FINITIZED MATHEMATICAL MODEL FOR IN-PLANE BEDDED CIRCULAR RINGS

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1. The substituting model

Let us substitute the elastically bedded circular ring loaded in plane by the following system:

Let us take a regular, n-sided polygon inscribed in the middle circle of radius a of the ring (Fig. 1). Let loads, reactions and relative displacements of

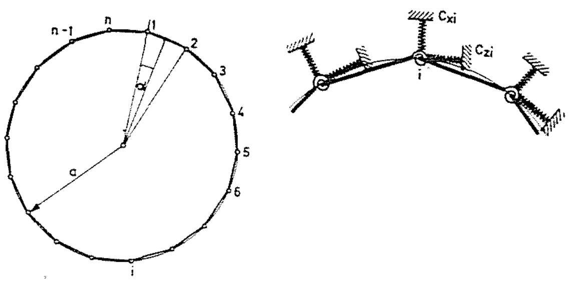

Fig. 1

the ring be concentrated at the polygon nodes connected by indeformable sides. With increasing n beyond all limits, this substituting model tends to the tested structure, and even a finite — although sufficiently high — n value causes an error admissible in engineering analyses.

Assuming the extension rigidity of the ring to be infinite results in a chain of bars with nodes elastically supported radially and normally to the radius, while at the nodes the bars are elastically, rather than moment-free, connected.

Let nodes be numbered $1, 2, \ldots, n-1, n$ and assign to each of them structural characteristics affected by a subscript corresponding to the node. Magnitudes of the same type are arranged in increasing node order by vectors of order n.

Vectors involved in the analysis denote the following nodal characteristics:

$\mathbf{u}$ — radial displacements;
$\mathbf{w}$ — displacements normal to the radius;
$\boldsymbol{\varphi}$ — relative angular rotation in the nodes;
$\mathbf{m}$ — nodal moments;
$\mathbf{n}$ — forces transmitted to the node from bars of higher order number;
$\mathbf{p}_x$ — radial external loads;
$\mathbf{p}_z$ — external loads normal to the radius;
$\mathbf{q}_x$ — concentrated radial bedding reactions;
$\mathbf{q}_z$ — bedding reactions normal to the radius.

Bedding and stiffness are described by diagonal matrices of order n, such as:

$\mathbf{D}$ — ring flexibility matrix;
$\mathbf{C}_x$ — radial bedding matrix;
$\mathbf{C}_z$ — matrix of bedding normal to the radius.

Based on stiffnesses and bedding coefficients considered as place-dependent, matrix diagonal elements are:

$$D_{ii} = (EI)_i \frac{n}{2\pi a} \tag{1}$$

$$C_{xii} = (C_x)_i \frac{2\pi a}{n}$$
$$C_{zii} = (C_z)_i \frac{2\pi a}{n} . \tag{2a, b}$$

Computations will further include unit matrix $\mathbf{E}$ of order n, and primitive cyclic permuting matrix $\boldsymbol{\Omega}$, again of order n, having 1 in the first right co-diagonal and in the lower left corner, all other elements being zero.

Bedding reactions and nodal displacements of the ring are related as

$$\mathbf{q}_x = -\mathbf{C}_x \cdot \mathbf{u}$$
$$\mathbf{q}_z = -\mathbf{C}_z \cdot \mathbf{w} \tag{3a, b}$$

the force pointing towards the positive displacement being considered as positive.

Nodal moments and relative rotations are related by:

$$\mathbf{m} = \mathbf{D} \cdot \boldsymbol{\varphi}. \tag{4}$$

2. The ring equation system

First, nodal displacements and rotations will be related (Fig. 2).

Inextensibility of connecting bars implied the following equality between the displacements:

$$-u_i \cdot \sin\frac{\pi}{n} + w_i \cos\frac{\pi}{n} = u_{i+1} \sin\frac{\pi}{n} + w_{i+1} \cos\frac{\pi}{n}. \tag{5a}$$

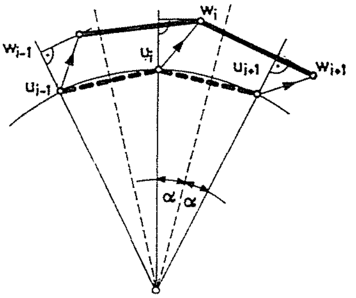

Fig. 2

Shortening $\alpha = \dfrac{\pi}{n}$ and using presented notations permits to replace (5a) by the vectorial equation:

$$(\mathbf{\Omega} + \mathbf{E})\mathbf{u} \cdot \sin\alpha + (\mathbf{\Omega} - \mathbf{E})\mathbf{w} \cdot \cos\alpha = 0. \tag{5b}$$

