TCHEBICHEV'S INEQUALITY IN THE CASE OF RANDOM VARIABLE OF SPECIAL DISTRIBUTION

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Abstract

In this paper it is proved by elemental methods that the following inequalities exist.

If the expected value of the probability variable ξ is 0, the variance is σ^2 and its density function satisfies the conditions

$$f(-x) = f(x), \quad f(x_1) \geqq f(x_2), \quad 0 \leqq x_1 < x_2 < +\infty$$

then in the case of any positive ε, inequality

$$\frac{\sigma^2}{\frac{9}{4}\varepsilon^2} \geqq P(|\xi| \geqq \varepsilon)$$

is valid.

If the expected value of the discrete probability variable ξ is 0, the square of its scattering is σ^2, then the following inequality applies

$$\frac{\sigma^2}{\frac{9}{4}x_{i-1}^2} \geqq P(|\xi| \geqq |x_i|). \quad (i = 1, 2, \ldots, n)$$

when conditions ensuring the concavity of distribution function and regarding its possible values and probability distribution are satisfied.

1. In the following it will be proved in an elementary way that in case the variance of random variable ξ exists and if it is valid for its density function that

$$f(-x) = f(x), \qquad f(x_1) \geqq f(x_2), \qquad 0 < x_1 < x_2, \tag{1}$$

then for any positive ε the following inequality holds

$$\frac{\sigma^2}{\frac{9}{4}\varepsilon^2} \geqq P(|\xi| \geqq \varepsilon) \tag{2}$$

For the proof of this, the following equality will be used as a starting basis

$$\int_0^\sigma (\sigma^2 - x^2) f(x)\, dx = \int_\sigma^\infty (x^2 - \sigma^2) f(x)\, dx \tag{3}$$

A lower limit is imposed on the integral on the right-hand side of equality (3) in the following way:

$$\int_{\sigma}^{\infty} (x^2 - \sigma^2)\, f(x)\, dx \geqq \int_{\sigma}^{3\lambda\sigma} (x^2 - \sigma^2) f(x)\, dx + \int_{3\lambda\sigma}^{\infty} (9\lambda^2 - 1)\sigma^2 f(x)\, dx, \qquad 3\lambda\sigma \geqq \sigma,$$

$$\int_{\sigma}^{\infty} (x^2 - \sigma^2)\, f(x)\, dx \geqq [(x^2 - \sigma^2)\, F(x)]_{\sigma}^{3\lambda\sigma} - 2 \int_{\sigma}^{3\lambda\sigma} x\, F(x)\, dx +$$

$$+ (9\lambda^2 - 1)\, \sigma^2(1 - F(3\lambda\sigma))$$

$$\int_{\sigma}^{\infty} (x^2 - \sigma^2) f(x)\, dx \geqq (9\lambda^2 - 1)\sigma^2 F(3\lambda\sigma) + (9\lambda^2 - 1)\sigma^2(1 - F(3\lambda\sigma)) - \int_{\sigma}^{3\lambda\sigma} x F(x)\, dx,$$

$$\int_{\sigma}^{\infty} (x^2 - \sigma^2)\, f(x)\, dx \geqq (9\lambda^2 - 1)\sigma^2 - 2 \int_{\sigma}^{3\lambda\sigma} x\; F(x)\, dx, \qquad 3\lambda\delta \geqq \sigma. \tag{4}$$

Instead of the expression on the left-hand side of inequality (4), the expression on the right-hand side of it is substituted into equality (3), so the following inequality is obtained

$$\int_{0}^{\sigma} (\sigma^2 - x^2)\, f(x)\, dx \geqq (9\lambda^2 - 1)\sigma^2 - 2 \int_{\sigma}^{3\lambda\sigma} x\; F(x)\, dx,$$

$$[(\sigma^2 - x^2)\; F(x)]_0^{\sigma} + 2 \int_{0}^{\sigma} x\, F(x)\, dx \geqq (9\lambda^2 - 1)\sigma^2 - 2 \int_{\sigma}^{3\lambda\sigma} x\; F(x)\, dx,$$

$$2 \int_{0}^{\sigma} x\, F(x)\, dx - \frac{\sigma^2}{2} \geqq (9\lambda^2 - 1)\, \sigma^2 - 2 \int_{\sigma}^{3\lambda\sigma} x\; F(x)\, dx,$$

$$2 \int_{0}^{3\lambda\sigma} x F(x)\, dx \geqq \left(9\lambda^2 - \frac{1}{2}\right) \sigma^2, \quad 3\lambda\sigma \geqq \sigma. \tag{5}$$

