CRITICAL SURVEY OF THE THEORY OF PLASTICITY*

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The Author was the first to publish in 1914, in a Hungarian review, the idea that the determination of the true load capacity of hyperstatic structures had to reckon with the residual steel strains. This true load capacity exceeding that according to the theory of elasticity, the residual strain might be reckoned with in the practical design of structures. In the meantime this problem had been discussed, expounded from several aspects, and experimentally tested. Let us have now a critical survey of the domain as a whole.

There are different denominations for the new design method. Theory of plasticity means a design method taking also residual deformations into consideration, as against the theory of elasticity relying on elastic deformations alone. It is also called ultimate load method (theory of plastic equilibrium), a term other than unambiguous, namely some authors e.g. STÜSSI mean by ultimate load the maximum load carried by the structure, while others such as F. BLEICH, MAIER-LEIBNITZ and the Author himself in an earlier publication, have meant the maximum load allowed in practice. The approach to this problem depends on certain fundamental principles. What is the goal of designing our structures? That is the serviceability in use. Taking uncertainties of manufacture, material characteristics and load into consideration, the structures have to be designed with a certain "safety" to failure. As stated at the Vienna Congress, the safety degree is a question of economy. On one hand, construction is expected to be inexpensive, on the other hand, the possible damage must not exceed the economy resulting from reduced cross sections. Thus, the higher the possible damage, the higher safety is required. These considerations make it clear why to be satisfied with a safety factor of 1.6 or 1.8 in cases where failure is unlikely else than in an excessive deflection, as against about 3 in cases where an excessive stress in the member would entrain instantaneous collapse without warning (e.g. buckling). Members likely to become unserviceable if excessively deformed are attempted to be given a satisfactory safety to excessive deformation rather than to failure. As a rule on the value of the allowable deflection, the load where the deflection is accelerated under monotonically increasing load could be considered as limit load (critical load or practical ultimate load). In tests by F. STÜSSI and C. F. KOLLBRUNNER [3] (Fig. 1) 1.71 t rather than 2.35 t should be taken as limit load of simple beams. Looked at from this aspect, also the conclusions drawn from these tests are slightly different, namely that the limit load (hence not the true load carry-

* Taken over from the Final Report of the I.A.B.S.E. Congress in Berlin, 1936, with the kind permission of International Association for Bridge and Structural Engineering.

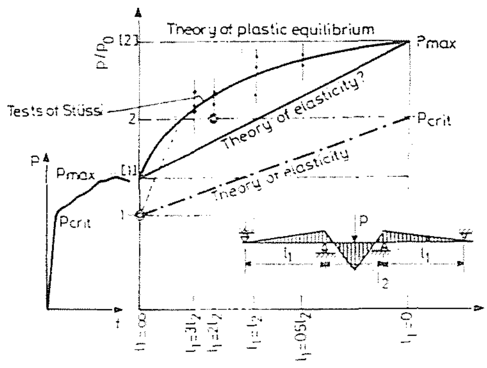

Fig. 1

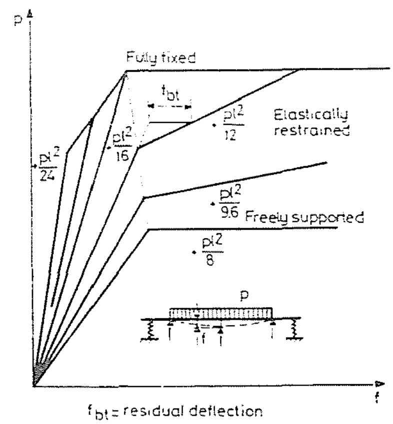

Fig. 2

ing capacity) is the double for elastically restrained beams* in any, even extreme cases. There is an exception for excessively soft restraints, namely then the elastic midspan deflections grow so fast at the yield point that the allowable values are reached before the beam starts to yield at the intermediate supports. Deflection curves of a uniformly loaded ideal-plastic beam with different restraints are seen in Fig. 2. In singular cases also deformations are seen to matter.

