ON THE CONVERGENCE OF GENERALIZED HAAR EXPANSIONS

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Summary

The classical convergence theorem of *Haar* (1909) yields an example of a non-trigonometric *Fourier* expansion which converges at any point of continuity to the basic function. (As it is well-known, the trigonometric *Fourier* series of a continuous function may be divergent at some points.) Considering the fact that in the last decades the importance of systems and series of *Haar*'s type has increased to a large extent in the theory and practice alike, the author has investigated since several years, how such systems and expansions can be characterized and generalized in the simplest and "most natural" manner. The present paper aims at briefly summarizing the obtained results.

1.

As it is well-known, the trigonometric *Fourier* expansion of a *Lebesgue* integrable function or, in particular, of a continuous function can be divergent at some points. *L. Fejér*'s classical summation theorem and its extensions aim to remedy this situation by considering other limiting processes than the *Cauchy* convergence.

On the other hand, A. Haar has shown in [5] that the insufficiency in question can be eliminated also by taking another suitable basic orthogonal system instead of the trigonometric one. In fact, the so-called *Haar* system which is defined in the interval (0, 1), apart from its points of discontinuity, by

$$\chi_0^{(0)}(t) \equiv 1; \quad \chi_\nu^{(\lambda)}(t) = \begin{cases} 2^{\nu/2} & \text{for} \quad (\lambda - 1)2^{-\nu} < t < \left(\lambda - \frac{1}{2}\right) 2^{-\nu} \\ -2^{\nu/2} & \text{for} \quad \left(\lambda - \frac{1}{2}\right) 2^{-\nu} < t < \lambda \cdot 2^{-\nu} \\ 0 & \text{for other} \quad t \end{cases} \tag{1}$$

with $\nu = 1, 2, \ldots$ and $1 \leq \lambda \leq 2^\nu$, yields an example of a complete orthonormal system of functions having two remarkable properties; 1) each function $f(t) \in L(0, 1)$ is the sum almost everywhere of its *Fourier* expansion formed with the system (1); 2) this so-called *Haar* expansion converges to $f(t)$ at all points of continuity, and the convergence is uniform in each closed interval where f is continuous. At the end of [5], *Haar* has mentioned still the possibility of constructing analogous further examples arising by other dyadic or triadic etc. divisions.

2.

The systems and series of *Haar*'s type are known to play an important role in the theory and applications of orthogonal series, especially by their close relation to the *Rademacher* and the ordinary or generalized *Walsh* systems (cf. [1]). In recent decades, however, these systems became useful expedients also in the numerical analysis and computing, furthermore in the theory of certain stochastic processes, with special regard to martingales. (Cf. e.g. [2], [4], [11], [12], [19], [21].) So it is understandable that several new theoretical statements on *Haar* systems and expansions have been published just since the beginning of the sixties, showing e.g. how far they can be characterized by the non-negativity of their *Dirichlet* kernels or by certain algebraic properties, further considering the sets on which a given subsystem of such a system is complete and some sets of uniqueness, or the convergence features of certain related "multiplicatively orthogonal" series, in the sense of ALEXITS, etc. (See [13] through [18] and [20].)

The present paper aims to give a brief survey of some pertinent results of the author which have been found since the sixties, motivated, on the one hand, by the rather sophisticated character of (1), and on the other hand, by the fact that all usual proofs of *Haar*'s convergence theorem are fairly complicated. (Cf. e.g. [5], 363—368 and [1], 47—50, where the properties of *Haar*'s kernel function are applied; or see [6], 120—122 where an "*ad hoc*" verification of three parts is given.) Next a *characterization* of (1) will be formulated by orthogonalization of some characteristic functions, which seems to be the "most natural" one, and the representation of (1) by means of characteristic functions will be shown to be of use for deducing Haar's results in a few lines. (Cf. Theorems I—II.)

Thereafter the question will be dealt with how to construct the largest class of orthogonal systems of *Haar*'s type for which the premised simplest treatment of (1) — based upon the connection with characteristic functions — can be extended. It will be given a *new complete orthogonal system* $\{\mu^{(r)}_{k_1 \dots k_r}(t)\}$ which 1° is associated with the most general system of partitions of a finite interval into (equal or unequal) subintervals; 2° has also a quite simple explicit representation and geometrical interpretation; 3° allows of a suitable "natural" characterization by means of characteristic functions; 4° enables to prove easily a strong *generalization of Haar's convergence theorem*. (Cf. Theorems III—IV.)

