INFLUENCE FUNCTION OF SKEW CIRCULAR RINGS ON ELASTIC BEDDING

By

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1. Survey of the literature

The elementary theory of bar systems suits the establishment of an ordinary system of differential equations for the deformations of a circular ring considered as a bar of curvilinear axis but because of algebraic computational difficulties arising in solving the system of differential equations there is hardly any practical computation method for analysing generalized circular ring problems.

A complete set of formulae refers to unbedded, common non-skew rings, rather common in practice. For instance, stress/strain relationships for annular members under concentrated loads and with different supports have been tabulated in [1], while [2] offers a set of stress/strain formulae for complete rings under loads of sine and cosine distribution, both referring to unbedded, non-skew rings. Also [3] presents a method for the analysis of non-skew circular rings bedded in their plane, normally to it, and against rotation, exposed to concentrated forces or to a group of forces. This method is adapted to influence functions. Its application is also presented in other references, e.g. in [4] for cooling tower foundations. Formulae for non-skew, bedded circular rings under loads of sine or cosine distribution are rather easy to deduce from formulae in [2] such as those for radial and annular bedding in [3]. There are no publications concerned with the direct analysis of circular rings on skew or general bedding.

These examinations have led to some important conclusions.

Analysis of loads and deformations in non-skew circular rings may be decomposed into that of loads and deformations in the ring plane, and that of loads and of resulting displacements — normal displacement, rotation, torsion — normal to the ring plane. Implicit condition of the decomposition is to neglect deformation components belonging to the — usually slight — change of the ring arc length, of course, provided there is two-way straight bending and non-skew bedding.

Ring stiffnesses and bedding stiffnesses act differently, depending on the load type. This is the most apparent for loads of sine or cosine distribution: until the number of complete waves n is low, bedding prevails in absorbing the loads — e.g. for $n = 0$ or 1, certain ring stiffnesses do not act at all — while for high n values, the bedding action becomes negligible.

The effect of bedding reactions themselves is different. The action of annular bedding is mostly negligible save for $n = 0$ or 1. In these latter two cases it is not, since for zero, the constant annular load can only be offset by annular bedding reaction, while for the load belonging to 1, causing rigid-body displacement, the values of displacement and normal force may be much affected by the annular bedding reaction, in these cases, however, reckoning with the action causes no difficulty because of the simplicity of stress/strain relationships, exactly due to the rigid-body displacements.

Force and reaction components being determinant for the effect of concentrated forces at both low and high n values, neither ring stiffnesses, nor bedding coefficients can be considered in themselves to prevail in the development of forces and reactions, it being much better described by dimensionless quotients of bedding coefficients multiplied by the respective powers of ring stiffnesses by the ring radius.

The lower the quotient, the higher the modes until the action of bedding prevails, the more the forces and reactions tend to those of rectilinear beams on elastic bedding, and the more the effect of concentrated forces can be considered as local.

These statements are partly valid to skew circular rings, the most essential difference being anyhow the inseparability of the effects of in-plane and normal loads because of skew bending and skew bedding. Neither introduction of a coordinate system of the cross section turned out of the ring plane provides an essential simplification of the analysis, since principal directions of bending and bedding do not necessarily coincide, but even if they do so, complexity of the involved geometry equations is prohibitive. The effect of ring and bedding stiffnesses is likely to be similar to that for skew circular rings, but the direct confrontation of rigidity constants is inexpedient because of the superposition of plane and normal effects.

2. Differential equation system of displacements

Geometry of the structure is seen in Fig. 1, where *1* and *2* are principal directions of inertia of the cross section, while the ring skewness is described by the angle α included between the positive directions of x and *1*.

Geometry equations of the stress/strain relationships being much simpler in (x, y)- than in $(1,2)$-systems, both deformations and internal forces will be

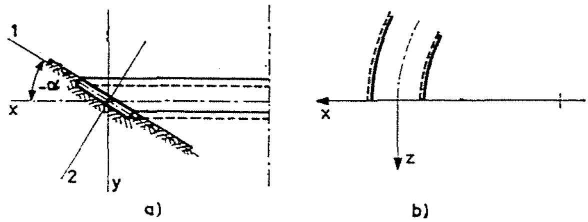

Fig. 1

described by components interpreted in the (x, y) coordinate system. Considering the curvature variation and moment vectors pointing to positive directions of x, y and of *1,2*, respectively, as positive (Fig. 2*a*), deformations and moments will be related as:

$$M_1 = D_1 \varkappa_1, \quad M_2 = D_2 \varkappa_2,$$
$$M_x = M_1 \cos\alpha + M_2 \sin\alpha$$
$$M_y = -M_1 \sin\alpha + M_2 \cos\alpha$$
$$\varkappa_1 = \varkappa_x \cdot \cos\alpha - \varkappa_y \cdot \sin\alpha$$
$$\varkappa_2 = \varkappa_x \cdot \sin\alpha + \varkappa_y \cdot \cos\alpha.$$

