EQUILIBRIUM CONDITION OF A RECTANGULAR CABLE NET STRETCHED OVER A RIGID FRAME

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1. Introduction

Big rooms are aptly roofed by so-called suspended roofs supported on cable nets fitting the ground plan.

Exact structural analysis of cable nets requires to determine the starting form, advisably starting from a rectangular net of a set of cables orthogonal in ground plan, stretched over a rigid frame of rectangular ground plan, to meet computer aspects. An expedient method for determining the equilibrium form of such a cable will be presented.

2. Fundamental equation of the cable net

Equilibrium form of the cable net of the layout seen in Fig. 1 is described by the matrix equation (details see p. 69 in [1]):

$$\frac{1}{a}\mathbf{C}_x\,\mathbf{Z}\,\mathbf{H}_x+\frac{1}{b}\mathbf{H}_y\,\mathbf{Z}\,\mathbf{C}_y=\mathbf{Q} \tag{1}$$

where

$$\mathbf{Q}=\mathbf{F}+\frac{1}{a}\mathbf{Z}_{0y}\,\mathbf{H}_x+\frac{1}{b}\mathbf{H}_y\,\mathbf{Z}_{0x}\,.$$

Letter symbols being:

$\mathbf{Z}$, $\mathbf{F}$ (m, n) (m, n)	matrices size $m\times n$ including height coordinates of internal nodes and of vertical nodal forces, respectively;
$\mathbf{C}_x$ $\mathbf{C}_y$ (m, m) (n, n)	tridiagonal matrices including second partial difference operators along x and y, respectively;
$\mathbf{H}_x$ $\mathbf{H}_y$ (n, n) (m, m)	diagonal matrices containing horizontal components of forces arising in cables along x and y, respectively;
a, b	ground-plan spacings of cables along y and x, respectively;
$\mathbf{Z}_{0x}$, $\mathbf{Z}_{0y}$	matrices size $m\times n$ containing prescribed height data of boundaries along x and y, resp., where only columns 1 and m, and rows 1 and n are non-zero.

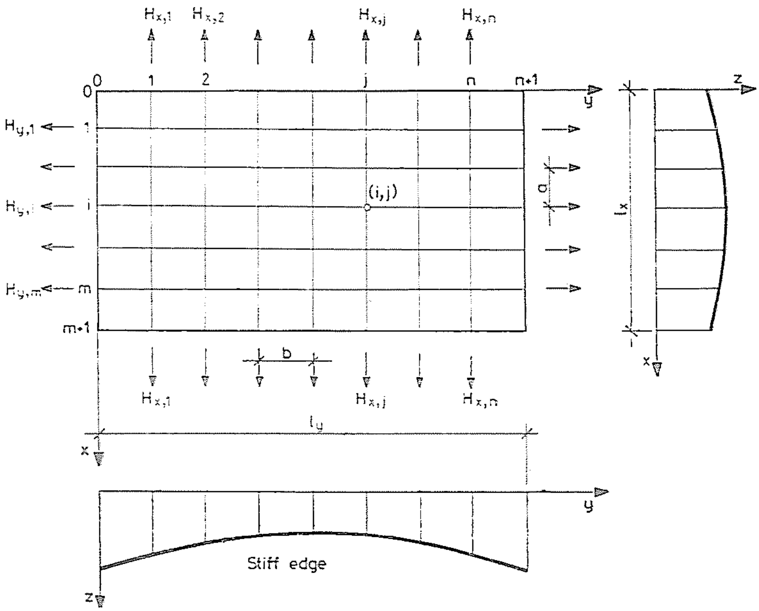

Fig. 1

3. Basic algorithm for solving (1)

Multiplying (1) by $\mathbf{H}_y^{-1}$ from the left and by $\mathbf{H}_x^{-1}$ from the right yields

$$\frac{1}{a}\,\mathbf{H}_y^{-1}\,\mathbf{C}_x\,\mathbf{Z}+\frac{1}{b}\,\mathbf{Z}\mathbf{C}_y\,\mathbf{H}_x^{-1}=\mathbf{H}_y^{-1}\,\mathbf{Q}\mathbf{H}_x^{-1}$$

brought by other notations to the form:

$$\frac{1}{a}\,\mathbf{A}_x\mathbf{Z}+\frac{1}{b}\,\mathbf{Z}\,\mathbf{A}_y=\widehat{\mathbf{Q}} \tag{2}$$

(where $\mathbf{A}_x=\mathbf{H}_y^{-1}\,\mathbf{C}_x$, $\mathbf{A}_y=\mathbf{C}_y\,\mathbf{H}_y^{-1}$ and $\widehat{\mathbf{Q}}=\mathbf{H}_y^{-1}\mathbf{Q}\mathbf{H}_x^{-1}$).

A rather expedient algorithm — first published by Szabó [2] — is available for solving equations type (2) (p. 71 in [1]).

