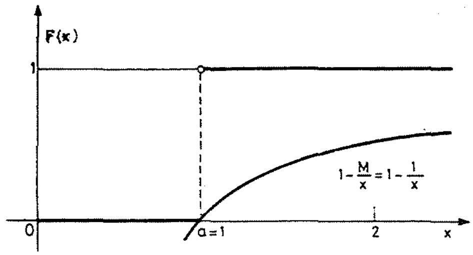

Fig. 2

For further investigations, the equality valid for the case of any arbitrary positive λ:

$$M = \lambda MP(\xi \geqq \lambda M) + \int_0^{\lambda M} x\, f(x)\, dx + \int_{\lambda M}^{\infty} (x - \lambda M) f(x) dx \qquad (2)$$

will be used for any arbitrary non-negative random variable of expected value M.

From this the followings are obtained:

$$M = \lambda MP(\xi \geq \lambda M) + \int_0^{\lambda M} x\, f(x)\, dx + \int_{\lambda M}^{2\lambda M} (x - \lambda M)\, f(x)\, dx +$$
$$+ \int_{2\lambda M}^{\infty} (x - \lambda M) f(x)\, dx,$$

$$M \geq \lambda MP(\xi \geqq \lambda M) + [xF(x)]_0^{\lambda M} - \int_0^{\lambda M} F(x)\, ax + [(x - \lambda M)F(x)]_{\lambda M}^{2\lambda M} -$$
$$- \int_{\lambda M}^{2\lambda M} F(x)\, ax + \lambda M(1 - F(2\lambda\, M)),$$

$$M \geq \lambda MP(\xi \geqq \lambda M) + \lambda MF(\lambda M) + \lambda F(2\lambda M) + \lambda M(1 - F(2\lambda M)) -$$
$$- \int_0^{2\lambda M} - F(x)\, dx,$$

$$M \geq \lambda MP(\xi \geq \lambda M) + \lambda MF(\lambda M) + \lambda M - \int_0^{2\lambda M} F(x)\, dx,$$

$$M \geqq \lambda MP(\xi \geqq \lambda M) + \lambda M(1 - F(\lambda M)) + 2\lambda MF(\lambda M) - \int_0^{2\lambda M} F(x)\, dx$$

$$M \geqq 2\, \lambda MP(\xi \geq \lambda M) + 2\lambda MF(\lambda M) - \int_0^{2\lambda M} F(x)\, ax, \qquad \lambda > 0 \qquad (3)$$

In this way, another proof of point 1 of inequality (13) has been gained. Since it has already been demonstrated that in the case of a concave distribution function

$$2\lambda MF(\lambda M) - \int_0^{2\lambda M} F(x)\,dx \geqq 0, \tag{4}$$

If without the use of the condition of concavity, it can be proved in some way (e.g. in a particular case, with the help of the empirical distribution function with some fixed λ (that inequality (4) is valid), then in this special case, the inequality:

$$\frac{M}{2 \cdot \lambda M} \geqq P(\xi \geqq \lambda M) \tag{5}$$

can also be applied.

Starting from equality (2), an inequality a bit more meaningful than inequality (3) can be yielded:

$$M = \lambda MP(\xi \geqq \lambda M) + \int_0^{\lambda M} x\, f(x)\ ax + \int_{\lambda M}^{4\lambda M} (x - \lambda M)\ f(x)\ dx +$$

$$+ \int_{4\lambda M}^{\infty} (x - \lambda M) f(x)\ dx,$$

$$M \geqq \lambda MP(\xi \geqq \lambda M) + [xF(x)]_0^{\lambda M} - \int_0^{\lambda M} F(x)\,dx +$$

$$+ [(x - \lambda M)\ F(x)]_{\lambda M}^{4\lambda M} - \int_{\lambda M}^{4\lambda M} F(x)\,dx + 3\lambda M(1 - F(4\lambda M)),$$

$$M \geqq \lambda MP(\xi \geqq \lambda M) + \lambda MF(\lambda M) + 3\lambda MF(4\lambda M) +$$

$$+ 3\lambda M(1 - F(4\lambda M)) - \int_0^{4\lambda M} F(x)\ ax,$$

$$M \geqq \lambda MP(\xi \geqq \lambda M) + \lambda MF(\lambda M) + 3\lambda M - \int_0^{4\lambda M} F(x)\ dx,$$

$$M \geqq \lambda MP(\xi \geqq \lambda M) + \lambda M - \lambda MF(\lambda M) + 2\lambda MF(\lambda M) + 2\lambda M - \int_0^{4\lambda M} F(x)\,dx,$$

$$M \geqq 2\lambda MP(\xi \geqq \varepsilon) + 2\lambda MF(\lambda M) + 2\lambda M - \int_0^{4\lambda M} F(x)\ ax. \tag{6}$$

After this, for the inequality (5) to be applied, it should be proved that

$$2\lambda MF(\lambda M) + 2\lambda M - \int^{4\lambda M} F(x)\ dx \geqq 0. \tag{7}$$

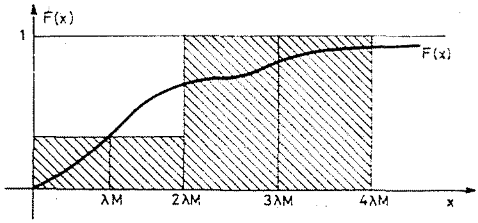

Fig. 3

This problem can be illustrated as in Fig. 3.

At last, let $\lambda(\varepsilon)$ be written as follows:

$$\lambda(\varepsilon) = \frac{\int_0^\varepsilon x f(x)\, dx}{\int_\varepsilon^\infty x f(x)\, dx} = \frac{\varepsilon F(\varepsilon) - \int_0^\varepsilon F(x)\, dx}{M - \varepsilon F(\varepsilon) + \int_0^\varepsilon F(x) dx}, \qquad \varepsilon > 0.$$

Function $\lambda(\varepsilon)$ is a monotonic increasing one, $\lambda(0) = 0$, $\lim_{\varepsilon \to \infty} \lambda(\varepsilon) = +\infty$, and obviously

$$\frac{M}{(\lambda(\varepsilon) + 1)\, \varepsilon} : \geqq P(\xi \geqq \varepsilon).$$

References

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* In Hungarian.