There are two means to respect in design the specified safety: either the load multiplied by the safety factor is reckoned with, or a limit stress divided by the safety factor is admitted, this latter being the common way. Thus, the ratio of limit to admitted stress would be the safety factor. This would be correct if stresses grew linearly up to the limit load but in fact this is often not true, in particular for hyperstatic structures (stress redistribution) as a rule.

* Simple beam: $P_{T'}=1.71$ t; $P_V=2.35$ t; $\frac{3.46}{1.71}=2.02$. Continuous beam: I=160–60·120; $P_{T'}=3.46$ t; $P_V=3.82$ t; $\frac{3.82}{2.35}=1.62$. $P_{T'}$ = ultimate load, P_V = working load.

Entering stresses multiplied by the safety factor into calculation would explain the stress redistribution arising anyhow beyond the admitted stress, hence critical only for estimating the safety rather than for the real stress.

In order to determine the theoretical value of the ultimate load of hyperstatic structures and to avoid the mathematical difficulties, a material with ideal characteristics, i.e. an idealized stress-strain diagram is introduced. Again, the cross section was assumed to remain plane in course of deformations, and the yield process to propagate from extreme fibres to the beam interior. According to this theory, a flexural cross section can undergo further deformation without moment increase if it has become plastic up to the neutral axis. To the plastic hinge effect an infinite deflection value belongs. This is impossible for mild steel because of strain hardening. This is why recently some researchers wanted a closer look into the process of plastic deformation, especially for cases with non-uniform stress field and yield phenomenon, when parts under lower stress delay the deformation of members in the plastic range (see recent theories of elasticity by W. KUNTZE [4], W. PRAGER [5] and J. FRITSCHE [6]). Observations, however, did not confirm this theory. Yield patterns are not delayed to the degree to cause the beam to yield at once up to the neutral axis. It is seen also in Fig. 230, p. 127 of "Plasticity of Structural Materials" by NÁDAI: yield was steadily spreading inwards.

Though, for I-beams, yield patterns are seen to appear at once on the flange. On the other hand, RINAGL [7] states the delay of yield to be erroneous in this concept, and to be attributed to an upper yield point, always manifest in bending, while in a tensile test it is negligible. The Author disagrees with Prof. RINAGL, namely he himself could observe yield delay for an uneven stress field in truss bars, to be discussed below. In all these, reckoning with real material characteristics leads to difficult computations. Since, however, the final goal is structural design rather than theoretical demonstration of test results, a simple computation method has to be found. This is possible by assuming a sharp transition from the elastic to the plastic range also in bending. MAIER-LEIBNITZ [8] showed how to solve simple problems by means of the true mo-