Provided matrices $\mathbf{A}_x$ and $\mathbf{A}_y$ may be produced in canonic forms $\mathbf{A}_x=\mathbf{U}_x\mathbf{\Lambda}_x\mathbf{U}_x^{-1}$ and $\mathbf{A}_y=\mathbf{U}_y\mathbf{\Lambda}_y\mathbf{U}_y^{-1}$, then Eq. (2) is solved to:

$$\mathbf{Z}=\mathbf{U}_x\,[\mathbf{M}\curlywedge(\mathbf{U}_x^{-1}\,\widehat{\mathbf{Q}}\mathbf{U}_y)]\,\mathbf{U}_y^{-1} \tag{3}$$

$\curlywedge$ being symbol of logical — or element-wise — multiplication of matrices (also termed in literature *Hadamard* product), $\mathbf{M}$ being a matrix to be formed from eigenvalues of $\mathbf{A}_x$ and $\mathbf{A}_y$, having as (i, j)th element:

$$[M_{i,j}] = \left[\frac{1}{\frac{\lambda_{x,i}}{a} + \frac{\lambda_{y,j}}{b}}\right].$$

This algorithm is particularly advantageous if spectral decomposition of coefficient matrices $\mathbf{A}_x$ and $\mathbf{A}_y$ is known or simple to establish. This is the case e.g. if horizontal components of cable forces arising in cables parallel in ground plan are equal, that is, if in Eq. (1), $[H_{x,j}] = H_x$ $(j = 1, 2, \ldots, n)$, $[H_{y,j}) = H_y$ $(i = 1, 2, \ldots, m)$. Namely then, solution (3) requires spectral decompositions $\mathbf{C}_x = \mathbf{C}_m = \mathbf{U}_m \mathbf{\Lambda}_m \mathbf{U}_m$ and $\mathbf{C}_y = \mathbf{C}_n = \mathbf{U}_n \mathbf{\Lambda}_n \mathbf{U}_n$ of matrices $\mathbf{C}_x$ and $\mathbf{C}_y$ alone, elements of the involved matrices being, however, known as formulae, and depend only on the matrix order. For the sake of completeness, these relationships are:

$$\begin{aligned} [U_{\mu;i,j}] &= \sqrt{\frac{2}{\mu+1}} \sin \frac{i \cdot j \cdot \pi}{\mu+1} \qquad (\mu = m, n) \\ [\lambda_{\mu,i}] &= 4 \sin^2 \frac{i \cdot \pi}{2(\mu+1)} \qquad (i, j = 1, 2, \ldots, \mu). \end{aligned} \tag{4}$$

In the actual case, matrices $\mathbf{H}_x$ and $\mathbf{H}_y$ contain different elements, imposing total spectral decomposition of — generally not symmetric — matrices $\mathbf{A}_x$ and $\mathbf{A}_y$ of size $(m \times m)$ and $(n \times n)$, respectively, much adding to the computation work in case of big sizes, even likely to yield complex eigenvalues and eigenvectors, requiring double storage space and complex arithmetics. In the following, an iteration method for solving (1) making use of advantages of the above algorithm, Eq. (3) with nothing but spectral decomposition of matrices $\mathbf{C}_x$ and $\mathbf{C}_y$ will be described.

4. The suggested method

4.1 *First variety of iteration*

Properly amplifying both left- and right-hand side of Eq. (1) yields

$$\frac{1}{a}\mathbf{C}_x(\mathbf{Z}\mathbf{H}_x + \mathbf{H}_y\mathbf{Z}) + \frac{1}{b}(\mathbf{Z}\mathbf{H}_x + \mathbf{H}_y\mathbf{Z})\,\mathbf{C}_y = \mathbf{Q}' \tag{5}$$

with

$$\mathbf{Q}' = \mathbf{Q} + \frac{1}{a}\mathbf{C}_x\,\mathbf{H}_y\,\mathbf{Z} + \frac{1}{b}\mathbf{Z}\mathbf{H}_x\,\mathbf{C}_y.$$

Considering, for a while, matrix $\mathbf{Z}' = \mathbf{Z}\mathbf{H}_x + \mathbf{H}_y\mathbf{Z}$ in the left-hand side of (4) as unknown, and the right-hand side to be known, then $\mathbf{Z}'$ may be directly expressed as:

$$\mathbf{Z}' = \mathbf{U}_x [\mathbf{M} \curlywedge (\mathbf{U}_x \mathbf{Q}' \mathbf{U}_y)] \mathbf{U}_y$$

yielding $\mathbf{Z}$ from $\mathbf{Z}'$ by another logical multiplication:

$$\mathbf{Z} = \mathbf{M}_1 \curlywedge \mathbf{Z}' = \mathbf{M}_1 \curlywedge \{\mathbf{U}_x [\mathbf{M} \curlywedge (\mathbf{U}_x \mathbf{Q}' \mathbf{U}_y)] \mathbf{U}_y\}. \tag{6}$$

In this solution, $\mathbf{U}_x$ and $\mathbf{U}_y$ are modal matrices of matrices $\mathbf{C}_x$ and $\mathbf{C}_y$, and $\mathbf{M}$ is a matrix developed from eigenvalues of the same matrices.

