INVESTIGATION OF A THREE-DIMENSIONAL SOIL CONSOLIDATION PROCESS INVOLVING CREEP

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Summary

The differential equation of the classic seepage problem changes upon creep. Examination of the classic seepage problem omits some factors such as presence of constitutional water, compressibility of pure water, viscosity of soil grains etc., sometimes rather significant for the soil mass settlement. The most important among these factors can always be taken into consideration without excessive mathematical complications. A model taking soil grain viscosity and presence of constitutional water into consideration has been presented. Numerical analyses using the obtained formulae may be economically computerized.

Development of the method to be presented was motivated by KÉZDI, Á.: "A Silo Foundation Story" (In Hungarian), Mélyépítéstud. Szle. No. 3, Vol. XXII. (1972). This paper, referred to under [1], will be kept in mind throughout the discussion. Professor Árpád Kézdi was also consulted in some items of the numerical problem.

Consolidation of the silo foundation can be considered as a problem of axial symmetry permitting one- or two-dimensional treatment. In this case, assuming identical consolidation coefficients in horizontal planes, the consolidation problem is described in a cylindrical coordinate system by:

$$\frac{\partial u(\varrho, z, t)}{\partial t} = a_\varrho^2\left[\frac{\partial^2 u}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial u}{\partial \varrho}\right] + a_3^2\frac{\partial^2 u}{\partial z^2} + f(\varrho, z, t). \tag{1}$$

Provided the examined soil mass rests on an impermeable layer and the construction itself rests directly on the soil, to generate a consolidation process requires to construct a sand pile system such as that seen in Figs 1a and 2a.

Elements of the sand pile system are axial to silos. Cylindrical sand pile radii $r = 0.3$ m, silo diameters $2R = 4.8$ m. No other than radial water flow is possible, making the problem one-dimensional. Because of symmetry, at silo contact surfaces:

$$\left.\frac{\partial u}{\partial \varrho}\right|_{\varrho=R} = 0.$$

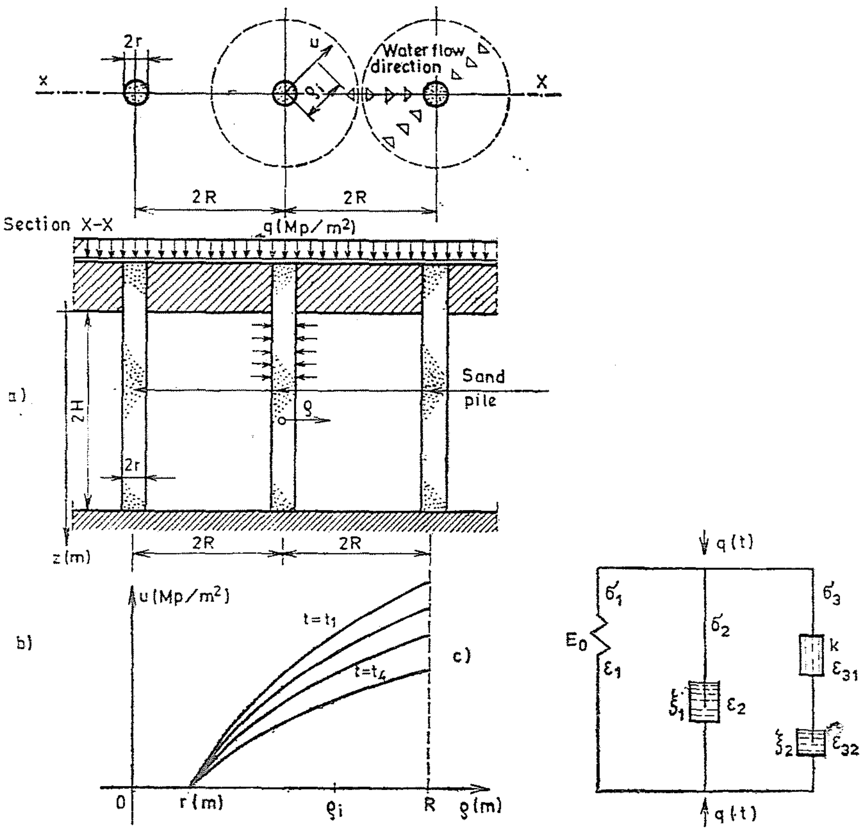

Fig. 1

Curves in Fig. 1b plotted in the system (u, ϱ) start from point $\varrho = r = 0.3$ m, because of zero neutral stress on the sand pile surface. Curves in the figure are isochronous.