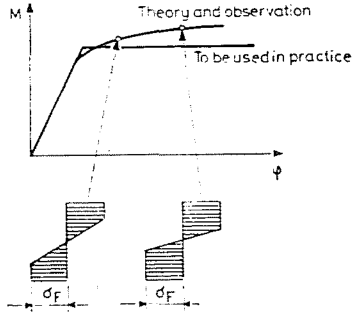

Fig. 3

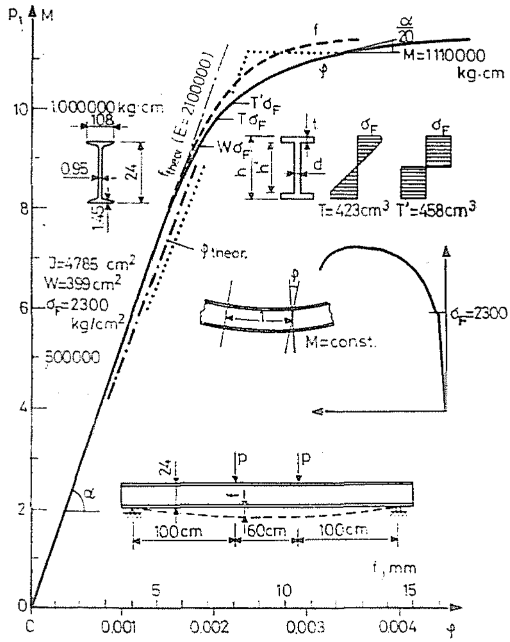

Fig. 4

ment-deformation method; a practical method has to rely on a simplified interpretation (Fig. 3). Maier-Leibnitz suggests to consider as ultimate moment that where the curvature of the deformation-moment diagram has a maximum. The Author suggests to consider as ultimate moment that where the residual deformation is twenty times the elastic one. For a deeper insight, the Author loaded an I-beam of about NP 24 ($W = 399$ cm), lackered to exhibit yield phenomena beyond the yield point. The bending curve remained about linear up to about $\sigma = 2250$ kg/cm² (Fig. 4). The flange in tension exhibited yield patterns at 2500 kg/cm² while the same appeared on the compressed flange — partly due to a local fault — already at $\sigma = M/W = 2120$ kg/cm². For $M/W = 2800$ kg/cm² that deformation rate was achieved, which was considered by the Author as the characteristic sign of the ultimate moment. The beam was removed from the bending tester, carefully inspected and photographed (Fig. 5). About half of the flange at the beam cross section exposed to a constant maximum moment exhibited yield patterns. Contrary to theoretical considerations, the yield pattern approached the neutral axis. The yield stress obtained on a tensile specimen cut out after the test from a load-free beam end was found to be 2300 kg/cm² with a very short yield deformation. This test argues for empirical rather than theoretical determination of the yield point. Ultimate moment and yield point seem to be else than simply related because of the effects of

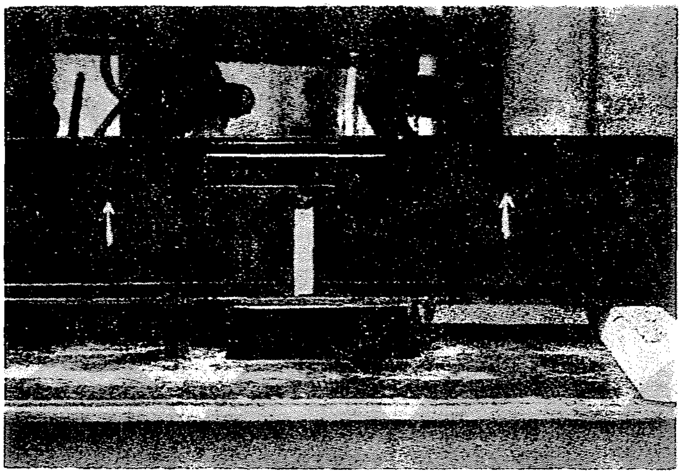

Fig. 5

the shape of the cross section and of material characteristics. Empirical determination of these ultimate moments for certain cross sections and steel types would eliminate difficulties of the assertion of the new design approach.* Having made up one's mind to calculate by using the idealized bending line ($M-\varphi$ diagram), the rules of structural analysis are as below.

1. Statically determinate structures in bending

The ultimate load capacity is exhausted only when the "beam" starts yielding, rather than at reaching the yield stress in the extreme fibre. The ultimate moment is not $M = W.\sigma_F$ but $M = T.\sigma_F$, T exceeding W by about 6 to 20% and has yet to be determined experimentally.

2. Statically determinate trusses

The computation remains unaltered. Secondary stresses resulting from the rigid connections of the bars at the nodes may be neglected. In compression, however, also in the plane of the truss, the theoretical bar length has to be considered as buckling length. Compressed bars have to be designed with a higher safety than have tensile bars, namely exceeding the buckling load may entrain collapse of the structure.

* KAZINCZY [9], KIST [10], FRITSCHE [11] and KUNTZE [4] have suggested methods to calculate the ultimate moments, which provide, however, lower ultimate moment value than the Author's tests did.