After this, an upper limit is imposed onto the integral on the left-hand side of inequality (5) with respect to the fact that the distribution function $F(x)$ of the random variable is concave over the section $(0, \infty)$ (Fig. 1)

The measuring number of the area under the section $[0, \lambda\sigma]$ of the distribution function $F(x)$ is smaller than that of the area under section $[0, \lambda\sigma]$ belonging to the secant $[0, \lambda\sigma]$ fitting the points P_1 with abscissa $\lambda\sigma$ and P_2 with abscissa $2\lambda\sigma$. And the measuring number of the area under section $[\lambda\sigma, 3\lambda\sigma]$ is smaller than that of the area under section $\overline{P_1', P_3}$ of the tangent line belonging to point P_2; the latter measuring number is identical with that of

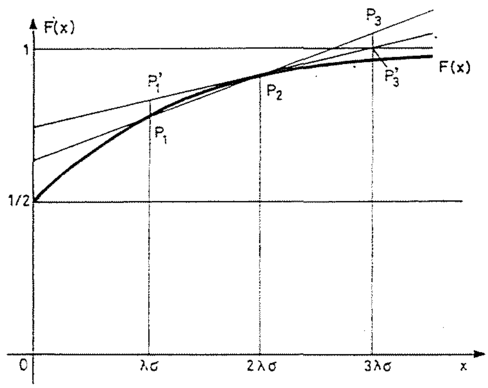

Fig. 1

the area under section $\overline{P_1 P_3}$ of the secant fitting the points P_1, P_2 due to the congruency of triangles $P_1\ P_2\ P_1'$ and $P_2\ P_3\ P_3'$. It is also obvious that the first coordinate of the centre of gravity of the plane-figure under section $\overline{P_1'\ P_3}$ is smaller than that of the centre of gravity of the plane-figure under section $\overline{P_1\ P_3'}$. With regard to the foregoing, the following can be expected:

$$2\int_0^{3\lambda\sigma} x\,F(x)\,dx \leqq \int_0^{3\lambda\sigma} x\left[F(2\lambda\sigma)+\frac{F(2\lambda\sigma)-F(\lambda\sigma)}{\lambda\sigma}(x-2\lambda\sigma)\right]dx, \qquad (6)$$

And now, the integral on the right-hand side of inequality (6) is calculated:

$$2\int^{3\lambda\sigma}\left[x\,F(2\lambda\sigma)-2\,\frac{F(2\lambda\sigma)-F(\lambda\sigma)}{\lambda\sigma}\,\lambda\sigma+\frac{F(2\lambda\sigma)-F(\lambda\sigma)}{\lambda\sigma}\,x^2\right]dx=$$

$$=2\int_0^{3\lambda\sigma}\left[x\,(2F(\lambda\sigma)-F\,(2\lambda\sigma))+\frac{F(2\lambda\sigma)-F(\lambda\sigma)}{\lambda\sigma}\,x^2\right]dx=$$

$$=2\left[(2F(\lambda\sigma)-F(2\lambda\sigma))\frac{x^2}{2}+\frac{F(2\lambda\sigma)-F(\lambda\sigma)}{\lambda\sigma}\,\frac{x^3}{3}\right]_0^{3\lambda\sigma}=$$

$$=2\lambda^2\sigma^2\frac{9}{2}\,F(2\lambda\sigma)=9\lambda^2\sigma^2\,F(2\lambda\sigma)\,.$$

Instead of the definite integral, this is substituted into inequality (5) and the following inequality is obtained:

$$9\lambda^2\sigma^2F(2\lambda\sigma)\geqq 2\int_0^{3\lambda\sigma} x\ \ F(x)\,dx \qquad (7)$$

Thereupon, from inequalities (4) and (6) the following inequality is obtained:

$$9\lambda^2\sigma^2 F(2\lambda\sigma) \geqq 2\int_0^{3\lambda\sigma} x\, F(x)\, dx \geqq 9\lambda^2\sigma^2 - \frac{1}{2}\sigma^2$$

$$9\lambda^2\sigma^2\, F(2\lambda\sigma) \geqq 9\lambda^2\sigma^2 - \frac{1}{2}\sigma^2 \tag{8}$$