Remark that the results at issue are discussed in the papers [7]—[9] and in the book [10] by the author partly within a wider framework, namely they are closely interlinked with some problems of approximation in the *Hilbert* space and L^p-spaces, respectively.

3.

Let us consider the simplest *complete* system of characteristic functions belonging to a dyadic partition of the interval [0, 1], i.e. the sequence

$$1;\ \bar{\chi}_1^{(1)},\ \bar{\chi}_2^{(2)};\ \bar{\chi}_2^{(1)},\ \bar{\chi}_2^{(2)},\ \bar{\chi}_2^{(3)},\ \bar{\chi}_2^{(4)};\ \ldots$$

where

$$\bar{\chi}_\nu^{(\lambda)}(t) = \chi_{\left[\frac{\lambda-1}{2^\nu},\ \frac{\lambda}{2^\nu}\right)}(t) \qquad (1 \leq \lambda \leq 2^\nu;\quad \nu = 0, 1, 2, \ldots).$$

These functions are clearly not linearly independent (we have $\bar{\chi}_\nu^{(\lambda)}(t) = \bar{\chi}_{\nu+1}^{(2\lambda-1)}(t) + \bar{\chi}_{\nu+1}^{(2\lambda)}(t)$), but so is the system obtained by removing all terms of the form $\chi_{\nu+1}^{(2\lambda)}(t)$.[1]

It is easy to verify by induction:

Theorem I.

Submitting the sequence

$$1;\quad \bar{\chi}_1^{(1)};\ \bar{\chi}_2^{(1)},\ \bar{\chi}_2^{(3)};\ \bar{\chi}_3^{(1)},\ \bar{\chi}_3^{(3)},\ \bar{\chi}_3^{(5)},\ \bar{\chi}_3^{(7)};\ \ldots$$

to the *Gram—Schmidt* orthogonalization process yields just *Haar*'s system (1) in the form:

$$\chi_0^{(0)}(t) \equiv 1;\quad \chi_\nu^{(\lambda)}(t) = 2^{\nu/2}[\chi_{d'_{\lambda\nu}}(t) - \chi_{d''_{\lambda\nu}}(t)] \qquad (\nu \geq 0,\ \lambda \geq 1), \tag{2}$$

where

$$d'_{\lambda\nu} = \left[(\lambda - 1)2^{-\nu},\ \left(\lambda - \frac{1}{2}\right)2^{-\nu}\right),\quad d''_{\lambda\nu} = \left[\left(\lambda - \frac{1}{2}\right)2^{-\nu},\ \lambda\cdot 2^{-\nu}\right).$$

Let now be $f(t) \in L(0, 1)$ and write for the *Haar* expansion of this function at a point x:

$$\gamma_0^{(0)}\chi_0^{(0)}(x) + \sum_{m=0}^{\infty}\sum_{k=1}^{2^m} \gamma_m^{(k)}\chi_m^{(k)}(x) \tag{3}$$

[1] Observe that this system (like the original one) is not orthogonal in [0, 1].

$$[\gamma_m^{(k)} = \int_0^1 f(t)\chi_m^{(k)}(t)dt];$$

the partial sum of (3) up to the term $\gamma_\nu^{(\lambda)}\chi_\nu^{(\lambda)}(x)$ will be denoted by $S_\nu^{(\lambda)}(x)$.

As an almost immediate consequence of (2) we deduce:

Theorem II.

The *Haar* expansion (3) converges to function f at each point $x \in [0, 1]$ where f equals the derivative f of its indefinite integral.[2]

Proof. Transforming $S_\nu^{(\lambda)}(x)$ by means of (2) into a linear combination of characteristic functions, a simple calculation yields:

$$(4)\qquad \begin{cases} S_\nu^{(\lambda)}(x) = 2^{\nu+1} \sum_{k=1}^{\lambda} [\chi_{d'_{k\nu}}(x) \int_{(d'_{k\nu})} f(t)dt + \\ + \chi_{d''_{k\nu}}(x) \int_{(d''_{k\nu})} f(t)dt] + \\ + 2^\nu \sum_{k=\lambda+1}^{2^\nu} \chi_{d_{k\nu}}(x) \int_{(d_{k\nu})} f(t)dt \quad (1 \leq \lambda \leq 2^\nu), \end{cases}$$

where $d_{k\nu} = d'_{k\nu} \cup d''_{k\nu} = [(k-1)2^{-\nu}, k \cdot 2^{-\nu})$, and the last term is to be replaced by 0 in the case $\lambda = 2^\nu$.