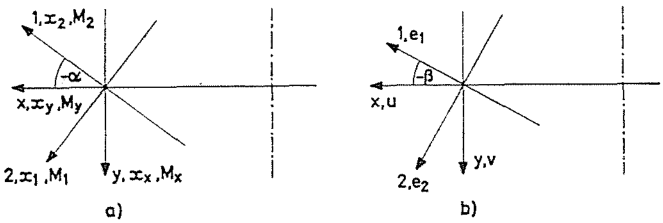

Fig. 2

Arranging the equations as an explicit relationship between M_x and M_y; and $\varkappa_x$ and $\varkappa_y$, resp.:

$$\left.\begin{aligned} M_x &= D_x \cdot \varkappa_x + D_{xy} \cdot \varkappa_y \\ M_y &= D_y \cdot \varkappa_y + D_{xy} \cdot \varkappa_x \end{aligned}\right\} \qquad (1.\text{a, b})$$

where

$$\left.\begin{aligned} D_x &= D_1 \cdot \cos^2\alpha + D_2 \cdot \sin^2\alpha \\ D_y &= D_1 \cdot \sin^2\alpha + D_2 \cdot \cos^2\alpha \\ D_{xy} &= (D_2 - D_1) \cdot \cos\alpha \cdot \sin\alpha. \end{aligned}\right\} \qquad (2.\text{a, b, c})$$

Relationships written for the principal directions of bedding can be transformed by a procedure similar to (2) for the determination of coefficients of bedding against normal and radial displacements. Be β the angle included between principal directions of bedding *1* and *2* and x and y, resp.; e_1 and e_2 the displacements in the principal directions; q_1 and q_2 the bedding reactions (Fig. 2*b*). The original bedding equations are (denoting by C the modified bedding coefficients of N/mm^2 dimension):

$$q_1 = C_1 \cdot e_1$$
$$q_2 = C_2 \cdot e_2.$$

Transformed equations:

$$\left.\begin{aligned} q_x &= -(C_x \cdot u + C_{xy} \cdot v) \\ q_y &= -(C_y \cdot v + C_{xy} \cdot u) \end{aligned}\right\} \qquad (3.\text{a, b})$$

with bedding coefficients:

$$\left.\begin{aligned} C_x &= C_1 \cdot \cos^2 \beta + C_2 \cdot \sin^2 \beta \\ C_y &= C_1 \cdot \sin^2 \beta + C_2 \cdot \cos^2 \beta \\ C_{xy} &= (C_2 - C_1) \cdot \cos \beta \cdot \sin \beta. \end{aligned}\right\} \qquad (4.\text{a, b, c})$$

Equations of annular bedding and of bedding against rotation remain inaffected:

$$q_z = -C_z \cdot w \qquad (5)$$

$$q_m = -C_m \cdot \vartheta. \qquad (6)$$

In Eq. (6), q_m stands for the value of the distributed bedding moment acting parallel to the ring strength axis divided by R, and ϑ for the magnified value of rotation about the strength axis multiplied by R. Reduction and magnification equalizes the dimension of C_m to that of the other bedding coefficients.

The system of ring strength equations — neglecting angular rotations due to shear — is complete by simply adding two further relationships: those between torque and twist, and between normal force and strain.

$$M_2 = T \cdot \varkappa_z \qquad (7)$$

$$N = EF \cdot \varepsilon_z\,. \qquad (8)$$

There are three projectional and three moment equations expressing equilibrium between forces acting on the ring part:

$$\left.\begin{aligned} &\frac{dQ_x}{ds} - \frac{N}{R} + p_x + q_x = 0 \\ &\frac{dQ_y}{ds} + p_y + q_y = 0 \\ &\frac{dN}{ds} + \frac{Q_x}{R} + p_z + q_z = 0 \end{aligned}\right\} \qquad (9.\text{a—c})$$

$$\left.\begin{aligned}
&\frac{dM_x}{ds} - \frac{M_z}{R} + Q_y = 0\\
&\frac{dM_y}{ds} - Q_x = 0\\
&\frac{dM_z}{ds} + \frac{M_x}{R} + R(m + q_m) = 0
\end{aligned}\right\} \qquad (10.\text{a—c})$$

m in Eq. (10.c) being the value of the outer, distributed moment-type load divided by R; derivatives in the equations have been derived with respect to arc lengths. Denoting derivatives with respect to $\varphi = s/R$ by comma superscript, combining Eqs (9.a) and (9.c) with (10.a) and (10.b), resp., permits to eliminate Q_x and Q_y to yield:

$$\left.\begin{aligned}
-M_x'' + M_z' + R^2 q_y &= -R^2 p_y\\
M_y'' - NR + R^2 q_x &= -R^2 p_x\\
M_y' + N'R + R^2 q &= -R^2 p_z\\
M_z' + M_x + R^2 q_m &= -R^2 m\,.
\end{aligned}\right\} \qquad (11.\text{a—d})$$

Geometry relationships of ring deformations and displacements are:

$$\left.\begin{aligned}
\varepsilon &= (u + w')\,\frac{1}{R}\\
\varkappa_x &= (-\vartheta + v'')\,\frac{1}{R^2}\\
\varkappa_y &= -(u + u'')\,\frac{1}{R^2}\\
\varkappa_z &= (\vartheta' + v')\,\frac{1}{R^2}\,.
\end{aligned}\right\} \qquad (12.\text{a—d})$$