Solution of the problem relies on linear algebra, in particular, on the so-called nodal line method (finite strip method) [2]. V. A. FLORIN [3] applied the method of finite differences (so-called nodal point method) to solve several problems. In applying the nodal point method for solving parabolic-type partial differential equations, difficulties arise from meeting the stability condition. Namely, application of the nodal point method imposes several determinations of function values of u of no other use.

This will be demonstrated on a one-dimensional problem of determining the u values meter by meter of depth for a soil consolidation coefficient $c = 0.08$ cm² sec ($\Delta h = 1$ m) from the stability condition

$$c \cdot \Delta t = 1/2 \qquad (2)$$

to yield

$$\Delta t = \frac{1}{2} \cdot \frac{10^3}{8 \cdot 86{,}4} \text{ days} \approx 1 \text{ day}.$$

Thus, daily variations of the u value have to be calculated to obtain the neutral stress value at a time of interest. For a silo system where another, increased load is applied after being erected, the neutral stress in the soil at the start of the second soil loading as initial value cannot be obtained else than by calculating it for every day of the waiting time. The applied nodal line method (rather popular today) consists essentially in discretizing variables ϱ and z, rather than time t in Eq. (1). Thus, the problem is approximated by a system of common, first-order differential equations containing independent variable t, to be solved by closed formulae, continuous in t, delivering the wanted function value at any time. The accuracy can be improved e.g. by systems in [2] and [4]. Maintenance of the continuity exactly of variable t may be explained, in addition to the above, by the stability problem. In the considered case, the solution stability is always granted.

Equation (1) is the classic equation of the consolidation process. Here it will be supplemented by involving the soil mass viscosity i.e. the creep effect, and the effect of constitutional water in the soil on the volume change, according to *Kelvin*'s rheological model.

Equations deduced from the model are:

$$\begin{aligned} &\text{a)} \quad \sigma_1 = E_0 \varepsilon_1 \\ &\text{b)} \quad \sigma_2 = \xi_1 \frac{\partial \varepsilon_2}{\partial t} \\ &\text{c)} \quad -k \nabla^2 \sigma_3 = \frac{\partial \varepsilon_{31}}{\partial t} \\ &\text{d)} \quad \sigma_3 = \xi_2 \frac{\partial \varepsilon_{32}}{\partial t}. \end{aligned} \tag{3}$$

Knowing that $\varepsilon_{31} + \varepsilon_{32} = \varepsilon_2 = \varepsilon_1$ we obtain:

$$-k \xi_2 \nabla^2 \sigma_3 + \sigma_3 = \xi_2 \frac{\partial \varepsilon_1}{\partial t}. \tag{4}$$

Equality

$$\sigma_1 + \sigma_2 + u = q(t) \tag{5}$$

being met, Eqs (3) yield, after some generalization, the wanted equation

$$\frac{\partial u}{\partial t}\left[1 + \frac{\xi_1}{\xi_2}\right] = k\left[E_0 \nabla^2 u + \xi_1 \frac{\partial}{\partial t} \nabla^2 u\right] + \dot{q} - \frac{E_0}{\xi_2} u \tag{6}$$

where $u(t)$ — neutral stress; ξ_1, ξ_2 — moduli of viscosity by volume; E_0 — displacement modulus by volume.

Remark. Analysis of the seepage equation being considered by most experts as sufficient since shear deformation appears at the instant of loading and is time-invariable, only Eq. (6) will be dealt with.

