NONDIMENSIONAL STABILITY TESTS OF WATER LEVEL CONTROL IN CANALS

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Introduction

Automation of water control is a significant and actual problem in water management, occurring both with irrigation and diversion canals and with drainage and seepage canals. The common method of automation is the control of water level in a given canal cross section. Control structures in controlled-level canal systems obtain information about changes in water resources of canal sections from water level changes in the given cross section. The obtained information is relied upon in changing the discharges in each canal to match the initial condition.

Difficulties arise in approximating the stability problem of water level controls from:

- nonlinear, partial differential equations describing flow conditions in canal sections;
- nonlinearities in operating mechanisms of control structures;
- complexity of water control systems.

There is internationally a scarcity of publications in this scope [3]. Hungarian references see under [6, 8]. These papers practically give only the analysis, rather than the synthesis of the problem.

Considering the complexity of the problem, the synthesis of stability of water level control seems to be the simplest by evaluating properly accomplished analyses. These analyses can be rather expedient in nondimensional models.

Nondimensional (generalized) expressions will be given for a test series suitable as an approach to the synthesized stability problem. The examined basic system consists of a controlled canal section and a control structure.

Notations:

q — discharge;
h — depth of flow;
h_k — mean depth of flow;
i — slope of water surface;

v — mean velocity of flow;
c — velocity of wave propagation;
b — width of flow surface;
f — area of flow cross section;
k — perimeter of flow cross section;
ϱ — bank slope;
t — time;
x — space coordinate of canal;
Δ — change;
l — length of canal;
$\mathbf{s}$ — $\partial/\partial t$-operator (1/s);
$\mathbf{p}$ — $\partial/\partial\tau$-operator (1);
Y — transfer function;
j — imaginary unit.

Subscript 0 refers to initial flow condition. Other notations are defined in the paper.

Description of controlled canal sections

The transient flow conditions in open canals are described by the *Saint-Venant* equations in a general case [4]. However, this paper discusses only controlled level canals having two essential features:
— they are prismatic;
— water levels change only within a limited operation range.

Due to these features, transient flow conditions can be described by the theory of small amplitude waves [4].

The linearized *Saint-Venant* equations are the following [4, 8]:

$$c_1^2 \frac{\partial^2 \Delta h}{\partial x^2} - \frac{\partial^2 \Delta h}{\partial t^2} - 2v_0 \frac{\partial^2 \Delta h}{\partial x\, \partial t} - c_0^2 i_0 \varkappa \frac{\partial \Delta h}{\partial x} - \frac{2 i_0 g}{v_0} \frac{\partial \Delta h}{\partial t} = 0 \tag{1}$$

$$b_0 \frac{\partial \Delta h}{\partial t} + \frac{\partial \Delta q}{\partial x} = 0 \tag{2}$$

where $c_1^2 = c_0^2 - v_0^2$

$$\varkappa = \frac{1}{h_{k_0}} \left(\frac{10}{3} - \frac{8}{3} \frac{\varrho h_{k_0}}{k_0} \right).$$

Let us introduce the following relative variables:

$\eta = \dfrac{\Delta h}{h_0}$ — relative water level change;

$\vartheta = \dfrac{\Delta q}{q_0}$ — relative discharge change;

$\xi = \dfrac{x}{l}$ — relative space coordinate;

$\tau = \dfrac{t}{t_0}$ — relative time, where t_0 is the period of natural oscillation in the canal: $t_0 = \dfrac{2lc_0}{c_1^2}$.