Absolute rotation of bar between nodes i and $i+1$ is based on the displacements:

$$\varphi_{i,i+1} = [(u_i - u_{i+1}) \cos\alpha + (w_i + w_{i+1}) \sin\alpha] \frac{1}{2a \cdot \sin\alpha}. \tag{6}$$

Relative rotation at node i, based on (6):

$$\varphi_i = [(2u_i - u_{1+1} - u_{i-1}) \cos\alpha + (w_{i+1} - w_{i-1}) \sin\alpha] \frac{1}{2a \cdot \sin\alpha} \tag{7a}$$

identical to vectorial equation:

$$\boldsymbol{\varphi} = (2\mathbf{E} - \mathbf{\Omega} - \mathbf{\Omega}^*)\mathbf{u} \cdot \frac{1}{2a \cdot \operatorname{tg}\alpha} + (\mathbf{\Omega} - \mathbf{\Omega}^*) \frac{1}{2a} \cdot \mathbf{w}. \tag{7b}$$

Multiplying Eq. (5b) by the transposed of $\mathbf{\Omega}$ and adding this latter to (5b) yields:

$$(2\mathbf{E} + \mathbf{\Omega} + \mathbf{\Omega}^*)\mathbf{u} \cdot \sin\alpha + (\mathbf{\Omega} - \mathbf{\Omega}^*)\mathbf{w} \cdot \cos\alpha = 0. \tag{8}$$

Combining (7b) and (8) permits to drop vector $\mathbf{w}$ from the formula for $\boldsymbol{\varphi}$:

$$\boldsymbol{\varphi} = \left[(2\mathbf{E} - \boldsymbol{\Omega} - \boldsymbol{\Omega}^*) \frac{1}{2a \cdot \operatorname{tg} \alpha} + (2\mathbf{E} + \boldsymbol{\Omega} + \boldsymbol{\Omega}^*) \frac{\operatorname{tg} \alpha}{2a}\right] \mathbf{u} \tag{9}$$

that may be considered as compatibility equation of the inextensible polygon ring. (4) and (9) suit to express the ring moment vector by means of vector $\boldsymbol{\varphi}$.

Next, displacing forces acting at nodes will be determined from the moments (Fig. 3).

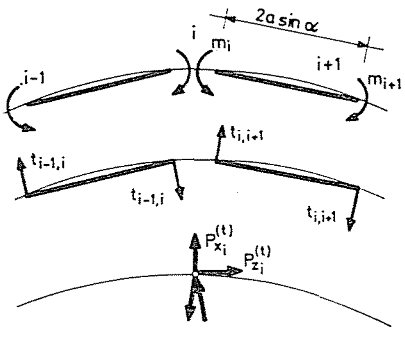

Fig. 3

Shear force of the bar length between nodes i and $i + 1$:

$$t_{i,i+1} = -(m_{i+1} - m_i) \frac{1}{2a \cdot \sin \alpha} . \tag{10}$$

Taking also the shear force in the preceding bar into account, shear causes the following displacing forces to act at node i:

$$p_{xi}^{(t)} = -(2m_i - m_{i+1} - m_{i-1}) \frac{\cos \alpha}{2a \cdot \sin \alpha} \tag{11a}$$

$$p_{zi}^{(t)} = +(m_{i+1} - m_{i-1}) \frac{\sin \alpha}{2a \cdot \sin \alpha} \tag{12a}$$

or, written as vectorial equations:

$$\mathbf{p}_x^{(t)} = -(2\mathbf{E} - \boldsymbol{\Omega} - \boldsymbol{\Omega}^*) \frac{1}{2a \cdot \operatorname{tg} \alpha} \mathbf{m} , \tag{11b}$$

$$\mathbf{p}_z^{(t)} = -(\boldsymbol{\Omega}^* - \boldsymbol{\Omega}) \frac{1}{2a} \mathbf{m} . \tag{12b}$$