Introducing the proper constants. Eq. (6) becomes:

$$\frac{\partial u}{\partial t} = k\left[\alpha \frac{\partial}{\partial t} \nabla^2 u + \beta \nabla^2 u\right] + \alpha_1 \dot{q}(t) + \beta_1 u. \tag{7}$$

Converted to axial coordinates $\left(\frac{\partial^2 u}{\partial z^2} = 0\right)$, by analogy to Eq. (1):

$$\frac{\partial}{\partial t}\left[u - k\alpha\left(\frac{\partial^2 u}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial u}{\partial \varrho}\right)\right] = k\beta\left[\frac{\partial^2 u}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial u}{\partial \varrho}\right] + \alpha_1 \dot{q} + \beta_1 u. \tag{8}$$

With notations applied in [5], finite differences

$$\frac{\partial^2 u_i}{\partial \varrho^2} \approx \frac{1}{\Delta \varrho^2}\left[u_{i-1} - 2u_i + u_{i+1}\right]$$

$$\frac{1}{\varrho_i}\frac{\partial u_i}{\partial \varrho} \approx \frac{1}{i\Delta\varrho}\,\frac{u_{i+1} - u_{i-1}}{2\Delta\varrho}, \quad (i = 1, 2, \ldots, n)$$

lead to equations of the form

$$\frac{\partial^2 u_i}{\partial \varrho^2} + \frac{1}{\varrho_i}\frac{\partial u_i}{\partial \varrho} \approx \left[\frac{1+n}{R}\right]^2\left[u_{i-1}\left(1 - \frac{1}{2i}\right) - 2u_i + u_{i+1}\left(1 + \frac{1}{2i}\right)\right].$$

Written for each i and involved in a matrix equation, (8) becomes:

$$\left[\mathbf{E} - \alpha k\left(\frac{1+n}{R}\right)^2 \mathbf{A}\right]\frac{du}{dt} = \left[k\beta\left(\frac{1+n}{R}\right)^2 \mathbf{A} + \beta_1 \mathbf{E}\right]u + \alpha_1 \dot{q} e. \tag{9}$$

Introducing notations

$$\mathbf{K} = \left[\mathbf{E} - \alpha k\left(\frac{1+n}{R}\right)^2 \mathbf{A}\right]^{-1}$$

$$\mathbf{M} = \mathbf{K}\left[k\beta\left(\frac{1+n}{R}\right)^2 \mathbf{A} + \beta_1 \mathbf{E}\right], \tag{10}$$

Eq. (9) may be written as an inhomogeneous differential equation with a first-order normal matrix coefficient:

$$\frac{du}{dt} = \mathbf{M}u + \alpha_1 \dot{q}\mathbf{K}e \tag{11}$$

taking boundary condition of the problem into consideration, thereby solution of (11) has only to meet initial condition $u_0 = u(t = 0)$. In compliance with results in [6], (11) is solved in the form:

$$u(t) = e^{\mathbf{M}t} u_0 + \alpha_1 \int_0^t \dot{q}(\tau) e^{\mathbf{M}(t-\tau)} \cdot \mathbf{K} e \, d\tau. \tag{12}$$

Matrix **A** in matrix **M** is of the form:

$$\mathbf{A} = \begin{bmatrix} -2 & 1.5 & 0 & \dots & 0 \\ 0.75 & -2 & 1.25 & \dots & 0 \\ 0 & 0 & 0 & \dots & \dfrac{2n+3}{2(n+1)} \\ 0 & 0 & 0 & \dots & -2 \end{bmatrix}$$

symmetrizable, permitting it to be written in canonic form.

For numerical analyses, solution (12) can be written by linear algebraic means. [6] presented matrix functions in canonic form as:

$$F(\mathbf{M}) = \sum_{k=1}^{M} F(\tau_k) v_k \cdot w_k^*, \tag{13}$$

where λ_k is k-th eigenvalue, v_k and w_k^* are k-th left- and right-hand-side eigenvectors, resp., of matrix **M**.

Solution (12) may be written as

$$u(t) = \sum_{k=1}^{n} e^{\lambda_k t} v_k \cdot w_k^* u_0 + \alpha_1 \sum_{k=1}^{n} \int_0^t e^{\lambda_k (t-\tau)} v_k \cdot w_k^* \mathbf{K} e \dot{q}(\tau) \, d\tau. \tag{14}$$

(For calculating λ_k, v_k and w_k^*, see [8].)

To accelerate the consolidation process, a gravel layer about 20 cm thick is inserted between the construction and the clayey soil, permitting the water two-way seepage, making the problem a two-dimensional one (Fig. 2a).