Substituting (12.a—d) into (1.a, b), (7) and (8), then the resulting equations, together with (3.a, b), (5) and (6), into the derived equilibrium equations (11.a—d) leads to the system of differential equations of displacements. Denoting the values of bedding coefficients C multiplied by R^4 by K; the coefficient $EF \cdot R^2$ by A, Eqs (12.a—d) become:

$$\left.\begin{aligned}
&-D_x(v^{IV} - \vartheta) + D_{xy}(u^{IV} + u'') + T(v'' + \vartheta'') - K_y v - K_{xy} u = -p_y \cdot R^4\\
&D_{xy}(v'' - \vartheta'') - D_y(u^{IV} + u'') - A(u + w') - K_x u - K_{xy} v = -p_x R^4\\
&D_{xy}(v''' - \vartheta') - D_y(u''' + u') + A(u' + w'') - K_z w = -p_z R^4\\
&T(v'' + \vartheta'') + D_x(v'' - \vartheta) - D_{xy}(u'' + u) - K_m \vartheta = -mR^4\,.
\end{aligned}\right\} \qquad (13.\text{a—d})$$

Ring displacements result from the solution of the system of differential equations; in knowledge of displacements, stresses may be obtained from geometry and strength relationships. Thereby the problem has been reduced to solving the system of differential equations (13.a—d).

3. Solution of the differential equation system

First and fourth terms of Eqs (13.a—d) are exempt from displacement function w, thereby, eliminating w from the combined second and third equations there is less by one unknown.

As a conclusion of preliminary observations, the effect of strains ε_z and of bedding stiffness K_z may be assumed to be negligible. Thus, omitting term $K_z \cdot w$ from (13.c) and adding the derivative of (13.b) yields:

$$-D_y(u^V + 2u''' + u') + D_{xy}(v^V + v''' - \vartheta''' - \vartheta') - Ku' - K_{xy}v' = \\ = -(p_z + p_x')R^4. \tag{14}$$

Neglect of strains ε_z means to consider relationships

$$u + w' = 0 \tag{15}$$

to prevail between normal and radial displacements. Introducing deformation and load vectors $\mathbf{u}$ and $\mathbf{p}$:

$$\mathbf{u} = \begin{bmatrix} u \\ v \\ \vartheta \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} (p_z + p_x') \cdot R^4 \\ p_y R^4 \\ m R^4 \end{bmatrix}$$

permits to relate them, in compliance with (13) and (14), by the operator matrix equation:

$$\begin{bmatrix} D_y\partial(\partial^2+1)^2 + K_x\partial & -D_{xy}\partial^3(\partial^2+1) + K_{xy}\partial & D\partial(\partial^2+1) \\ -D_{xy}\partial^2(\partial^2+1) + K_{xy} & D_x\partial^4 - T\partial^2 + K_y & -(D_x + T)\partial^2 \\ D_{xy}(\partial^2+1) & -(D_x + T)\partial^2 & -T\partial^2 + D_x + K_m \end{bmatrix} \mathbf{u} = \mathbf{p}. \tag{16}$$

(16) differs from a differential equation with symmetrical operator matrix by assuming the equation corresponding to the first row as a once derived one. Therefore the general solution of the equation system differs from that of a system of differential equations with symmetric operator matrix by the

general solution of the inhomogeneous differential equation

$$\begin{bmatrix} D_y(\partial^2+1)+K_x & -D_{xy}\partial^2(\partial^2+1)+K_{xy} & D_{xy}(\partial^2+1) \\ -D_{xy}\partial^2(\partial^2+1)+K_{xy} & D_x\partial^4-T\partial^2+K_y & -(D_x+T)\partial^2 \\ D_{xy}(\partial^2+1) & -(D_x+T)\partial^2 & -T\partial^2+D_x+K_m \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} \quad (17)$$

where k is an arbitrary constant. Particular inhomogeneous solution of (17) is advisably sought for in the form $\mathbf{u}_0 =$ const., eliminating terms containing derivatives, hence, rather than a differential equation, solution of the linear inhomogeneous algebraic equation system for $\mathbf{u}_0$ becomes:

$$\begin{bmatrix} D_y + K_x & |K_{xy} & D_{xy} \\ K_{xy} & K_y & 0 \\ D_{xy} & 0 & D_x + K_m \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \\ \vartheta \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}. \quad (18)$$

