A GENERALIZATION OF THE MAIN THEOREM OF THE PROJECTIVE MAPS IN TWO-DIMENSIONAL REAL PLANES

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Summary

This paper proves J. Bognár's conjecture that if the range of a transformation of the real projective plane is the whole plane and this transformation holds the collinearity of any three points of the plane then this transformation is a one to one mapping.

J. BOGNÁR *(Department of Geometry, Eötvös L. University)* raised the problem if the map f of the next properties is a collineation:

1. f is a map of the (real) projective plane Σ onto the other projective plane Σ';

2. if A, B, C are three points of a line of Σ then there is a suitable line of Σ' which contains the images $f(A), f(B), f(C)$.

According to the original one the next theorem proves the answer to be "yes":

Theorem: Let f be a map of the real projective plane into the same plane with the property of holding collinearity (cf. 2) and fixed points A, B, C, E of general position. Thereby f is an identical map.

Proof: Let us denote the intersection point of lines (AE) and (BC) by $(AE) \cap (BC) = U$ and two other intersection points by $(BE) \cap (AC) = V$ and $(CE) \cap (AB) = W$.

Let $R \cup \{\infty\}$ be denoted by $\overline{R}$, where R is the set of real numbers and ∞ is out of R and define the function $\overline{f} : \overline{R} \to \overline{R}$ as for $P \in (AC)$; if $f(P) \neq C$ and $P \neq C$ then $\overline{f}((ACPV)) = (ACf(P)V)$ where $(QRST)$ denotes the double ratio of Q, R, S and T, and in any other case, be $\overline{f}(x) = \infty$.

Remark: Replacing C and V by B and W, respectively, the definition of $\overline{f}$ provides the same function because denoting $(BC) \cap (VW)$ by R and $(AB) \cap (RP)$ by P', R is evidently a fixed point of f so not only $(ACPV) =$ $= (ABP'W)$ but because of holding collinearity, $(ACf(P)V) = (ABf(P')W)$.

Notations; In case of $a \in \bar{R}$-$\{0\}$, denote $a \cdot \infty$ by ∞; $\frac{a}{\infty}$ by 0, and $\frac{a}{0}$ by ∞.

Lemma 1: If λ and $\mu \in \bar{R}$, $\bar{f}(\lambda) \neq 0$ and $\neq \infty$ then $\bar{f}\left(\frac{\mu}{\lambda}\right) = \frac{\bar{f}(\mu)}{\bar{f}(\lambda)}$.

Proof of lemma 1: Be $P, Q \in AC$ namely $(ACPV) = \lambda$ and $(ACQV) = \mu$. Denoting $(WP) \cap (BC)$ by T, $T \neq B, C$ for $\bar{f}(\lambda) \neq 0, \infty$ $\lambda \neq 0, \infty$, and for the same reason $f(T) = (f(P)W) \cap (BC) \neq B, C$. Denoting $(QT) \cap (AB)$ by S because of the projectivity of the centre T it holds that $(ABSW) = (ACQP) = \frac{(ACQV)}{(ACPV)} = \frac{\mu}{\lambda}$ and for the projectivity of centre $f(T)$, $(ABf(S)W) = (ACf(Q)f(P)) = \frac{(ACf(Q)V)}{(ACf(P)V)} = \frac{\bar{f}(\mu)}{\bar{f}(\lambda)}$, leading, in compliance with the remark after the definition of function f, to:

$$\bar{f}\left(\frac{\mu}{\lambda}\right) = \frac{\bar{f}(\mu)}{\bar{f}(\lambda)} \text{ as expected.}$$

Lemma 2: For any $\lambda \in \bar{R}$, $\bar{f}\left(\frac{1}{\lambda}\right) = \frac{1}{\bar{f}(\lambda)}$.

Proof of lemma 2: If $\bar{f}(\lambda) \neq 0$ or ∞ then the equality is evident from lemma 1 and from $\bar{f}(1) = (ACf(V)V) = 1$ considering λ as $(ACPV)$. If $\bar{f}(\lambda) = 0$ or ∞ then with notations of lemma 1 requiring $Q = V$ it also holds that $(ABSW) = \frac{\mu}{\lambda} = \frac{1}{\lambda}$ but $\bar{f}\left(\frac{1}{\lambda}\right)$ will be of order ∞ or 0 for $Q = V$ as stated in this lemma.

Lemma 3: If $\bar{f}(\lambda) \neq 0, \infty$ then $\bar{f}(\lambda \cdot \mu) = \bar{f}(\lambda) \cdot \bar{f}(\mu)$.

Proof of lemma 3: Using the previous lemmas
$\bar{f}(\lambda\mu) = \bar{f}\left(\frac{\mu}{1/\lambda}\right)$, where $\bar{f}\left(\frac{1}{\lambda}\right) = \frac{1}{\bar{f}(\lambda)} \neq 0, \infty$ thus, the equation holds.

Lemma 4: $\lambda > 0$ involves $\bar{f}(\lambda) > 0$ or $\bar{f}(\lambda) = 0$ or $\bar{f}(\lambda) = \infty$.

Proof of lemma 4: As there is μ, $\mu^2 = \lambda$ so $\mu = \lambda \cdot \frac{1}{\mu}$ in conformity with lemma 3, $\bar{f}(\lambda) = 0$ or $\bar{f}(\lambda) = \infty$ or $f(\mu) = \bar{f}(\lambda) \frac{1}{\bar{f}(\mu)}$, implying the statement of this lemma.

Let us complete the proof of the theorem with the help of lemma 4! Consider points A, B and C constituting the triangle as system of projective coordinates with the unity point E. Because of holding collinearity, the points of binary fraction coordinates are fixed points of map f and simply the fact has to be verified that f cannot change separation on a line of triangle ABC. In an indirect way, assume that among four points, the first, the second, the third and the fourth are A, B, X and V, respectively, and it holds that $(ABXV) > 0$ but $(f(A)f(B)f(X)f(V)) < 0$. But it is inconsistent with lemma 4 because with the same notation, $\lambda = (ABXV)$ would imply the simultaneity of $\lambda > 0$ and $\bar{f}(\lambda) < 0$. This completes the proof of the theorem.

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