Denoting by d_x the only subinterval figuring in (4) for which $x \in d_x$, we have[3]

$$S_\nu^{(\lambda)}(x) = |d_x|^{-1} \int_{(d_x)} f(t)dt.$$

Hence the assertion follows for $\nu \to \infty$ at once.

We add that our expression for $S_\nu^{(\lambda)}(x)$ implies obviously also its uniform convergence to $f(x)$ in any interval $[x_1, x_2] \subset [0, 1]$ where f is continuous.

4.

As far as the question on *generalization* is concerned, in [7]—[8] the following family of orthogonal systems has been investigated.[4]

Starting from any finite basic interval $I_0 = [a, b]$, consider a partition of I_0 into mutually disjoint (not necessarily equal) subintervals, say

$$I_0 = \bigcup_{k=1}^{N} I_k.$$

[2] As well-known, this condition is fulfilled almost everywhere and in particular at any point of continuity. For details, see [10], 309—310.

[3] Here and later, we use the notation $|.|$ for the length of an interval.

[4] The system (12) in [7] differs from the system (3.1) in [8] only by notation.

Then divide each I_k further into union of some disjoint I_{kl}, each subinterval I_{kl} into union of some disjoint I_{klm}, etc.; in formulae:

$$I_k = \bigcup_{l=1}^{N_k} I_{kl} \, (1 \leq k \leq N), \quad I_{kl} = \bigcup_{m=1}^{N_{kl}} I_{klm} \quad (1 \leq k \leq N, \quad 1 \leq l \leq N_k), \ldots$$

Let us assume that the number of subintervals is always > 1, and that the partitions

$$I_{k_1} \ldots {}_{k_\nu} = \bigcup_{k_{\nu+1}=1}^{N_{k_1 \ldots k_\nu}} I_{k_1} \ldots {}_{k_\nu \, k_{\nu+1}}$$

satisfy the condition $\max |I_{k_1 \ldots k_{\nu+1}}| \to 0$. Write now [cf. (2)]:

$$(5) \quad \begin{cases} \mu_0^{(0)}(t) \equiv 1 \\ \mu_0^{(r)}(t) = \sum\limits_{p=r+1}^{N} [|I_p| \chi_{I_r}(t) - |I_r| \chi_{I_p}(t)] & (1 \leq r \leq N-1) \\ \mu(_k^{(r)} t) = \sum\limits_{p=r+1}^{N_r} [|I_{kp}| \chi_{I_{kr}}(t) - |I_r| \chi_{I_{kp}}(t)] & \begin{pmatrix} 1 \leq k \leq N \\ 1 \leq r \leq N_k - 1 \end{pmatrix} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Plainly, in general:

$$\mu_{k_1 \ldots k_\nu}^{(r)}(t) = \begin{cases} \sum\limits_{p=r+1}^{N_{k_1 \ldots k_\nu}} |I_{k_1 \ldots k_\nu p}| & \text{for} \quad t \in I_{k_1 \ldots k_\nu r} \\ -|I_{k_1 \ldots k_\nu r}| & \text{for} \quad t \in \bigcup\limits_{p=r+1}^{N_{k_1 \ldots k_\nu}} I_{k_1 \ldots k_\nu p} \\ 0 & \text{for other} \quad t. \end{cases}$$

Therefore, each term of (5) has a very simple geometrical meaning: $\mu_{k_1 \ldots k_\nu}^{(r)}(t)$ $(r \geq 1)$ is a step function with at most three intervals of constancy and with a zero-valued integral on I_0. (See Figure 1; the rectangles over and below the t-axis are congruent.)