Determinant of Eq. (18):

$$(D_y + K_x)\,(D_y + K_m)K_y - K_{xy}^2(D_x + K_m) - D_{xy}^2 K_g$$

being non-zero for real stiffnesses and bedding, the solution $\mathbf{u}_0 =$ const. does exist. Merging the reciprocal of the determinant with k:

$$u_0 = k \cdot K_y(D_x + K_m),$$

$$v = -k \cdot K_{xy}^2(D_x + K_m),$$

$$\vartheta = -k \cdot D_{xy} \cdot K_{xy}.$$

This system of displacements cannot, however, develop without changing the arc length, hence $\mathbf{u}_0$ is missing from the general solution. Thus, it can be stated that, after having integrated the first equation in the set (16), the general solution of the inhomogeneous differential equation system obtained by replacing $k = 0$ can be considered as the general solution of the original equation system, thereby Eqs (16) have been reduced to the equation system with symmetrical operator matrix:

$$\begin{bmatrix} D_y(\partial^2+1)^2+K_x & -D_{xy}\partial^2(\partial^2+1)+K_{xy} & D_{xy}(\partial^2+1) \\ -D_{xy}\partial^2(\partial^2+1)+K_{xy} & D_x\partial^4-T\partial^2+K_y & -(D_x+T)\partial^2 \\ D_{xy}(\partial^2+1) & -(D_x+T)\partial^2 & -T\partial^2+D_x+K_m \end{bmatrix} \begin{bmatrix} u \\ v \\ \vartheta \end{bmatrix} =$$

$$= \begin{bmatrix} (p_x + \int p_z d\varphi)R^4 \\ p_y R^4 \\ m R^4 \end{bmatrix}. \quad (16.a)$$

Thereby also the interpretation of $\mathbf{p}$ is affected by having replaced the first term with its integrated. Theoretically, the procedure of solving the differential equation system may first involve determination of the operator matrix determinant:

$$\begin{aligned}\det(\,) = {} & [D_y(\partial^2+1)^2 + K_x][(D_x\partial^4 - T\partial^2 + K_y)(-T\partial^2 + D_x + K_m) - \\ & -(D_x+T)^2\partial^4] + \{2D_{xy}(\partial^2+1)(D_x+T)\partial^2[D_{xy}\partial^2(\partial^2+1) - K_{xy}] - \\ & - D_{xy}^2(\partial^2+1)^2(D_x\partial^4 - T\partial^2 + K_x) + [D_{xy}\partial(\partial^2+1) - K_{xy}]^2 \times \\ & \times (T\partial^2 - D_x - K_m)\}\,.\end{aligned}$$

The second step will be to determine the fundamental system

$$\det(\mathcal{F}) = 0$$

of the general solution of the homogeneous linear differential equation system of order ten:

$$\det(f_i) = 0 \qquad (i = 1, \dots 10)$$

defined by the determinant. It is sufficient to seek linear independent basic functions f_i in form

$$f_i = e^{\lambda_i \cdot \varphi}.$$

Namely, the tenth-power algebraic equation of coefficient λ_i delivers — in case of real ring and bedding stiffnesses — two different real roots, and in addition, eight different complex roots, thus, there is no degenerating homogeneous solution.

The next step will be to substitute functions f_i belonging to respective λ_i values for determining basic solution vectors $\boldsymbol{\xi}_i = e^{\lambda i \varphi} \cdot \mathbf{x}_i$ by solving the homogeneous linear equation system left after dropping functions $e^{\lambda i \varphi}$ encountered in every left-hand side term, using an adequate condition of normality. Complex functions $e^{\lambda i \varphi}$ are interconjugated in basic solution vectors $\boldsymbol{\xi}_i$ of the homogeneous differential equation system, permitting them to be rearranged into a fundamental system containing trigonometric and hyperbolic, or trigonometric and exponential functions of real argument. The real root pair may have real hyperbolic functions as concomitant. Based on the solution vector system obtained with functions of real argument, influence function vectors η_x, η_y, η_M belonging to unit loads $P_x = 1, P_y = 1, P_z = 1, M = 1$ can be determined as linear combinations of vectors of the basic system, leading to the displacement vector belonging to an arbitrary load $\mathbf{p}(\varphi)$ in the definite integral form.

$$\begin{aligned}\mathbf{u}(\varphi) = \int_0^{2\pi} \Big\{ & \eta_x(\alpha)\Big[p_x(\varphi-\alpha) + \int_0^{\varphi} p_z(\varphi-\alpha)\,d\zeta\Big] + \\ & + \eta_y(\alpha)\,p_y(\varphi-\alpha) + \eta_M(\alpha)\cdot m(\varphi-\alpha)\Big\} R d\alpha\,. \qquad (19)\end{aligned}$$

Based on the displacement vector component, the wanted internal forces can be expressed intermediating geometry conditions substituted into physical equations.

Practical application of the presented method hits serious difficulties.

Solution of the algebraic equation of tenth power corresponding to the characteristic differential equation of order ten delivers rather complex, though parametrically expressible roots for non-skew circular rings when the equation can be decomposed into two products of sixth and fourth power. For skew circular rings there is no likelihood of a similar parametric expression for the roots, the numerical solution by separating the complex root pairs is little illustrative, of problematic stability, while determination of solution vectors of real arguments and of influence vectors as well as calculation of the specified defined integrals are rather labour consuming.

To avoid at least part of these difficulties and inconvenients, a special method such as that presented below had to be developed.