3. Calculation of rivet connections

Just as hitherto, the total bar force is assumed to be uniformly distributed among all the connecting rivets. Practice and experience have perfectly confirmed the theory of plasticity. The connecting rivets or welded joints should, however, be designed for the maximum admissible rather than for the calculated bar force in order to have bars rather than joints yielded due to a higher than ultimate stress. To distribute secondary stresses in the bars themselves, rigid connections are advisable.

4. Analysis of continuous beams

For beams with a constant rolled cross section, moments M_0 in each span have to be determined as for simple beams, and the closing line has to be located to equalize negative and positive moments. Then the beam has to be designed for the maximum moment calculated in such a manner.

For beams with cross sections adapted to the course of moments by means of flange plates, calculation according to the theory of plasticity is essentially meaningless. If, however, economical reasons argue for the new method, the closing line can be deliberately located so as to minimize production costs. It has to be considered as a rule that negative moments may arbitrarily be reduced, while yield at mid-beam is associated with large deflections. For live loads, the maximum moments have first to be determined according to the theory of elasticity, then the closing line may be arbitrarily shifted in order to equalize the maximum moments [12, 13].

A major achievement of the theory of plasticity is the possibility to ignore residual support subsidences. On the other hand, effects of displacements of the elastic supports have to be taken into consideration.

Rolling and shrinkage stresses may be neglected, as against stresses arising from uneven heating in use [13].

Calculating with a more significant moment redistribution, in particular, when the middle cross section is in yield, the compressed flange is advisably made the stronger, to have the yield process in the tensile flange.

5. Structures of bars with bending stiffness (frames)

Several Authors stated that yield of n cross sections of a statically indeterminate framework with n redundancies does not cause failure. The problem may be considered as if having hinges at these sections, acted upon by constant moments. Earlier the Author was of the same opinion [14] but now he suggests a modification. To cause instability of a structure, as many hinges have to arise as to produce a kinematic chain. During the displacement of the structure

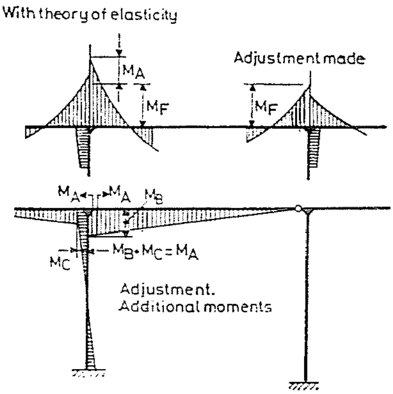

Fig. 6

these hinges turn only in a given sense. Thereby the plastic hinge acts as a hinge only in one direction, while in the other direction it behaves as a perfectly elastic member. Thus, plastic hinges turning in the opposite direction as those in the kinematic chain do not act as hinges. This is why in a hyperstatic structure with n redundancies the yield point will be exceeded at more than n spots before becoming unstable. A framework can safely support a given load when a possible moment line satisfying the condition of equilibrium with external forces nowhere exceeds the value $M = T\sigma_{\text{adm}}$. An exacter procedure may be established by analogy to the *Cross* method. First, moments are determined according to the theory of elasticity. At sections where moments have to be reduced, the structure has to be considered as cut through, balanced by introducing additional unloading moments. At sections with reduced moments, and expected to develop moments, hinges are introduced (Fig. 6). The main advantage of the theory of plasticity is the possibility to control the moments, protecting thereby delicate cross sections from excessive stresses. In general, the most important member of a framework is the column. Weakening the beams at the joints may spare the column, namely thereby, after having reached the ultimate moment at the joint, the beam cannot transfer additional moments to the column. Thus, a harmless yield of the beam at the joint may save the column from hazardeous deflections.