Dividing Eq. (1) by $\dfrac{c_1^2 h_0}{l^2}$ and Eq. (2) by q_0/l, the following nondimensional linear equation system is obtained:

$$\frac{\partial^2\eta(\xi,\tau)}{\partial\xi^2} - \frac{c_1^2}{4c_0^2}\frac{\partial\eta^2(\xi,\tau)}{\partial\tau^2} - \frac{v_0}{c_0}\frac{\partial^2\eta(\xi,\tau)}{\partial\xi\,\partial\tau} -$$

$$- \frac{c_0^2 i_0 l}{c_1^2 h_{k_0}}\left(\frac{10}{3} - \frac{8}{3}\frac{\varrho h_{k_0}}{k_0}\right)\frac{\partial\eta(\xi,\tau)}{\partial\xi} - \frac{i_0 l_g}{v_0 c_0}\frac{\partial\eta(\xi,\tau)}{\partial\tau} = 0 \tag{3}$$

$$\frac{c_1^2 b_0 h_0}{2c_0 f_0 v_0}\frac{\partial\eta(\xi,\tau)}{\partial\tau} + \frac{\partial\vartheta(\xi,\tau)}{\partial\xi} = 0\,. \tag{4}$$

Substituting:

$$\frac{v_0}{c_0} = Fr\,; \quad \frac{i_0 l}{h_{k_0}} = R\,; \quad \frac{10}{3} - \frac{8}{3}\frac{\varrho h_{k_0}}{k_0} = M\,; \quad \frac{f_0}{b_0 h_0} = N$$

transforms Eqs (3) and (4) to:

$$\frac{\partial^2\eta}{\partial\xi^2} - \frac{1-Fr^2}{4}\frac{\partial^2\eta}{\partial\tau^2} - Fr\frac{\partial^2\eta}{\partial\xi\,\partial\tau} - \frac{RM\,\partial\eta}{1-Fr^2\,\partial\xi} - \frac{R}{Fr}\frac{\partial\eta}{\partial\tau} = 0 \tag{5}$$

$$\frac{1-Fr^2}{2Fr\,N}\frac{\partial\eta}{\partial\tau} + \frac{\partial\vartheta}{\partial\xi} = 0\,. \tag{6}$$

Applying operator $\mathbf{p} = \partial/\partial\tau$ transfers Eqs (5) and (6) to the operator range:

$$\frac{d^2H(\xi,\mathbf{p})}{d\xi^2}\,\frac{1-Fr^2}{4}\left(\mathbf{p}^2H(\xi,p) - \mathbf{p}\eta_0(\xi) - \eta_0'(\xi)\right) -$$

$$- Fr\left(\mathbf{p} - \frac{dH(\xi,\mathbf{p})}{d\xi} - \eta_0(\xi)\right) - \frac{RM}{1-Fr^2}\frac{dH(\xi,\mathbf{p})}{d\xi} - (\mathbf{p}H(\xi,\mathbf{p}) - \eta_0(\xi)) = 0 \tag{7}$$

$$\frac{1-Fr^2}{2Fr\,N}(\mathbf{p}H(\xi,\mathbf{p}) - \eta_0(\xi)) + \frac{d\theta(\xi,\mathbf{p})}{d\xi} = 0\,. \tag{8}$$

Supposing permanent initial flow condition, the general solutions of Eqs (7) and (8) are [4]:

$$H(\xi, \mathbf{p}) = C_1(\mathbf{p}) \exp(r_1(\mathbf{p})\xi) + C_2(\mathbf{p}) \exp(r_2(\mathbf{p})\xi) + \eta_0(\xi) \tag{9}$$

$$\Theta(\xi, \mathbf{p}) = A(\mathbf{p}) C_1(\mathbf{p}) \exp(r_1(\mathbf{p})\xi) + B(\mathbf{p}) C_2(\mathbf{p}) \exp(r_2(\mathbf{p})\xi) \tag{10}$$

where

$$A(\mathbf{p}) = -\frac{1 - Fr^2}{2FrN} \frac{\mathbf{p}}{r_1(\mathbf{p})}$$

$$B(\mathbf{p}) = -\frac{1 - Fr^2}{2FrN} \frac{\mathbf{p}}{r_2(\mathbf{p})}$$

$$r_{1,2}(\mathbf{p}) = \frac{RM}{2(1 - Fr^2)} + \frac{Fr}{2}\mathbf{p} \pm \frac{1}{2}\left(\mathbf{p}^2 - \mathbf{p}\left(\frac{2RMFr}{1 - Fr^2} + 4\frac{R}{Fr}\right) + \right.$$
$$\left. + \frac{R^2M^2}{(1 - Fr^2)^2}\right)^{1/2} = \frac{RM}{2(1 - Fr^2)} + \frac{Fr}{2}\mathbf{p} \pm \frac{1}{2}F(\mathbf{p}).$$