4. Trigonometric series for solving the differential equation system

Let us consider first the displacements of a ring exposed to cyclic loads:

$$\begin{aligned} p_x(\varphi) &= p_{xn} \cdot \cos n\varphi \\ p_y(\varphi) &= p_{yn} \cdot \cos n\varphi \\ p_z(\varphi) &= p_{zn} \cdot \sin n\varphi \\ m(\varphi) &= m_n \cdot \cos n\varphi . \end{aligned} \tag{20}$$

Configuration of Eq. (16.a) is a hint that also deformation functions for loads assumed according to (20) will be cyclic functions:

$$\begin{aligned} u(\varphi) &= u_n \cdot \cos n\varphi \\ v(\varphi) &= v_n \cdot \cos n\varphi \\ w(\varphi) &= w_n \cdot \sin n\varphi \\ \vartheta(\varphi) &= \vartheta_n \cdot \cos n\varphi . \end{aligned} \tag{21}$$

Substituting (20) and (21) into the system of differential equations (16.a) and reducing by cos $(n\varphi)$ yields for the amplitudes u_n, v_n, ϑ_n, of the deformation functions the linear equation system:

$$\begin{bmatrix} D_y(n^2-1)^2+K_x & -D_{xy}n^2(n^2-1)+K_{xy} & -D_{xy}(n^2-1) \\ -D_{xy}n^2(n^2-1)+K_{xy} & D_xn^4+Tn^2+K_y & (D_x+T)n^2 \\ -D_{xy}(n^2-1) & (D_x+T)n^2 & Tn^2+D_x+K_m \end{bmatrix} \begin{bmatrix} u_n \\ v_n \\ \vartheta_n \end{bmatrix} = \begin{bmatrix} p_{xn}-\dfrac{p_{zn}}{n} \\ p_{yn} \\ m_n \end{bmatrix} \cdot R^4 \quad (22)$$

The coefficient matrix can be considered as sum of two matrices, one for the inherent stiffnesses, and the other for the bedding stiffnesses:

$$\begin{bmatrix} D_y(n^2-1)^2 & -D_{xy}n^2(n^2-1) & -D_{xy}(n^2-1) \\ -D_{xy}n^2(n^2-1) & D_xn^4+Tn^2 & (D_x+T)n^2 \\ -D_{xy}(n^2-1) & (D_x+T)n^2 & Tn^2+D_x \end{bmatrix} + \begin{bmatrix} K_x & K_{xy} & \\ K_{xy} & K_y & \\ & & K_m \end{bmatrix} = \mathbf{D}_n+\mathbf{K} \quad (23)$$

The problem of the unbedded ring leads to that for $\mathbf{K}=0$. Provided non-zero terms of $\mathbf{K}$ refer to effective bedding stiffnesses, matrix (22) is invariably regular irrespective of $\mathbf{D}_n$, so that, for an arbitrary load vector, to every n a defined solution belongs. Regularity of matrix $\mathbf{D}$ is conditioned, independent of n, by $T \neq 0$, that is, it requires torsional rigidity. Also, regularity of matrix $\mathbf{D}$ is conditioned, independent of n, by non-zero principal bending stiffnesses D_1 and D_2. In this case D_x and D_y cannot be zero either.

Existence of regularity conditions independent of n automatically provides for the regularity of D_n for $n \geq 2$. For $n = 0$ or 1, $\mathbf{D}_n$ is singular, irrespective of the actual stiffness values. Singularity refers to the possibility of displacement systems defined by displacement vectors in the zero field of the singular matrix to occur without internal forces to arise. Zero field of the singularity of the case $n = 0$ includes vectors describing rigid-body displacement parallel to, and rigid-body rotation around, the axis of revolution of the ring. Also displacement vectors in the zero field of singularity of the case $n = 1$ describe rigid-body displacements: translation, and rotation about the axis, in the ring plane. Other than rigid-body displacements are, however, outside the zero field of matrix $\mathbf{D}_n$.

In consequence of becoming singular, loads that do not compose a system of force equilibrium cannot be transferred to the structure without reckoning with the bedding, while equilibrium force systems can be borne by the ring without intervention of bedding. Loads composing a non-equilibrium force system are, in case of $n = 0$, uniform normal loads with a resultant parallel to the rotation axis, and uniform annular loads with moment vector resultants parallel to the rotation axis; while for $n = 1$, loads with resultants in the ring plane, and with moment vector resultants in the ring plane. A perfect complementarity can be demonstrated between displacement vectors in the zero fields, and load vectors describing loads not to be balanced without a bedding.

Decomposition according to (23) is also supported by the n-dependence of elements in $\mathbf{D}$, as against those in $\mathbf{K}$. This difference explains why the action of inherent stiffnesses and of bedding stiffnesses varies for rather different n values. Until n is low — as in most of practical cases — $\mathbf{K}$ controls the sum of both matrices. With an increasing number of waves, elements of $\mathbf{D}_n$ rapidly increase — and so does the norm of $\mathbf{D}_n$, causing this latter to prevail in the sum.