In this case, solution of Eq. (1) has to meet [beside condition $f(\varrho, z, t) = 0$] the following conditions:

$$\begin{aligned} &\text{a)} \quad \nabla u(\varrho, z, t = 0) = \nabla u_0 \\ &\text{b)} \quad u(\varrho, z = 0, t) = 0 \\ &\text{c)} \quad u(\varrho = r, z, t) = 0 \\ &\text{d)} \quad \left. \frac{\partial u}{\partial z} \right|_{z=2H} = 0 \\ &\text{e)} \quad \left. \frac{\partial u}{\partial \varrho} \right|_{\varrho=R} = 0. \end{aligned} \tag{15}$$

Solution of this problem using *Bessel* functions is found in [8] and [9]. Application of the lattice point method see in [3].

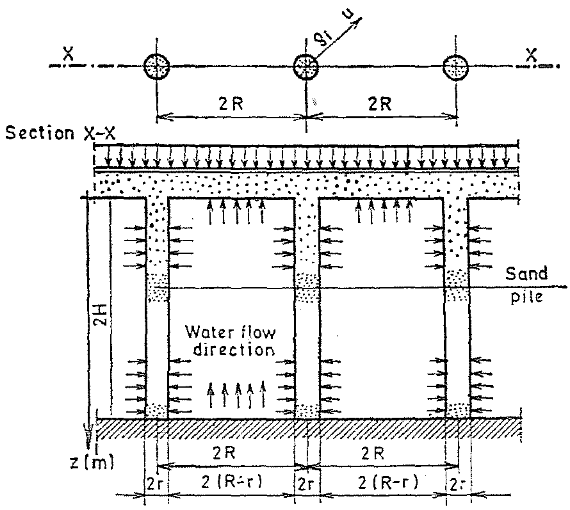

Fig. 2

Replacing partial derivatives with respect to variables ϱ and z by finite differences leads to the ordinary differential equation system:

$$\frac{d\mathbf{U}(t)}{dt} = \mathbf{M}\mathbf{U}(t) + \mathbf{U}(t)\mathbf{A}^*. \tag{16}$$

According to [11], solution of (16) is of the form:

$$\mathbf{U} = e^{\mathbf{M}t} \cdot \mathbf{u}_0 \cdot e^{\mathbf{A}^*t}. \tag{17}$$

In the one-dimensional case, elements of vector $u_0 = u(t = 0)$ are directly obtained from the load, while in the two-dimensional case, matrix $\mathbf{U}_0$ results as solution of a *Dirichlet* problem. Partial differential equation

$$\nabla^2 U = 0 \tag{18}$$

has to be solved by means of the nodal point method seen also in [4].

Numerical determination of solution [17] may rely on canonic forms of matrices **M** and **A***.

Let us consider now the concrete numerical problem. Its data being taken from [1] where data for characteristics of our rheological model are missing, the numerical problem will be presented for Eq. (1).

A water saturated silt layer 12 m thick has been the subsoil to bear a $15 \times 30 \times 40$ m system of 18 silos. The surface of the load-bearing soil is considered to be watertight. The silos are built in the slipform system. Reckoning with about one third of subsidence to occur during construction, at the bottom level of the layer — the worst position — there was a neutral stress

$u \approx 5.6$ Mp/m² by the end of construction. To accelerate the consolidation process, a sand-pile system was built. A few months after the end of the construction the neutral stress dropped to about zero, permitting the silos to be filled. Filling was intermittent, with partial emptying each three months, so filling was complete after 11 months applying a load of 20 Mp/m². The process paralleling the construction is described by the solution of Eq. (1) at an initial condition $u(\varrho, t = 0) = 0$ (omitting dead load). Boundary conditions are $u(\varrho = r, t) = 0$ on the sand pile surface, and

$$\left.\frac{\partial u}{\partial \varrho}\right|_{\varrho=R} = 0 \text{ below the silo edge.}$$

Accordingly, solution of (1):

$$u(t) = \beta \frac{e^{\mathbb{A}t} - \mathbb{E}}{\mathbb{A}} e \tag{19}$$

$\mathbb{E}$ being unit matrix.