$C_1(\mathbf{p})$ and $C_2(\mathbf{p})$ are integration constants depending on boundary conditions. In determining boundary conditions, the two cases of control basic system have to be considered [6] (Fig. 1):

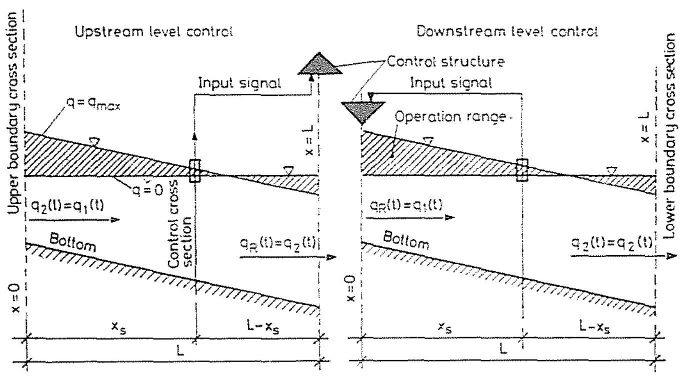

Fig. 1. Schemes of water level control systems

— in case of upstream level control the disturbing effects (input signals) as discharge changes set up in the upper boundary section. Control structures operate in the opposite (lower) boundary sections. Their discharge changes represent the modifying effects, namely:

$$\vartheta_Z(\xi = 0, \tau) = \vartheta_1(\tau) \to \Theta_1(\mathbf{p})$$

$$\vartheta_R(\xi = 1, \tau) = \vartheta_2(\tau) \to \Theta_2(\mathbf{p})$$

— in cases of downstream level control the loci of disturbing and modifying effects are opposite to the previous ones, namely:

$$\vartheta_Z(\xi = 1, \tau) = \vartheta_2(\tau) \to \Theta_2(\mathrm{p})$$

$$\vartheta_R(\xi = 0, \tau) = \vartheta_1(\tau) \to \Theta_1(\mathrm{p}).$$

Considering these boundary conditions, the general solutions are:

$$H(\xi, \mathrm{p}) = \frac{-2FrN}{(1 - Fr^2)(1 - \exp(-F(\mathrm{p})))} \frac{1}{\mathrm{p}} (\Theta_1(\mathrm{p})(r_2(\mathrm{p}) \exp(r_2(\mathrm{p})\xi) -$$
$$- r_1(\mathrm{p}) \exp(-r_1(\mathrm{p})(1 - \xi) - r_2(\mathrm{p})) + \Theta_2(\mathrm{p}) \times$$
$$\times \left(r_1(\mathrm{p}) \exp(-r_1(\mathrm{p})(1 - \xi) - r_2(\mathrm{p}) \exp(-r_1(\mathrm{p}) + r_2(\mathrm{p})\xi)\right)) \quad (11)$$

$$\Theta(\xi, \mathrm{p}) = \frac{1}{1 - \exp(-F(\mathrm{p}))} \left(\Theta_1(\mathrm{p})\left(\exp(r_2(\mathrm{p})\xi)\right) - \exp\left(-r_1(\mathrm{p})(1 - \xi) +\right.\right.$$
$$\left.\left. + r_2(\mathrm{p})\right) + \Theta_2(\mathrm{p})\left(\exp\left(-r_1(\mathrm{p})(1 - \xi) - \exp(-r_1(\mathrm{p}) + r_2(\mathrm{p})\xi\right)\right)\right). \quad (12)$$

Solution (11) is already a possibility to describe the nondimensional transfer functions of controlled canals:

— the transfer function of relative water level change ($\eta_1(\xi, \tau)$) due to relative discharge changes ($\vartheta_1(\tau)$):

$$Y_{H1}(\xi, \mathrm{p}) = \frac{-d_1(\xi)}{1 - \exp(-F(\mathrm{p}))} \frac{1}{\mathrm{p}} (L_1(\mathrm{p})\, I_1(\xi, \mathrm{p})\, E_1(\xi, \mathrm{p}) -$$
$$- L_2(\mathrm{p})\, I_2(\xi, \mathrm{p})\, E_2(\xi, \mathrm{p})) \quad (13)$$

— the transfer function of relative water level change ($\eta_2(\xi, \tau)$) due to relative discharge changes ($\vartheta_2(\tau)$):

$$Y_{H2}(\xi, \mathrm{p}) = \frac{-d_2(\xi)}{1 - \exp(-F(\mathrm{p}))} \frac{1}{\mathrm{p}} (L_2(\mathrm{p})\, I_3(\xi, \mathrm{p})\, E_3(\xi, \mathrm{p}) -$$
$$- L_1(\mathrm{p})\, I_4(\xi, \mathrm{p})\, E_4(\xi, \mathrm{p})) \quad (14)$$

where

$$d_1(\xi) = \frac{FrN}{1 - Fr^2} \exp \frac{RM\xi}{2(1 - Fr^2)}; \quad d_2(\xi) = \frac{FrN}{1 - Fr^2} \exp \frac{-RM(1 - \xi)}{2(1 - F_r^2)};$$

$$L_1(\mathrm{p}) = \frac{RM}{1 - Fr^2} + Fr\,\mathrm{p} - F(\mathrm{p}); \quad L_2(\mathrm{p}) = \frac{RM}{1 - Fr^2} + Fr\mathrm{p} + F(\mathrm{p})$$

$$I_1(\xi, \mathrm{p}) = \exp\left(\frac{\xi}{2}(\mathrm{p} - F(\mathrm{p}))\right); \quad E_1(\xi, \mathrm{p}) = \exp\left(-\frac{\xi}{2}(1 - Fr)\,\mathrm{p}\right)$$

$$I_2(\xi, \mathrm{p}) = \exp\left(\frac{1}{2}(2 - \xi)(\mathrm{p} - F(\mathrm{p}))\right); \quad E_2(\xi, \mathrm{p}) = \exp\left(-\frac{1}{2}(2 - \xi)(1 + Fr)\mathrm{p}\right)$$

$$I_3(\xi,\mathbf{p}) = \exp\left[\frac{1}{2}(1-\xi)(\mathbf{p}-F(\mathbf{p}))\right]; \quad E_3(\xi,\mathbf{p}) = \exp\left[-\frac{1}{2}(1-\xi)(1+Fr)\mathbf{p}\right]$$

$$I_4(\xi,\mathbf{p}) = \exp\left[\frac{1}{2}(1+\xi)(\mathbf{p}-F(\mathbf{p}))\right]; \quad E_4(\xi,\mathbf{p}) =$$

$$= \exp\left[-\frac{1}{2}(1+\xi)\left(1+\frac{1-\xi}{1+\xi}\right)\mathbf{p}\right].$$

Description of control structures

The other basic element of water level control system is the control structure. Its discharge is described by: [8]

$$q_R(t) = a_0\, m(t)\, \left(z_f(t) - z_a(t)\right)^{\delta} \tag{15}$$

where

a_0 — constant area of a typical flow section in the structure;
$m(t)$ — discharge coefficient, changing according to the dynamic features of controller;
$z_f(t)$ and $z_a(t)$ — stages of head- and tailwater, resp.;
δ — constant, depending on the features of flow.

In this paper the general description of discharge change in control structures is omitted to get simpler approach [8]. The simplifying condition is the following:
— the effect of changes in head- and tailwater levels can be neglected in comparison to the discharge change caused by changes in the controlled water level.