It can be shown that the normed version at (5) is such a *generalization of Haar's orthonormal system*, which is complete on $L(I_0)$ and yields a "natural"

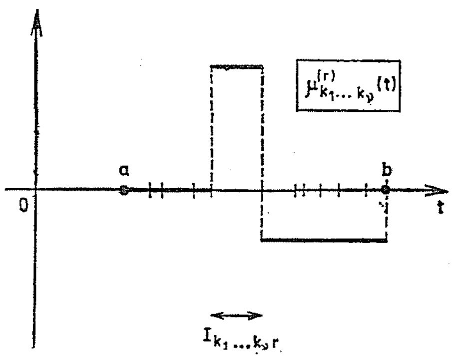

Fig. 1

extension in the sense that also the counterpart of Theorem I (with the above most general interval partition instead of the dyadic one) holds. Namely, introducing the notation $(\tilde{g}_1 | \tilde{g}_2, \ldots, \tilde{g}_n)$ for the system of functions which issues from $(g_1, g_2, \ldots, g_n)$ by *orthogonalization* and after by discarding the first term $||g_1||^{-1}g_1$, we obtain

Theorem III.

The sequence of functions

$$\tilde{\chi}_{I_0};\ (\tilde{\chi}_{I_0}|\tilde{\chi}_{I_1}, \ldots, \tilde{\chi}_{I_{N-1}});\ (\tilde{\chi}_{I_k}|\tilde{\chi}_{I_{k,1}}, \ldots, \tilde{\chi}_{I_{k,N_k-1}}) \qquad (1 \leq k \leq N); \ldots$$

is identical with (5), apart from order and eventual constant factors. More precisely: this system is the normed variant of (5).

The verification is based on a new property of the *Gram* determinant of certain characteristic functions and some general facts on orthogonalized systems. (Cf. [8], 245—248.)

Finally, the *Fourier* series associated with (5) will be shown to have the same outstanding convergence properties as the original *Haar* expansion. Namely,

Theorem IV.

The *Fourier* expansion of any function $f \in L(I_0)$ in terms of the system (5) represents f at every point $x \in I_0$, where the function is equal to the derivative of its indefinite integral. So this expansion converges to $f(x)$ almost everywhere and, in particular, in all points of continuity; the convergence is, besides, uniform in any closed subinterval of I_0 where f is continuous.

Proof. (Cf. [7], 311, (III).) Let $S_M = S_M(x)$ be the partial sum of the *Fourier* series in question and let $i_1, i_2, \ldots, i_\varkappa$ denote the subintervals of I_0 in which $S_M(x)$ is constant. Then S_M (as a step function) can be written clearly in the form

$$S_M(x) = \sum_{n=1}^{\varkappa} c_n \chi_{i_n}(x), \tag{6}$$

where coefficients c_n depend on function f.

If $f \in L^2(I_0)$, the minimum property of the partial sums of an arbitrary *Fourier* series yields immediately

$$c_n = |i_n|^{-1} \int_{(i_n)} f(t)dt \qquad (n = 1, 2, \ldots, \varkappa); \tag{7}$$

and this holds also in case of $f \in L(I_0)$, $f \notin L^2(I_0)$, because any such function is the $L(I_0)$-limit of suitably chosen functions $\{f_q\} \subset L^2(I_0)$, so that each *Fourier* coefficient of f is the limit for $q \to \infty$ of the corresponding *Fourier* coefficient of f_q.

(6)—(7) imply that there is a (uniquely determined) subinterval i_x among the i_n for which $x \in i_x$ and

$$S_M(x) = |i_x|^{-1} \int_{(i_x)} f(t)dt.$$

Therefore, $S_M(x) \to f(x)$ for $\max |I_{k_1 \dots k_{\nu+1}}| \to 0$ ($\nu = 2, 3, \dots$), provided that

$$\lim_{h \to 0} [h^{-1} \int_x^{x+h} f(t)dt] = f(x) \qquad (h \gtrless 0),$$

which is just the main point of the above assertion.

The remaining part of the theorem can be gotten in the same manner as its analogue for the classic *Haar* expansion.

5.

Remark that a further extension of the definition of (5) to arbitrary measurable sets instead of intervals is easy to realize. Note also that the generalizations indicated by *Haar* himself in [5] are contained in the construction of (5), but *Haar*'s processes allow of no simple explicit representations for the resulting systems.