Hence, the followings are yielded:

$$\frac{\sigma^2}{2} \geqq 9\lambda^2\sigma^2(1 - F(2\lambda\sigma))$$

$$\frac{\sigma^2}{2} \geqq \frac{9}{4}(2\lambda\sigma)^2\,(1 - F(2\lambda\sigma))$$

$$\frac{\sigma^2}{2} \geqq \frac{9}{8}(2\lambda\sigma)^2 \cdot 2(1 - F(2\lambda\sigma))$$

$$\frac{\sigma^2}{\frac{9}{4}(2\lambda\sigma)^2} \geqq P(|\xi| \geqq 2\lambda\sigma)) \quad \lambda > \frac{1}{3}$$

At last, with the use of $2\lambda\sigma = \varepsilon$, the following inequality is obtained:

$$\frac{\sigma^2}{\frac{9}{4}\varepsilon^2} \geqq P(|\xi| \geqq \varepsilon), \quad \varepsilon > \frac{2}{3}\sigma \tag{10}$$

And now, if $\frac{1}{\sqrt{3}}$ is substituted for λ in inequality (9), then the followings are yielded for $F\left(\frac{2}{\sqrt{3}}\sigma\right)$:

$$\frac{1}{\frac{9}{4}\cdot\frac{4}{3}} \geqq 2\left(1 - F\frac{2}{\sqrt{3}}\sigma\right)\Big)$$

$$\frac{1}{6} \geqq 1 - F\left(\frac{2}{\sqrt{3}}\sigma\right)$$

$$F\left(\frac{2}{\sqrt{3}}\sigma\right) \geqq \frac{5}{6} \tag{11}$$

After this, let the trivial equality

$$\sigma^2 = P(|\xi| \geqq \varepsilon) + \sigma^2 - 2\varepsilon^2(1 - F(\varepsilon)) \tag{12}$$

be considered, and in this way, let a lower limit imposed on function

$$G(\varepsilon) = \sigma^2 - 2\varepsilon^2(1 - F(\varepsilon)), \quad 0 < \varepsilon \leqq \frac{2}{\sqrt{3}}\sigma, \tag{13}$$

be defined.

Due to the concavity of $F(x)$:

$$F(\varepsilon) \geqq \frac{1}{2} + \frac{1}{2\sqrt{3\sigma}}\varepsilon = \tilde{F}(\varepsilon), \quad 0 < \varepsilon \leqq \frac{2}{\sqrt{3}}\sigma \tag{14}$$

This can also be read off in Fig. 2.

With this in view:

$$\sigma^2\, 2\varepsilon^2(1 - F(\varepsilon)) \geqq \sigma^2 - 2\varepsilon^2\left(1 - \frac{1}{2} - \frac{1}{2\sqrt{3}\,\sigma}\varepsilon\right) = H(\varepsilon) \quad 0 < \varepsilon < \frac{2}{\sqrt{3}}\sigma.$$

Now, the minimum of the function

$$H(\varepsilon) = \sigma^2 - \varepsilon^2 + \frac{1}{\sqrt{3}\,\sigma}\varepsilon^3 \qquad 0 < \varepsilon < \frac{2}{\sqrt{3}}\sigma \tag{15}$$

will be defined. This is the lower limit on $G(\varepsilon)$:

$$H'(\varepsilon) = -2\varepsilon + \frac{\sqrt{3}}{\sigma}\varepsilon^2 = -2\varepsilon\left(1 - \frac{\sqrt{3}}{2\,\sigma}\varepsilon\right),$$

and hence, it is yielded that $H'(\varepsilon) < 0$ is valid over the whole domain, so

$$\min H(\varepsilon) = H\left(\frac{2}{\sqrt{3}}\sigma\right) = \sigma^2 - \frac{4}{3}\sigma^2 + \frac{1}{\sqrt{3}\,\sigma}\frac{8}{3\sqrt{3}} \cdot \sigma^3,$$

$$\min H(\varepsilon) = \left(1 - \frac{4}{3} + \frac{8}{9}\right)\sigma^2 = \frac{5}{9}\sigma^2.$$