In this case the relative discharge change of the control structure is:

$$\vartheta_R(t) = \frac{\Delta q_R(t)}{q_0}\,\frac{\partial \dfrac{m}{m_0}}{\Delta h_s}\,\Delta h_s = \frac{\partial \mu}{\partial \eta_s}\,\Delta\eta_s(t) \tag{16}$$

where

$\mu = \dfrac{m}{m_0}$ — relative change in discharge coefficient;

$\Delta\eta_s = \dfrac{\Delta h_s}{h_0}$ — relative change in the controlled water level:

The change $\Delta\eta_s(t)$ is caused by two effects:

— by disturbing effect $\vartheta_2(t)$, $\eta_b(t)$ and
— by modifying effect $\vartheta_R(t)$ caused by $\Delta\eta_s(t)$.

Considering these the *Laplace* transform of the relative discharge change of the control structure is:

$$\Theta_R(\mathbf{s}) = Y_{Rs}(\mathbf{s})\,(H_{sb}(\mathbf{s}) + H_{rs}(\mathbf{s})) = Y_{Rs}(\mathbf{s})\,(H_{sb}(\mathbf{s}) + \Theta_R(\mathbf{s})\,Y_{rs}(X_s, \mathbf{s})). \quad (17)$$

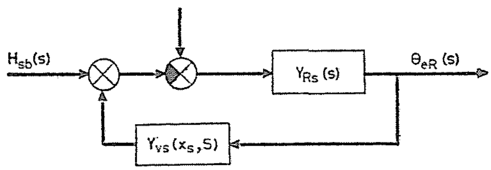

Fig. 2. Flowchart of water level control

From solution (17) the resultant relative discharge change of the control structure is (Fig. 2):

$$\Theta_{eR}(\mathbf{s}) = \frac{Y_{Rs}(\mathbf{s})}{1 - Y_{Rs}(\mathbf{s})\,Y_{Vs}(x_s, \mathbf{s})}\,H_{sb}(\mathbf{s}) \quad (18)$$

where

$Y_{Rs}(\mathbf{s})$ — transfer function of control structure relating to controlled level change;

$Y_{Vs}(\mathbf{s})$ — transfer function of feedback.

The transfer function $Y_{Rs}(\mathbf{s})$ can usually be approximated as:

— in case of proportional-type controller:

$$Y_{Rs}(\mathbf{s}) = \frac{A_R}{(1 + T_1 s)}\,e^{-T_H \mathbf{s}}$$

— in case of integrating-type controller:

$$Y_{Rs}(\mathbf{s}) = \frac{A_R}{T_i \mathbf{s}(1 + T_1 s)}\,e^{-T_H \mathbf{s}}.$$

Substituting:

$$\mathbf{s} = \frac{\partial}{t_0 \partial \frac{t}{t_0}} = \frac{1}{t_0}\frac{\partial}{\partial \tau} = \frac{1}{t_0}\mathbf{p}$$

$$Y_{Rs}(\mathbf{p}) = \frac{A_R}{1 + \tau_1 \mathbf{p}}\,e^{-\tau_H \mathbf{p}}$$

or

$$Y_{Rs}(\mathbf{p}) = \frac{A_R}{\tau_i\,\mathbf{p}(1 + \tau_1 \mathbf{p})}\,e^{-\tau_H \mathbf{p}} \quad (19)$$

where

A_R — proportionality coefficient;

$\tau_1 = \frac{T_1}{t_0}$ — relative time constant of the storage element;

$\tau_i = \frac{T_i}{t_0}$ — relative time constant of the integrating element;

$\tau_H = \frac{T_H}{t_0}$ — relative time constant of dead-time element.

Transfer functions (19) are already nondimensional. A further simplifying condition is needed to determine the nondimensional form of feedback transfer function $Y_{\iota s}(x_s;s)$:

— total reflection is supposed in the boundary cross section opposite to the controller of the controlled canal. (The case of stability.)

