A RELATION BETWEEN INVERSION NUMBER AND LEXICOGRAPHIC ORDERING OF PERMUTATIONS

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Summary

Relation between serial number of permutations in lexicographic order and inversion number of the considered permutations is examined. The finally resulting formula is an application of writing an arbitrary integer number in factorial notation.

The problem mentioned above is important in both theoretical and applied mathematics.

The aim of this paper is to determine the number of inversions in an arbitrary permutation on the basis of the serial number of the permutation in lexicographic order, without writing down the permutation itself.

The suggested theorem is introduced by some notations and definitions and then it will be proved.

Let P_N denote the set of permutations of the finite set N. For any $\pi \in P_N$, $I(\pi)$ denotes the number of inversion in π. Further let $l(\pi)$ denote the serial number of permutation $\pi \in P_N$ in the lexicographic list of all permutations of N.

Definition: Let $1 = b_0, b_1, b_2, \ldots$ be a monotonically increasing sequence of natural numbers. Any natural number n can be written in a unique form as

$$\sum_{i=0}^{k} a_i b_i$$

where the a_i are non-negative integer numbers and for every i, $a_i b_i < b_{i+1}$. The set $a_k, \ldots, a_0$ is called the form of n in the system of base numbers $(b_1, b_2, \ldots)$ while numbers a_i are the digits of the form in question. If needed, number n is allowed to be written in the form $0, 0, \ldots, 0, a_k, \ldots, a_0$ (in case it seems useful to write all numbers by as many numerals. The case $b_1 = b_2 = \ldots = 10$ is the common decimal writing of the natural numbers.)

Definition: In the special case $b_i = (i+1)!$, the writing in the system of base numbers $(2!, 3!, \ldots)$, is called briefly factorial notation.

Theorem: Let $\pi \in P_N$ be an arbitrary permutation and the form of

$$l(\pi) - 1$$

be $a_k, \ldots, a_0$ in factorial notation.

Then

$$I(\pi) = \sum_{t=0}^{k} a_t. \tag{1}$$

Proof: The theorem will be proved by induction on the number $|N|$ of elements in the set N. Cases $|N| = 1, 2$ are trivial.

Let $|N| = n > 2$ and for the sake of simplicity, $N = \{1, 2, \ldots, n\}$. Let us consider a permutation $\pi = (i, j_2, \ldots, j_n) \in P_N$. As the first element in π is i, therefore $(i-1)(n-1)! \leq l(\pi) - 1 < i(n-1)!$, consequently the first digit of (1) in factorial notation is $i - 1$ (the total number of digits $n - 1$; starting a number with zeros is allowed). Let us consider the function

$$f(k) = \begin{cases} k, & \text{if} \quad k < i \\ k-1, & \text{if} \quad k > i \end{cases}$$

defined on the set $N/\{i\}$. Then

$$l(\pi) - 1 = (i-1)(n-1)! + (l(\pi') - 1), \tag{2}$$

where π' stands for the permutation

$$((f(j_2), \ldots, f(j_n)) \in P_{N'}, \qquad (N' = \{1, 2, \ldots, n-1\}).$$

As i forms an inversion with $i - 1$ elements on the first place of π (namely with elements $1, \ldots, i-1$) therefore

$$I(\pi) = i - 1 + I((j_2, \ldots, j_n)). \tag{3}$$

On the other hand,

$$I((j_2, \ldots, j_n)) = I((f(j_2), \ldots, f(j_n))) = I(\pi'). \tag{4}$$

Since $\pi' \in P_{N'}$ and $N' = n - 1$, we can assume by induction:

$$I(\pi') = \sum_{t=0}^{n-3} c_t \tag{5}$$

where $c_{n-3}, \ldots, c_0$ means $l(\pi') - 1$ in the factorial notation.

According to (2) and taking also (5) into consideration:

$$l(\pi) - 1 = (i-1)(n-1)! + \sum_{t=0}^{n-3} c_t(t+1)!.$$

Hence $l(\pi) - 1$ is seen to be written in factorial notation as $i - 1, c_{n-3}, c_{n-4}, \ldots, c_0$.

Its digits sum up to

$$i - 1 + \sum_{t=0}^{n-3} c_t \overset{(5)}{=} (i-1) + I(\pi') \overset{(4)}{=} i - 1 + I((j_2, \ldots, j_n)) = I(\pi).$$

Q.E.D.

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