The complete force system due to internal forces acting at the nodes is obtained by taking the effect of unknown normal forces acting on the bars into consideration. Displacing components of the normal forces of both bars joining at node i are (Fig. 4):

$$p_{xi}^{(n)} = -(n_{i,i+1} + n_{i,i-1}) \cdot \sin \alpha,$$

$$p_{zi}^{(n)} = (n_{i,i+1} - n_{i,i-1}) \cdot \cos \alpha; \tag{13a, b}$$

that is:

$$\mathbf{p}_x^{(n)} = -(\mathbf{E} + \mathbf{\Omega}^*) \sin \alpha \cdot \mathbf{n},$$

$$\mathbf{p}_z^{(n)} = (\mathbf{E} - \mathbf{\Omega}^*) \cos \alpha \cdot \mathbf{n}. \tag{13c, d}$$

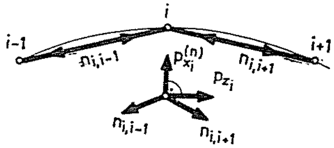

Fig. 4

Utilizing Eqs (3), (4), (5), (7), (11), (12) and (13), equilibrium equation system of polygon ring nodes is:

$$(2\mathbf{E} - \mathbf{\Omega} - \mathbf{\Omega}^*) \frac{1}{2a \cdot \mathrm{tg}\,\alpha} \mathbf{D} \left[(2\mathbf{E} - \mathbf{\Omega} - \mathbf{\Omega}^*) \frac{1}{2a \cdot \mathrm{tg}\,\alpha} \mathbf{u} + \right.$$
$$\left. + (\mathbf{\Omega} - \mathbf{\Omega}^*) \frac{1}{2a} \mathbf{w} \right] + (\mathbf{E} + \mathbf{\Omega}^*) \sin \alpha \cdot \mathbf{n} + \mathbf{C}_x \mathbf{u} = \mathbf{p}_x, \tag{14a}$$

$$(\mathbf{\Omega}^* - \mathbf{\Omega}) \frac{1}{2a} \mathbf{D} \left[(2\mathbf{E} - \mathbf{\Omega} - \mathbf{\Omega}^*) \frac{1}{2a \cdot \mathrm{tg}\,\alpha} \mathbf{u} + (\mathbf{\Omega} - \mathbf{\Omega}^*) \frac{1}{2a} \mathbf{w} \right] -$$
$$- (\mathbf{E} - \mathbf{\Omega}^*) \cos \alpha \cdot \mathbf{n} + \mathbf{C}_z \mathbf{w} = \mathbf{p}_z, \tag{14b}$$

$$(\mathbf{\Omega} + \mathbf{E})\mathbf{u} \cdot \sin + (\mathbf{\Omega} - \mathbf{E})\mathbf{w} \cdot \cos \alpha = 0. \tag{14c}$$

Introducing matrices

$$\frac{1}{(2a \cdot \mathrm{tg}\,\alpha)^2} (2\mathbf{E} - \mathbf{\Omega} - \mathbf{\Omega}^*) \mathbf{D} (2\mathbf{E} - \mathbf{\Omega} - \mathbf{\Omega}^*) + \mathbf{C}_x = \mathbf{K}_{11},$$

$$\frac{1}{4a^2 \cdot \mathrm{tg}\,\alpha} (2\mathbf{E} - \mathbf{\Omega} - \mathbf{\Omega}^*) \mathbf{D} (\mathbf{\Omega} - \mathbf{\Omega}^*) = \mathbf{K}_{12},$$

$$\frac{1}{4a^2} (\mathbf{\Omega}^* - \mathbf{\Omega}) \mathbf{D} (\mathbf{\Omega} - \mathbf{\Omega}^*) + \mathbf{C}_z = \mathbf{K}_{22}, \tag{15a—e}$$

$$(\mathbf{E} + \mathbf{\Omega}^*) \sin \alpha = \mathbf{K}_{13},$$

$$-(\mathbf{E} - \mathbf{\Omega}^*) \cos \alpha = \mathbf{K}_{23}$$

permits to write Eqs (14) in hypermatrix form:

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \mathbf{K}_{13} \\ \mathbf{K}_{12}^* & \mathbf{K}_{22} & \mathbf{K}_{23} \\ \mathbf{K}_{13}^* & \mathbf{K}_{23}^* & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \\ \mathbf{n} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_x \\ \mathbf{p}_z \\ \emptyset \end{bmatrix}. \tag{16}$$