Deformations due to an arbitrary, symmetric load will be determined as follows:

First, load vector elements are expanded into *Fourier* series. Symmetry causes the *Fourier* series to contain only cosine terms in case of p_x, p_y and m, while for p_z, only sine terms. Including load components for an arbitrary n into a vector $\mathbf{p}_n$ permits to replace the load by the vector series

$$\mathbf{p} = \sum_{n=0}^{\infty} \begin{bmatrix} p_{nx} - \dfrac{p_{nz}}{n} \\ p_{ny} \\ m_n \end{bmatrix} \cos n\varphi \tag{24}$$

implying $\dfrac{p_{nz}}{n} = 0$ for $n = 0$.

The displacement vector will be obtained in a similar form:

$$\mathbf{u} = \sum_{n=0}^{\infty} \begin{bmatrix} u_n \\ v_n \\ \vartheta_n \end{bmatrix} \cos n\varphi = \sum_{n=0}^{\infty} (\mathbf{D}_n + \mathbf{K})^{-1} \begin{bmatrix} p_{nx} + \dfrac{p_{nz}}{n} \\ p_{ny} \\ m_n \end{bmatrix} \cos n\varphi . \tag{25}$$

For determining displacements due to loads of arbitrary distribution, the applied cosine series has to be replaced by cosine and sine series. Zeroth term of the sine series describing the antimetric load part being meaningless, summing has to start at $n = 1$. For the antimetric load part alone:

$$\hat{\mathbf{p}} = \sum_{n=1}^{\infty} \begin{bmatrix} \hat{p}_{nx} - \dfrac{\hat{p}_{nz}}{n} \\ \hat{p}_{ny} \\ \hat{m}_n \end{bmatrix} \sin n\varphi , \tag{24.a}$$

$$\hat{\mathbf{u}} = \sum_{n=1}^{\infty} (\mathbf{D}_n + \mathbf{K})^{-1} \begin{bmatrix} \hat{p}_{nx} - \dfrac{\hat{p}_{nz}}{n} \\ \hat{p}_{ny} \\ \hat{m}_n \end{bmatrix} \sin n\varphi . \tag{25.a}$$

A peculiarity of Eq. (25.a) is to contain a matrix to be inverted, identical to the matrix in Eq. (25).

Determination of the vector series in (25) — although a simple, stereotype task — is too laboursome to do else than in a computer. Some special cases will be presented below, where simplification of the problem permits manual computation — using nomograms or desk calculators.

5. Unbedded, skew circular ring under equilibrium force system

The general method of analysing rings exposed to arbitrary equilibrium force systems starts by generalizing the influence line concept adapted to our problem.

In the following, influence line will be considered such as that having ordinatae showing the effect elicited by unit forces P_x, P_y, unit moment M, and the balacing distributed loads $(p_{x0}, p_{x1}, \hat{p}_{x1})$, $(p_{y0}, p_{y1}, \hat{p}_{y1})$, and m_0, m_1, $\hat{m}_1'$, respectively, above the ordinate.

Fourier series of load groups defined as above are simple to produce.

Fourier series of a concentrated force P at $\varphi = 0$:

$$p(\varphi) = \frac{P}{2R\pi} + \frac{P}{R\pi} \sum_{n=1}^{\infty} \cos n\varphi \,. \tag{26}$$

Omitting the zeroth and first terms of the *Fourier* series and replacing P by 1 results in the *Fourier* series of the modified unit load acting at the origin.

Omitting terms in **K** from (22) and inverting the equation yields for the displacements:

$$\left.\begin{aligned}
u_n &= \frac{R^4}{D_x D_y - D_{xy}^2} \left[D_x p_{xn} + D_{xy}(p_{yn} - m_n)\right] \frac{\cos n\varphi}{(n^2-1)^2} \\
v_n &= \frac{R^4}{D_x D_y - D_{xy}^2} \left\{\left[D_{xy} p_{xn} + D_y p_{yn} + (D_y + F) m_n\right] \frac{\cos n\varphi}{(n^2-1)^2} + \right. \\
&\qquad \left. + F \cdot p_{yn} \frac{\cos n\varphi}{n^2(n^2-1)^2}\right\} \\
\vartheta_n &= \frac{R^4}{D_x D_y - D_{xy}^2} \left\{\left[-D_{xy} p_{xn} + (D_y + F) p_{yn} + D_y m_n\right] \frac{\cos n\varphi}{(n^2-1)^2} + \right. \\
&\qquad \left. + F \cdot m_n \frac{n^2 \cos n\varphi}{(n^2-1)^2}\right\}
\end{aligned}\right\} \tag{27}$$

where $F = \dfrac{D_x D_y - D_{xy}^2}{T}$.

Deformations resulting from unit loads can be produced in form of *Fourier* series written by means of the formulae above.