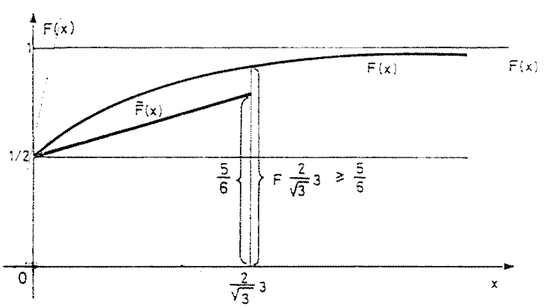

Fig. 2

If min $H(\varepsilon)$ is substituted for $G(\varepsilon)$ in equality (12), it is yielded:

$$\sigma^2 \geqq \varepsilon^2 P(|\xi| \geqq \varepsilon) + \frac{5}{9}\sigma^2$$

and hence, the inequality

$$\frac{\sigma^2}{\frac{9}{4}\varepsilon^2} \geqq P(|\xi| \geqq \varepsilon), \quad 0 < \varepsilon < \frac{2}{\sqrt{3}}\sigma \tag{16)}$$

is obtained.

Thereupon, on the basis of inequalities (10) and (16), it is resulted that in case the expected value of random variable ξ is zero, its variance is σ^2 and its density function satisfies the conditions

$$f(-x) = f(x), \qquad f(x_1) \geqq f(x_2), \qquad 0 < x_1 < x_2 < +\infty$$

then the inequality

$$\frac{\sigma^2}{\frac{9}{4}\varepsilon^2} \geqq P(|\xi| \geqq \varepsilon) \tag{17}$$

is valid for any positive ε.

2. After this, the case will be dealt with when the density function of random variable ξ is not an even function, but it will be supposed further on, too, that it reaches its maximum at the point $x = 0$. Besides, random variable ξ is supposed to have a dispersion equal to σ $M(\xi) = 0$ and a distribution function convex over the section $(-\infty, 0)$ and it is concave over the section $(0, +\infty)$.

In Fig. 3, the diagram of the density function satisfying the condition mentioned above is plotted in a full line, and its mirror-image related to the straight-line $x = 0$ is plotted in dashed line.

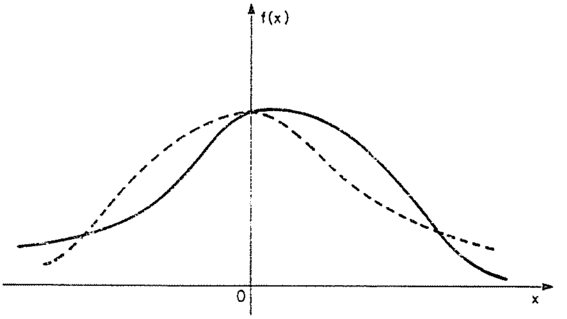

Fig. 3

From Fig. 3 it can be read off that the function

$$g(x) = \frac{1}{2}\,[f(x) + f(-x)] \qquad -\infty < x < +\infty \tag{1}$$

satisfies the conditions associated with the evenness and concavity — used in the previous proof — and it is obvious, too, that $g(x)$ is a density function, besides the following apply:

$$\sigma^2 = 2\int_0^\infty x^2 \frac{1}{2}(f(x) + f(-x))\,dx, \tag{2}$$

$$P(|\xi| \geqq \varepsilon) = 2\int_\varepsilon^\infty \frac{1}{2}(f(x) + f(-x))\,dx. \tag{3}$$

After this, it is obvious that the inequality corresponding to inequalities (5) and (7) in point 1 is yielded in the same way as in point 1. So if the conditions mentioned at the start of point 2 are satisfied, inequality (17) is valid, i.e. it can be written that the expected value of random variable ξ is zero, its variance is σ^2, its distribution function is convex over the section $(0, +\infty)$ and it is concave over the section $(-\infty, 0)$, then for any positive ε the inequality

$$\frac{\sigma^2}{\frac{9}{4}\varepsilon^2} \geqq P(|\xi| \geqq \varepsilon) \tag{4}$$

is valid.

It should be noted that the inequality

$$\frac{\sigma^2}{\frac{9}{4}\,x_{i-1}^2} \geqq P(|\xi| \geqq x_i) \tag{5}$$

is also valid for the case of a discrete random variable if function $\widetilde{F}(x)$ is concave. This can be proved in the same way as the inequality:

$$\frac{M}{2 \cdot x_{i-1}} \geqq P(\xi \geqq x_i)$$

was proved in our paper "Markov's inequality in the case of random variable of concave distribution".

References

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