3. Solution of the equation system

Provided matrices $\mathbf{C}_x$, $\mathbf{C}$ and $\mathbf{D}$ in Eqs (15a—c) stand for constant bedding and stiffness values independent of the i value, matrices are products of the unit matrix by a scalar coefficient each. Then the scalar coefficients can be factored out, matrices $\mathbf{K}_{11}, \ldots, \mathbf{K}_{23}$ are interchangeable in multiplication. From the theory of hypermatrices with interchangeable elements it is known that — keeping the invertibility of hyper-elements in mind — any operation referred to ordinary matrices of scalar elements may be performed on hypermatrices consisting of such blocks. Thus, applying the Cramer rule, formulae for $\mathbf{u}$, $\mathbf{w}$ and $\mathbf{n}$ can be made explicit. Provided, however, $\mathbf{D}$ or $\mathbf{C}_x$ and $\mathbf{C}_y$ cannot be expressed as multiples of the unit matrix, solution of vectorial equation (16) requires to solve a linear equation system of size $3n$.

4. Yield of ring and bedding

The relationships above lead to a step-by-step method of a finite number of steps for taking the yield of ring and bedding into consideration, provided yield conditions are conform to equalities $|m_i| = m_{pl}$ and $|q_{x_i}| = q_{xpl}$ or $|q_{z_i}| = q_{zpl}$, respectively.

Let us start from the analysis of a perfectly elastic ring on perfectly elastic bedding. Let the load intensity increase gradually from zero to the plastic limit. Until then, relationships for the elastic ring on elastic bedding are valid, $\mathbf{D}$, $\mathbf{C}_x$ and $\mathbf{C}_z$ are multiples of the unit matrix. Beyond the plastic limit, load intensity increase is not accompanied by a further increase of bedding reaction or nodal moment. Therefore, upon the next load increment, the structure behaves as if at the yield spot plastic displacement were uninhibited. Its mathematical counterpart is to zero the diagonal element in $\mathbf{D}$, $\mathbf{C}_x$ or $\mathbf{C}_y$ for the yield constraint.

The incremental displacement due to load increment exactly producing the next yield has to be determined using the modified matrix. Reckoning with the modified matrix does not require to invert it again since the inverse of a matrix modified by a diad can be determined from the inverse of the original matrix by means of the well-known Sherman—Morrison formula.

There are two possibilities of establishing the system of relationships after the second yield. In the first, simpler case, the load increment to be calculated after modification of the yield constraint produces a displacement increment of the same sign as the previous ones, now the load increment can be directly determined from the intensity needed for the next yield. The other, infrequent though not exclusible possibility is when the load increment calculated after the second modification produces a displacement increment over the former yield constraint of the opposite sign as before. Now, the constraint becomes elastic again so that for calculating the load increment, the diagonal element zeroed in the previous step has to be reestablished. The subsequent load increments will be calculated as before but for every previously zeroed element it has to be checked whether the displacement increment has the same sign as the previous one, and if not, for calculating the load increment, the original diagonal element has to be reestablished.

At the ultimate plastic load capacity of the structure, the modified matrix becomes singular, hence deformation may increase without further load increment. This deformation increment is the typical displacement system in the plastic failure mechanism of the structure.

The outlined mathematical algorithm is rather cumbersome, and since, in course of taking consecutive yields into consideration, the interchangeability of hyperelements in the matrix of relationships, initially accessible to semi-analytic treatment, vanishes, use of a computer is imperative.

Application of the algorithm may be attempted for taking the displacement-dependent bedding coefficients into consideration, then, in calculating displacement increments related to load increments, the corresponding diagonal elements are modified not only because of yield but also of displacements intervened in preceding steps. Modification by a high number of diads being rather labour-consuming, in this event the complete set of equations has to be solved for every load increment. Reliability of computation results much depends on what the load increments are, therefore in applying this method, nonlinearity of the bedding requires also the step before the predicted yield to be replaced by several steps.

Application of an inconsiderate relationship between displacement and bedding reaction renders, even for small load increments, the solutions of the problem irreal or impossible. On the other hand, in lack of published results on, and experience with, numerical calculation, preconditions of stability of the computation and of the reality of results cannot be enounced.

Summary

A substitutive mathematical model is presented for the analysis of rings of elastic or elastoplastic material and bedding. This model based on the direct finitization of the structure permits rather simple analysis of in-plane loaded rings of constant stiffness on uniform elastic bedding, and offers a rather illustrative method for taking ring or bedding yield into consideration.

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* In Hungarian.