For instance, angular rotations due to radial unit load:

$$\vartheta(\varphi) = \frac{R^4}{D_x D_y - D_{xy}^2} \cdot \frac{-D_{xy}}{R\pi} \sum_2^\infty \frac{\cos n\varphi}{(n^2-1)^2} \tag{28.a}$$

or displacements in their direction due to normal unit load:

$$v(\varphi) = \frac{R^4}{D_x D_y - D_{xy}^2} \left[\frac{D}{R\pi} \sum_2^\infty \frac{\cos n\varphi}{(n^2-1)^2} + \frac{F}{R\pi} \sum_2^\infty \frac{\cos n\varphi}{n^2(n^2-1)^2} \right]. \tag{28.b}$$

Specified summations can be produced in closed form (6):

$$\left.\begin{aligned}
\chi(\varphi) &= \sum_2^\infty \frac{\cos n\varphi}{n^2(n^2-1)^2} = \left(1 + \frac{\cos\varphi}{2}\right)\left[\frac{\pi^2}{6} - \frac{\pi\varphi}{2} + \frac{\varphi^2}{4}\right] - \\
&\qquad - \frac{3\sin\varphi}{2}\,\frac{\pi-\varphi}{2} - \frac{23\cos\varphi}{16} - 1 \\
\psi(\varphi) &= \sum_2^\infty \frac{\cos n\varphi}{(n^2-1)^2} = -\chi(\varphi)'' \\
\xi(\varphi) &= \sum_2^\infty \frac{n^2\cos n\varphi}{(n^2-1)^2} = -\psi(\varphi)''\,.
\end{aligned}\right\} \tag{29}$$

Deformations of a ring loaded by an arbitrary equilibrium force system — except rigid-body displacements and terms arising from non-rigid-body displacement components for $n < 2$ — can be determined from (27) and (29).

Substituting functions $\chi(\varphi)$, $\psi(\varphi)$ and $\xi(\varphi)$ into sums defined by analogy to (28.a, b) yields a solution with closed formulae for the displacements rather convenient for the slowly converging definition series of $\xi(\varphi)$ and $\psi(\varphi)$. In conformity with the theorem of exchangeability, displacement functions can be directly interpreted as displacement influence functions. In doing so, φ in the argument will be replaced by $(\alpha - \varphi)$ where φ and α indicate the points of force and of action provided the absolute value of the thereby modified argument is higher than π, it has to be modified by 2π, with respect to the limit of validity of the formula. Stress/strain influence lines will be obtained by combining derivatives of displacement influence functions with respect to α in conformity with geometry and strength equations, fixing α in the combination to cope with the point of action. Now, stresses of the ring exposed to an equilibrium force system can be produced as definite integrals corresponding to (19). The effect of components balancing unit loads is missing from equilibrium loads, strains and stresses due to non-rigid-body displacements for $n < 2$ can be assigned to the outcome by simple calculation.

The deformation vector takes the simple form:

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 \cdot \cos\varphi + \hat{\mathbf{u}}_1 \cdot \sin\varphi + \int_{-\pi}^{+\pi} \mathbf{\Psi}(\alpha)\, \mathbf{p}(\alpha + \varphi) \cdot R\, d\alpha \tag{30}$$

where

$$\mathbf{\Psi} = \frac{R^4}{(D_x D_y - D_{xy}^2)} \begin{bmatrix} D_x\psi & D_{xy}\psi & -D_{xy}\psi \\ D_{xy}\psi & D_y\psi + F\chi & (D_y + F)\psi \\ -D_{xy}\psi & (D_y + F)\psi & D_y\psi + F\xi \end{bmatrix}$$

χ, ψ and ξ being functions determined by (29), $\mathbf{u}_0$, $\mathbf{u}_1$, and $\hat{\mathbf{u}}_1$ are rigid-body deformations and those calculated from equilibrium force system components for $n = 0_1$ and $n = 1$, respectively. Transformation of the variable in (30) $\mathbf{\Psi}(\alpha) \cdot \mathbf{p}(\alpha - \varphi)$ instead of $-\mathbf{\Psi}(\varphi - \alpha) \cdot \mathbf{p}(\alpha)$ was simply imposed by the limit of validity of the functions in matrix $\mathbf{\Psi}$.

Application of function matrix $\mathbf{\Psi}$ permits to state the problem of displacements of rings on elastic bedding as an integral equation.

6. Influence lines of non-skew circular rings on elastic bedding

Deformations of a non-skew circular ring are much simpler to determine than those in the general case, since zeroing D_{xy} and K_{xy} permits separate consideration of in-plane and normal displacements.

The method deduced for calculating displacements of unbedded rings permits simple indication of non-skew ring displacements under equilibrium force systems. Also reckoning with the bedding is simpler, permitting to use formulae for the deflections of unbedded rings.

Let us consider first the system of displacements due to radial unit force acting at $\varphi = 0$. Because of rectilinear bending, it is composed exclusively of deformations in the ring plane. Displacement function for mode $n \geq 2$:

$$u(\varphi)_n = u_n \cdot \cos n\varphi = \frac{R^4}{(n^2 - 1)^2 D_y} \frac{1}{R\pi} \cos n\varphi .$$