Spectral decomposition of matrix $\mathbb{A}$ yields for the i-th element of vector (19):

$$u_l(i) = \beta \sum_{k=1}^{5} \frac{e^{\lambda_k t} - 1}{\lambda_k} v_{ik} \left(\sum_{j=1}^{5} w_{kj}^{*}\right). \tag{20}$$

The q value being given as 8 Mp/m², the value of β has to be calculated from (20) so that the max. neutral stress in [1] may be $u \approx 5.6$ Mp/m². Thus, the value in (18) has been calculated at several times in the period from 10 to 25 days — as in [1] —, the optimum was found to be 12 days, arguing for $\beta = 20$. Neutral stresses imposed by the construction for different building rates are seen in Fig. 3, together with the timely course of consolidation after

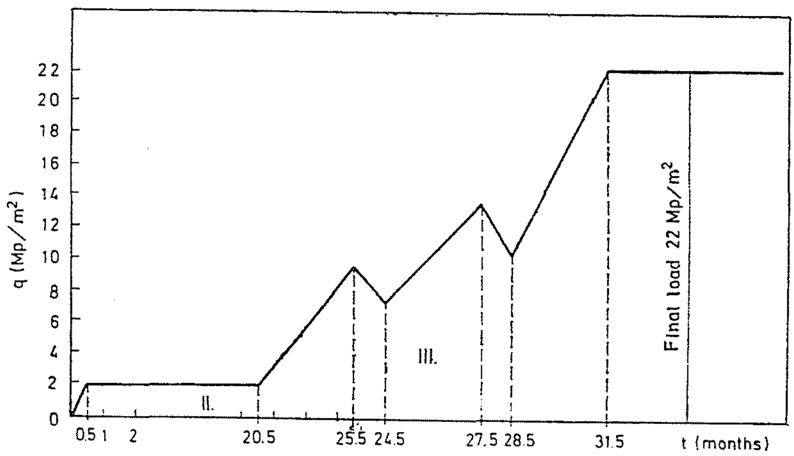

Fig. 3. Load diagram. I. During construction. II. Left to consolidate. III. Silo loading

the building was complete. Consolidation coefficient contained in λ_k was taken as 0.08 cm²/sec after [1].

Now, the u_I value after 12 days was taken as initial value in analyzing the process. The time where the value

$$u_{II}(i) = \sum_{k=1}^{5} e^{\lambda_k t} v_{ik}\Big(\sum_{j=1}^{5} w_{kj}^{*} u_I(j)\Big), \quad (i = 1, 2, \ldots, 5) \tag{21}$$

was lower than 10^{-2} Mp/m² was computer determined as start of linear silo loading, to become 7.5 Mp/m² after three months. At this time unloading followed for a month, at the rate of filling. In the next three months, a load of 6.5 Mp/m² was applied on the silo system, followed by one month of unloading by 3.5 Mp/m². In the subsequent three months, the missing 12 Mp/m² were added (Fig. 4). Neutral stresses developed during filling are seen in Fig. 5. The applied mathematical formulas are combinations of (20) and (21).

Remark 1. Computations raised unloading problems. How to assume the neutral stress decrease?

a) Unloading may be assumed to be instantaneous, causing the development of negative neutral stresses, still increasing after the second unloading. The problem becomes two-dimensional if a draining layer is applied under the construction.

Now, the negative neutral stress develops slower, a case obviously impossible in practice.

b) Unloading may be protracted in time, causing neutral stresses to be considered as negative. In this case, neutral stresses due to unloading have

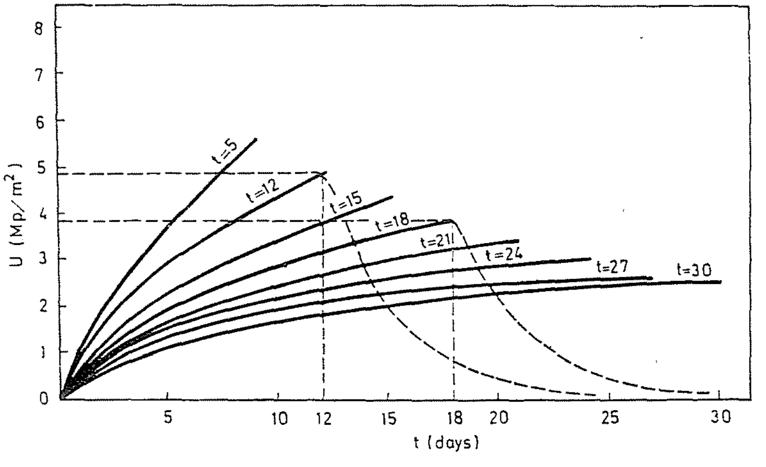

Fig. 4. $t = 12$ has been calculated as the admissible shortest time of completing a construction
——— neutral stress values under the same load developing during different times
– – – – consolidation in time