We remind still of the fact: *Franklin* [3] has altered the definition of (1) in such a way that the system obtained there consists of *everywhere continuous* (linear) functions. A similar procedure can be given for the system (5), too.

As mentioned succintly in the introduction, the essential background of the above results is the deeper use of a (recently detected) close connection between some complete systems of characteristic functions and orthogonal expansions of *Haar*'s type. This connection is just the source of some applications in the theory of $L^p (p \geq 1)$ spaces ([7], [9]). Additional results on the topic, in particular, about a new class of linear metric spaces, will be published elsewhere.

References

1. ALEXITS, G.: Convergence Problems of Orthogonal Series. Pergamon Press, New York—Oxford—London—Paris, 1961, 350 pp.
2. BUTZER, P. L. and WAGNER, H. J.: Walsh-Fourier Series and the Concept of a Derivative. Applicable Analysis, *3* (1973), 29—46.
3. FRANKLIN, P.: A Set of Continuous Orthogonal Functions. Math. Annalen, *100* (1928), 522—529.

4. GIBBS, J. E. and IRELAND, B.: Walsh Functions and Differentiation — Applications of Walsh Functions. Proc. Sympos. Naval Research Lab., Washington, D.C., 1974, 147—176.
5. HAAR, A.: Zur Theorie der orthogonalen Funktionensysteme. Math. Annalen, *69* (1910). 331—371.
6. KACZMARZ, S. and STEINHAUS, H.: Theorie der Orthogonalreihen. (2. edition.) Chelsea Publ. Co., New York, 1951, 296 pp.
7. MIKOLÁS, M.: Über die Approximation durch Treppenfunktionen in L^p. Proc. Conf. Constructive Theory of Functions (Budapest, 1969), Akad. Kiadó, 307—314.
8. MIKOLÁS, M.: Über die Charakterisierung orthogonaler Systeme von Haarschem Typ. Publicationes Math., *19* (1972), 239—248.
9. MIKOLÁS, M.: Orthogonal Systems of Characteristic Functions and Hilbert Space. Resultate der Mathematik — Mathematical Results, *1* (1978), 195—206.
10. MIKOLÁS, M.: Real Functions and Orthogonal Series. (In Hungarian.) Tankönyvkiadó, Budapest, 1978, 394 pp.
11. NELSON, P. I.: A Class of Orthogonal Series Related to Martingales. Annals Math. Stat., *41* (1970), 1684—1694.
12. NEVEU, J.: Discrete-parameter Martingales. North-Holland and American Elsevier Publ., Amsterdam—Oxford—New York, 1975, 236 pp.
13. PRICE, J. J.: Orthonormal Sets with Non-negative Dirichlet Kernels. Trans. Amer. Math. Soc., *95* (1960), 256—262.
14. PRICE, J. J.: Orthonormal Sets with Non-negative Dirichlet Kernels, II. Trans. Amer. Math. Soc., *100* (1961), 153—161.
15. PRICE, J. J. and ZINK, R. E.: On Sets of Completeness for Families of Haar Functions. Trans. Amer. Math. Soc., *119* (1965), 262—269.
16. PRICE, J. J.: An Algebraic Characterization of Certain Orthonormal Systems. Proc. Amer. Math. Soc., *19* (1968), 268—273.
17. SCHIPP, F.: On a Generalization of the Haar System. Acta Math. Acad. Sci. Hung., *33* (1979), 183—188.
18. SOX, J. L. and HARRINGTON, W. J.: A Class of Complete Orthogonal Sequences of Step Functions. Trans. Amer. Math. Soc., *157* (1971), 129—135.
19. SPLETTSTÖSSER, W. and WAGNER, H. J.: Eine dyadische Infinitesimalrechnung für Haar-Funktionen. Zeitschr. Angew. Math. Mech., *57* (1977), 527—544.
20. WADE, W. R.: Sets of Uniqueness for Haar Series. Acta Math. Acad. Sci. Hung., *30* (1977), 265—281.
21. WAGNER, H. J.: Ein Differential- und Integralkalkül in der Walsh-Fourier-Analysis mit Anwendungen. Westdeutscher Verlag, Köln—Opladen, 1973, 73 pp.

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