6. Trusses

Externally hyperstatic trusses are designed as beams and frameworks. Yield phenomena are restricted to a part of a bar. Redistribution cannot, however, be made else than with tensile bars, namely the resistance of a compressed bar abruptly drops after buckling, as stated by the Author in Liège [9]. Recently, E. CHWALLA [15] has reconsidered this problem and experimentally confirmed the drop of the compressive strength. For internally hyperstatic trusses, according to the theory of elasticity, often not all the bars can be fully stressed such as for that in Fig. 7 where, according to the theory of elasticity, part system B cannot be fully utilized. In this respect, the theory of plasticity is economically more advantageous by permitting full use of all the bars. Normally, such structures are easy to design. The statically superfluous tensile bars are omitted and replaced by known forces $F \cdot \sigma_{\mathrm{adm}}$. Hence tensile bars with the highest stresses, which start to yield the first, have to be omitted,

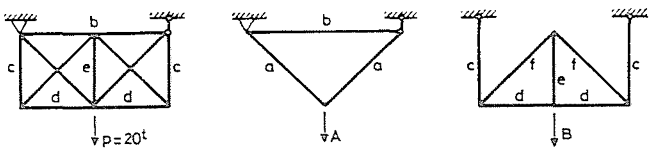

Fig. 7

either by simple consideration or involving the theory of elasticity. The cross sections have to be adjusted to let always tensile bars yield, and never compressed bars buckle.

Live loads require special methods such as that by E. MELAN [16], with the comment that no plastic deformation in compressed bars is admissible.

To check theoretical considerations on the theory of plasticity for trusses, some tests have been made, to be briefly outlined below. Two kinds of internally hyperstatic trusses, namely welded and riveted, have been tested, while tests by G. GRÜNING and E. KOHL [17] concerned externally hyperstatic trusses, where the tensile bars with the maximum stresses were made to eye bars, inhibiting to draw conclusions on usual nodal joints. The form of the tested truss specimens with sizes and results is seen in Fig. 8. The truss may be considered as consisting of two basic systems A and B. Resistances of systems A

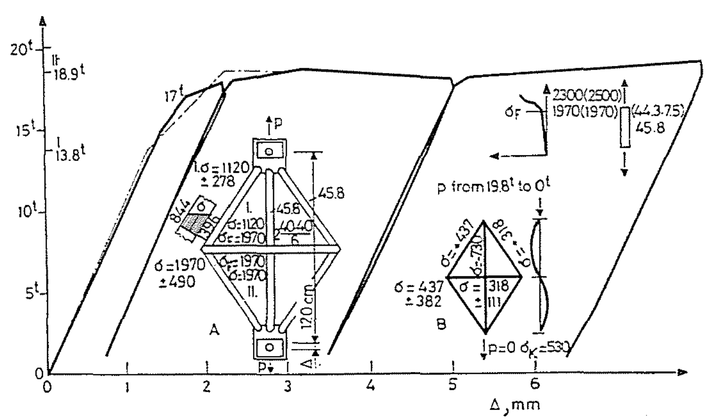

Fig. 8

and B each have been plotted in terms of the imposed elongations. P_I and P_{II} are indicated as "first" and "second" ultimate load (= ultimate load capacity), respectively. After unloading, in the systems remained residual stresses (residual forces in Fig. 9).

Strength test of the applied structural material showed the band steel to be excessively soft, and to have a very wide yield range with increasing stresses. The yield point was first reached in the vertical tensile bar (first ultimate load). Under additional loads, stresses in this vertical bar remained constant and grew only in the other bars until the yield point (second ultimate load). Theoretical secondary stresses indicated in Fig. 8 are actually considered to vanish in yield. In unloading, the truss behaved as perfectly elastic, residual stresses are seen in Fig. 8. But the vertical bar does not cope with the residual stresses of 830 kg/cm^2, it being made of band steel buckling already at 530 kg/m^2.