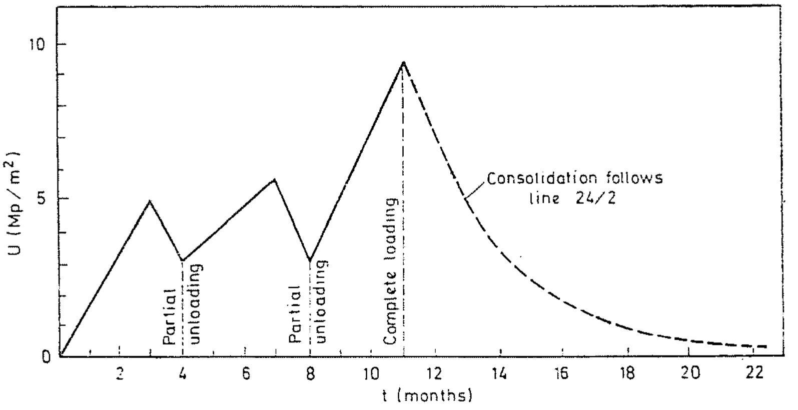

Fig. 5. Silo filling intermetting two partial emptyings

been deduced from those prevalent at the instant of unloading, considered to be more realistic. To our knowledge, this important fact has not yet been described in literature.

The cases above have all been computed all along, and our relevant program is available.

Another alternative in solving the problem was to consider filling as instantaneous. Still other alternatives tested were filling linearly varying with time, for different time intervals, examining also consolidation processes. The problem was also solved for a varying consolidation coefficient where the soil mass got compacted during construction and rest time, as seen in Fig. 6.

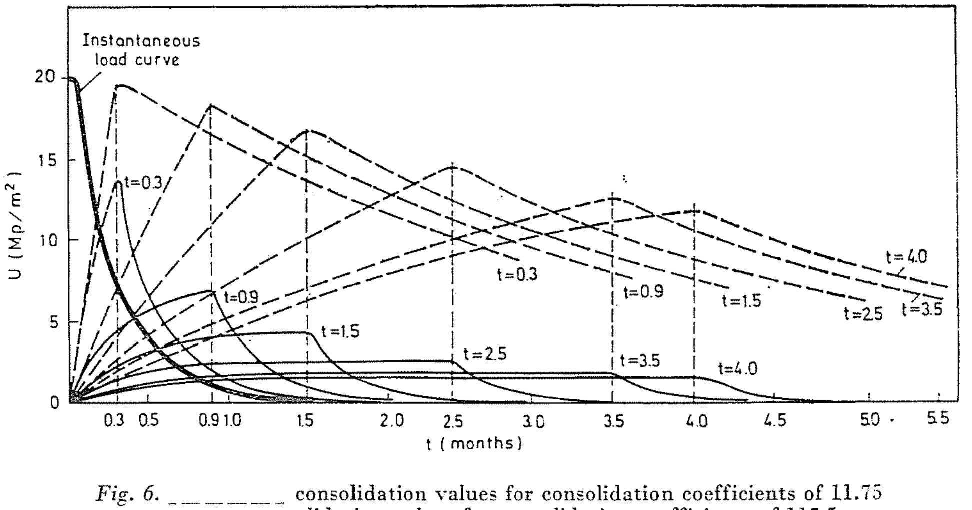

Fig. 6. _ _ _ _ _ _ consolidation values for consolidation coefficients of 11.75
——— consolidation values for consolidation coefficients of 117.5

Remark 2. Depth is irrelevant to one-dimensional problems involving sand piles.

Remark 3. In a two-dimensional problem, neutral stress in the impermeable layer is half that in the one-dimensional problem.

Remark 4. Subject of [12] is the same as that of this paper. Its Eq. (8) is formally similar to our Eq. (7). In our case, this equation has been derived from a rheological model. In [12], it was written assuming separability of primary and secondary consolidation processes. Methods of solving these equations are quite different, our method seems to be more advantageous for numerical calculation.

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