It follows from the foregoing that:

— in cases of upstream level control:

$$Y_{vs}(x_s, s) \rightarrow Y_{H2}(\xi_s, p) \tag{20}$$

— in cases of downstream level control;

$$Y_{\iota s}(x_s, s) \rightarrow Y_{H1}(\xi, p). \tag{21}$$

Using relationships (18) to (21) the nondimensional resultant transfer function of control structures takes the form:

$$Y_{eR}(p) = \frac{Y_{Rs}(p)}{1 - Y_{Rs}(p)\, Y_H(\xi_r, p)} . \tag{22}$$

Stability of water level control

Due to the complexity of transfer function (22) the *Nyquist* criterion is suitable for stability tests [1, 6, 8].

From transfer function (22) the loop transfer function of the control system is:

$$Y(\xi_s, p) = -Y_{Rs}(p)\, Y_H(\xi, p). \tag{23}$$

Loop transfer function (23) can be translated to the frequency range by substituting $p = j\nu$:

$$Y(\xi_s, j\nu) = -Y_{Rs}(j\nu)\, Y_H(\xi, j\nu) \tag{24}$$

where

$$\nu = \frac{\omega}{\omega_0} = \frac{2lc_0}{c_1^2}\,\omega \quad \text{— resonance factor;}$$

ω — frequency;

ω_0 — natural frequency of the canal section.

It is demonstrable that under certain conditions the characteristic equation $1 - Y(\xi_s, \mathbf{p}) = 0$ has no positive real part root [1], making the simplified *Nyquist* criterion adequate to test stability.

According to the criterion the control system is stable if the following condition is realized:

$$Re\,\{Y(\xi_s, j\nu)\} > 1{,}0 \quad \text{if} \quad Im\,\{Y(\xi_s, j\nu)\} = 0$$

in all cases where $0 \leq \nu \leq \infty$ (Fig. 3).

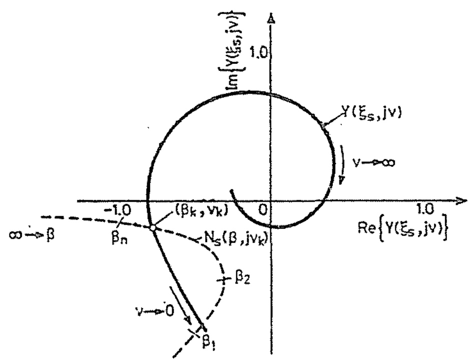

Fig. 3. Nyquist curve and curve of the describing function

According to the foregoing, the stability of a water level control system is unequivocally determined by five nondimensional dynamic canal parameters (Fr, R, M, N, ξ_s) and max. four nondimensional dynamic control structure parameters (A_R, τ_1, τ_i, τ_H).

If a control structure has essential nonlinearities, the method of describing functions can be applied [2]. In this case the nonlinear elements can be approximated by describing functions substituting transfer functions. Resultant frequency function of the control structure is:

$$Y_{eR}(\xi_s, j\nu) = \frac{N_s(\beta, j\nu)\, Y_{Rs}(j\nu)}{1 - N_s(\beta, j\nu)\, Y_{Rs}(j\nu)\, Y_H(\xi_s, j\nu)} \tag{26}$$

where

$N_s(\beta, j\nu)$ — nondimensional describing function of nonlinear element;

β — amplitude of relative change in the controlled level;

$Y_{Rs}(j\nu)$ — frequency function characterizing linear signal transmission of the control structure.

Using the denominator of (26) the nonlinear stability can be determined as: [2]

$$-Y_{Rs}(j\nu)\,Y_H(\xi_s, j\nu) = \frac{-1}{N_s(\beta, j\nu)}\,. \tag{27}$$

Equality (27) also yields the relative parameters of steady-state oscillation (β_k, ν_k) in case of a convergent critical cycle.

Summary

Nondimensional expressions are presented for the dynamic description of controlled-level canal sections and control structures.

These expressions involve max. nine nondimensional parameters to test the stability of a given system. Introducing the describing function method, also nonlinear stability can be tested.

The given expressions permit to carry out a test series, by varying nine parameters. Evaluating the test results yields practical design stability conditions.

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