This buckling appeared also in the specimen. The first yield lines near the middle of the vertical bar appeared at $P = 14$ t, but actually it began to yield only at 17 t. The specimen suffered a significant deformation while only small bar portions yielded (Fig. 10). Hence plastic strain is restricted to certain spots where a certain value of strain is reached. Elongation of a mild steel bar has to be realized according to Fig. 11, where K_I and K_{II} are different imposed elongations. Lines e and p represent elastic and plastic strains, respectively. The ultimate load (second ultimate load) agrees with the theoretical value, pointing to the irrelevance of welding shrinkage stresses to the load capacity. They only affect the beginning of the force redistribution.

Shrinkage stress values have been determined by the Author on specimens observed for elongations at different spots during welding and cooling,

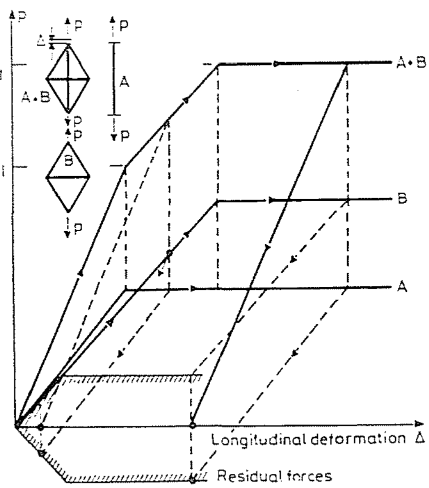

Fig. 9

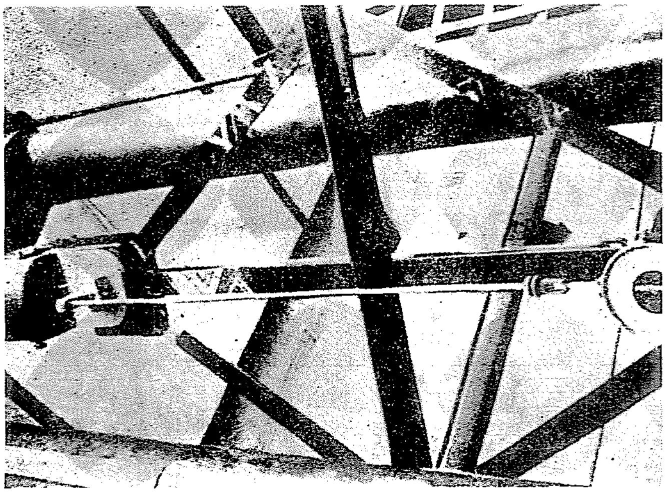

Fig. 10

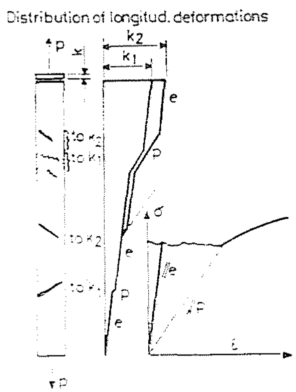

Fig. 11

exhibiting shrinkage stresses of 900 kg/cm^2. No delay of yield phenomena, hence an upper yield point, could be observed: skew bars with significant secondary stresses yielded where mean stresses reached the yield point. Hence, these tests seem to confirm the recent theory of plasticity. On the other hand, no test performed gave hint to the previous yield condition. These kinds of tests will be published by the Author in a detailed report.

A similar truss was made with rivets (Fig. 12). The somewhat higher yield point of the applied band steel resulted in a higher maximum load than that of the welded beam (20.4 t compared to 19.1 t). At the first loading, rivets got somewhat loosened. Further loads elicited elastic behaviour. In spite of rivet holes, the yield point was achieved in the entire cross section.

These tests lead to the following conclusions. Shrinkage stresses in hyperstatic welded trusses affect only the beginning of force redistribution rather than the critical load value. Remind that shrinkage stresses add to the primary stresses in tensile bars and reduce those in compressed bars (option of the technology process).