Assuming the effect of bedding in direction w to be negligible, bedding acts only as a load of a distribution like that of, and counteracting, displacements u. Taking the bedding equation into consideration, u_n is determined by:

$$u_n \cdot \cos n\varphi = \frac{R^4}{(n^2 - 1)^2 D_y} \left(\frac{1}{R\pi} - u_n C_x \right) \cos n\varphi$$

expressing coefficient u_n — introducing short form $\varrho^4 = \frac{R^4 \cdot C_x}{D_y} = \frac{K_x}{D_y}$:

$$u_n = \frac{1}{R\pi C_x} \frac{\varrho^4}{(n^2-1)^2 + \varrho^4}. \tag{31}$$

Displacement functions belonging to the zeroth and first terms of the Fourier series of concentrated force:

$$u_0 = 0$$

$$u_1 = \frac{1}{R\pi C_x} \cos \varphi .$$

The complete displacement function:

$$u(\varphi) = \frac{1}{R\pi C_x} \sum_{n=1}^{\infty} \frac{\varrho^4 \cos n\varphi}{(n^2-1)^2 + \varrho^4}. \tag{32}$$

The sum can be obtained in closed form (6) utilizing the identity:

$$\sum_{1}^{\infty} \frac{\cos n\varphi}{n^2 + a^2} = \frac{\pi}{2a} \frac{\operatorname{ch} a(\pi - \varphi)}{\operatorname{sh} a\pi} - \frac{1}{2a^2} \qquad 0 \le \varphi \le 2\pi. \tag{33}$$

Introducing complex parameter $a = \sqrt{i\varrho^2 - 1}$, term to be summed in (32) can be decomposed into two interconjugated partial fractions of the same structure as that in (33). The sum in closed form:

$$\sum_{n=1}^{\infty} \frac{\varrho^4 \cos n\varphi}{(n^2-1)^2 + \varrho^4} = -\varrho^2\pi \operatorname{Im} \left(\frac{\operatorname{ch} a(\pi - \varphi)}{a \cdot \operatorname{sh} a\pi} \right) - \frac{\varrho^4}{1 - \varrho^4}. \tag{34}$$

Introducing notation $a = \lambda + i\mu$, term Im ( ) in (34) can be decomposed into terms of real arguments containing terms $\operatorname{ch} \lambda\varphi \cdot \cos \mu\varphi$, $\operatorname{ch} \lambda\varphi \cdot \sin \mu\varphi$ and $\operatorname{sh} \lambda\varphi \cdot \cos \mu\varphi$, $\operatorname{sh} \lambda\varphi \cdot \sin \mu\varphi$.

The closed-form displacement function yields an influence function in the usual way, hence by replacing φ by coupled variables $(\alpha - \varphi)$, with derivatives delivering the stress influence lines by fixing α indicating the point of action.

Analysis of the outcome is rather instructive. Provided $\varrho^4 \gg 1$, that is $K_x \gg D_y$, the λ and μ values tend to each other and to $\frac{\sqrt{2}}{2}\varrho$, hence also displacement functions tend to those of an elastically bedded straight-axed beam.

Influence functions of a ring under normal force, elastically bedded in its plane, will be obtained alike, neglecting twists. The only difference is that in writing the unit force acting at $\varphi = 0$, Fourier term $n = 0$ involves deflection

$$v_0 = \frac{1}{2R\pi C_y}.$$

Thus, introducing $\varrho^4 = \frac{K_y}{D_x}$, the total deflection becomes:

$$v(\varphi) = \frac{1}{2R\pi C_y} + \frac{1}{R\pi C_y} \sum_{n=1}^{\infty} \frac{\varrho^4 \cos n\varphi}{(n^2 - 1)^2 + \varrho^4}. \tag{35}$$

Reckoning with twists makes the formulae of *Fourier* terms of the displacement function more complicated:

$$v_0 = \frac{1}{2R\pi C_y}$$

$$v_n = \frac{1}{R\pi K_y} \frac{\cos n\varphi}{\frac{(n^2+1)^2 n^2}{\varrho^4(\beta + n^2)} + 1}$$

where $\beta = \frac{D_x}{T}$. Decomposition into partial fractions permits to determine function v in closed form as a similar expression although in a much more complicated way than before. Decomposition requires to determine square root exponent form of the equation

$$n^6 + 2n^4 + (\alpha^4 + 1)n^2 + \alpha^4\beta = 0$$

thus, indirectly, root exponent form of cubic equation

$$z^3 + 2z^2 + (\alpha^4 + 1)z + \alpha^4\beta = 0.$$

Numerical analyses of circular plate rings with $\beta \approx 0.3 \sim 0.5$ showed the neglect of the twist effect to result in as little as a few per mille — at most a few per cent — displacement errors, still further reduced by neglecting the effect of bedding against rotation. Of course, stresses are affected more seriously.

Summary

A method has been developed for the determination of stresses and strains of elastically bedded skew circular rings, i.e. such where the principal directions of cross-sectional inertia are not in the mid-plane of the ring and do not coincide with the rotation axis. Elastic bedding is considered as skew bedding in a meaning that displacements normal to the ring plane may

cause bedding stresses parallel to the ring plane, and *vice versa*. For the case of simplicity, approximations made in the theory of strength of curvilinear bars are considered to be admissible, and effects of the inhibited cambering of cross sections are neglected.

Methods have been presented for the analysis of both bedded and unbedded skew circular rings.

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* In Hungarian.