In riveted hyperstatic trusses the plastic strain starts at joints, friction somewhat increases the force eliciting this phenomenon. A similar influence would arise from the increase of the yield point at rivet hole edges or from strain hardening due to the mode of riveting. For small bar lengths even a slight flexibility of the joint may bring about force redistribution. Bar connections have to be strong enough to have the whole bar yielding before they fail. Ultimate load of a riveted truss is that obtained according to the theory of plasticity, considering cross sections not to be weakened by rivet holes, provided no compressed bar buckles. In view of the high residual deformations, the practically applicable ultimate load is obtained by deducing the rivet holes, and reckoning with force redistribution. Then the safety will always exceed that for welded trusses calculated with full cross sections.

In addition to trusses, also riveted steel beams have been tested by the Author. Simply supported beams were loaded at third points, and deflection angles of central beam parts under a constant moment determined. The results have been plotted in Fig. 13. In determining the moment of inertia, the rivet holes were not deduced. The measured deflection somewhat exceeded that obtained by the use of the value $E = 2100$ t/cm^2, while the deflection at unloading (elastic recovery) was in good agreement. After two days of rest, the yield point increased by 6% and the beam behaved purely elastically. For the

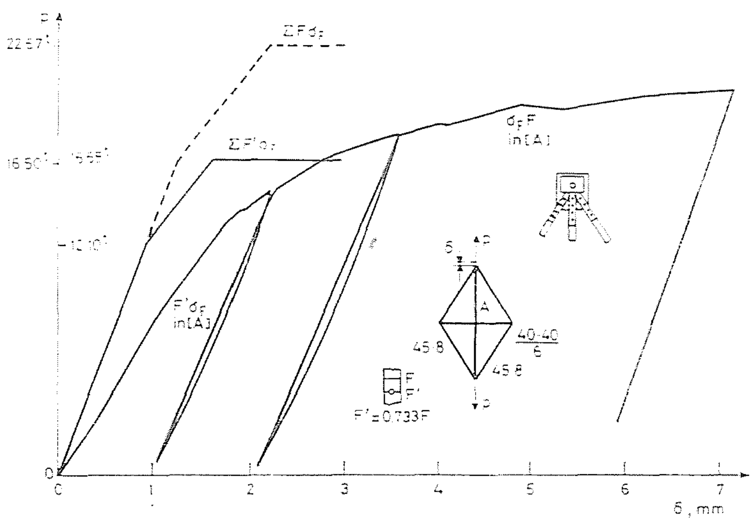

Fig. 12

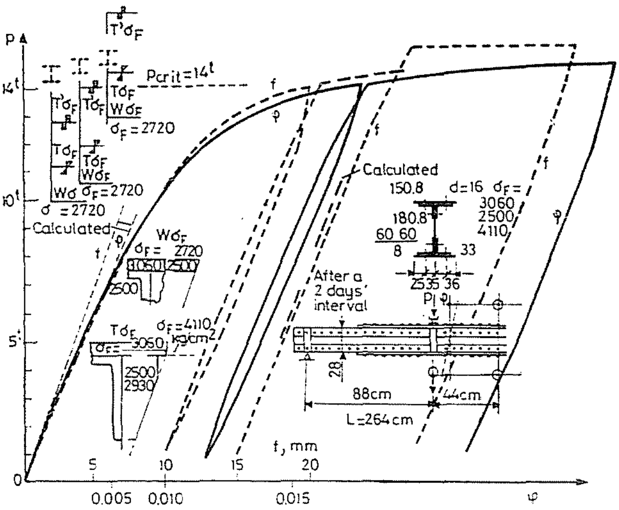

Fig. 13

suggested assumption $d\sigma/d\varepsilon = (1/20)E$, the critical load was found to be 14 t. Comparison of this test value with different concepts has been plotted in Fig. 13 where the lowest yield point was taken as 2500 kg/cm² resulting in a corresponding maximum stress of 2720 kg/cm² in the extreme fibre of the flange. The ultimate moment $T \cdot \sigma_{\mathrm{adm}}$ was determined from the condition of the flange at yield (Fig. 13). In this test another unknown is the value to be assumed for

Table 1

Welded I-beam			Riveted I-beam d = 16 mm	
σ_F kg/cm²	Cross section mm		Cross section mm	σ_F kg/cm²
2680	152.6 · 13	Compressed flange	152 · 12.8	2680
2620	155 · 7.7	Tensile flange	154 · 7.7	2590
2750	60 · 60 · 6.1	4 L	60 · 60 · 6.1	2780
4280	182 · 8.2	Web	183 · 8.6	4060
1 513 000		Critical moment kgcm according to the test	**1 266 000**	
Tension 1 180 000	Compression 1 420 000	$W\sigma_F$ (σ_F = flange)	Tension 1 170 000	Compression 1 140 000
		$W\sigma_F$ rivet holes deduced	**965 000**	**1 135 000**
		$W\sigma_F$ rivet holes deduced also in the web	906 000	1 087 000
1 644 000		$T\sigma_F$ overall cross section	1 632 000	
		$T\sigma_F$ rivet holes deduced	**1 387 000**	
		$T\sigma_F$ rivet holes deduced also in the web	1 266 400	
1 513 000		$T\sigma_F$ of flanges and angle steel = $W\sigma_F$ of web	rivet holes deduced **1 259 000**	

rivet holes. To clarify this problem, comparative tests have been made by the Author on riveted and welded beams of the same section and material. Results have been compiled in Table 1.

Also a riveted beam continuous over three supports has been tested (Fig. 15). Deflections exceeded the calculated values, even after unloading. The web yielded in shear between the central support and the loading point (Fig. 14), experimentally confirming the theoretical statement by STÜSSI [18] that shear stresses definitely increase upon the propagation of yield from the beam edge to a certain depth, even if to a somewhat lower degree.

It may be ascribed to the rapid moment decrease endangering only a short portion of the beam restrained in displacement by the adjacent beam parts. In final account, the maximum load upon perfect moment redistribution was characterized by the ultimate moment $T' \cdot \sigma_F$.

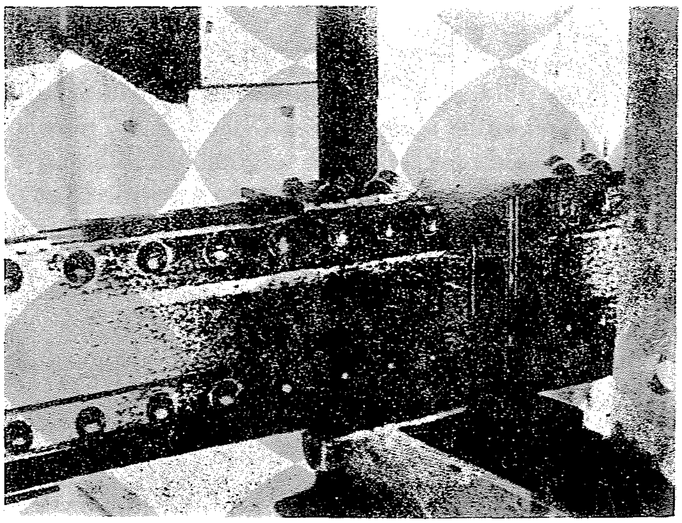

Fig. 14

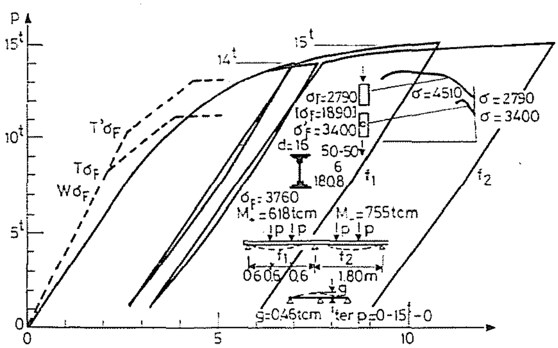

Fig. 15

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