Matrix $\mathbf{M}_1$ will be formed from elements of matrices $\mathbf{H}_x$ and $\mathbf{H}_y$ following the rule:

$$[\mathbf{M}_{1;i,j}] = \left[\frac{1}{\mathbf{H}_{y,i} + \mathbf{H}_{x,j}}\right].$$

Solution by iteration will make use of Eqs (5) and (6) so that in the first step, only $\mathbf{Q}$ in the right-hand side of (5) will be taken into consideration, computing the pertaining matrix $\mathbf{Z}$ from (5), using it to compute the new right-hand side of (5) and again calculating $\mathbf{Z}$ from (6) for the obtained matrix $\mathbf{Q}'$ etc. Formulating the first and the ν-th iteration steps:

$$\left.\begin{array}{ll} 1^\circ & \mathbf{Q}_1' = \mathbf{Q} \\ & \mathbf{Z}_1 = \mathbf{M}_1 \curlywedge \{\mathbf{U}_x [\mathbf{M} \curlywedge (\mathbf{U}_x \mathbf{Q}_1' \mathbf{U}_y)] \mathbf{U}_y\} \\ & \vdots \\ \nu^\circ & \mathbf{Q}_\nu' = \mathbf{Q} + \frac{1}{a} \mathbf{C}_x \mathbf{H}_x \mathbf{Z}_{\nu-1} + \frac{1}{b} \mathbf{Z}_{\nu-1} \mathbf{H}_x \mathbf{C}_y \\ & \mathbf{Z}_\nu = \mathbf{M}_1 \curlywedge \{\mathbf{U}_x [\mathbf{M} \curlywedge (\mathbf{U}_x \mathbf{Q}_\nu' \mathbf{U}_y)] \mathbf{U}_y\} \end{array}\right\}. \tag{7}$$

In every iteration step, the obtained matrix $\mathbf{Z}_\nu$ is resubstituted into (1) to see what a load the determined net shape can balance:

$$\mathbf{Q}_{e\nu} = \frac{1}{a} \mathbf{C}_x \mathbf{Z}_\nu \mathbf{H}_x + \frac{1}{b} \mathbf{H}_y \mathbf{Z}_\nu \mathbf{C}_y \tag{8}$$

and the iteration is considered as complete if some norm of difference matrix

$$\mathbf{Q} - \mathbf{Q}_{e\nu}$$

is less than a specified value.

4.2 *Accelerated iterative solution*

Numerical experience with the method under 4.1 shows it to converge but the convergence is much accelerated according to the following considerations: Adding Eq. (1) in a zeroed form:

$$\mathbf{O} = \mathbf{Q} - \frac{1}{a}\mathbf{C}_m \mathbf{Z} \mathbf{H}_x - \frac{1}{b}\mathbf{H}_y \mathbf{Z} \mathbf{C}_n$$

to (4) yields, after arranging:

$$\frac{1}{a}\mathbf{C}_x(\mathbf{Z}\mathbf{H}_x + \mathbf{H}_y\mathbf{Z}) + \frac{1}{b}(\mathbf{Z}\mathbf{H}_x + \mathbf{H}_y\mathbf{Z})\mathbf{C}_y =$$

$$= 2\mathbf{Q} - \frac{1}{a}\mathbf{C}_x(\mathbf{Z}\mathbf{H}_x - \mathbf{H}_y\mathbf{Z}) + \frac{1}{b}(\mathbf{Z}\mathbf{H}_x - \mathbf{H}_y\mathbf{Z})\mathbf{C}_y. \tag{9}$$

Performing iteration steps under 4.1 for Eqs (6) and (9) leads, after starting with $\mathbf{Q}'_1 = 2\mathbf{Q}$, to the ν-th step:

$$\left.\begin{aligned} \mathbf{Q}'_\nu &= 2\mathbf{Q} - \frac{1}{a}\mathbf{C}_x(\mathbf{Z}_{\nu-1}\mathbf{H}_x - \mathbf{H}_y\mathbf{Z}_{\nu-1}) + \frac{1}{b}(\mathbf{Z}_{\nu-1}\mathbf{H}_x - \mathbf{H}_y\mathbf{Z}_{\nu-1})\mathbf{C}_y \\ \mathbf{Z}_\nu &= \mathbf{M}_1 \wedge \{\mathbf{U}_x[\mathbf{M} \wedge (\mathbf{U}_x\mathbf{Q}'_\nu\mathbf{U}_y)]\mathbf{U}_y\}. \end{aligned}\right\} \tag{10}$$

Iteration is finished as indicated under 4.1.

5. Numerical analyses

Applicability of the presented method has been tested numerically. To this aim, computer programs have been made for iterations (7) and (10). In both iterations, the program forms value

$$\alpha = \frac{\max|Q_{i,j} - Q_{ei,j}|}{\max|Q_{i,j}|}$$

each iteration step, and the iteration is considered as complete for $\alpha < 10^{-4}$.

The program has been written in ALGOL-60 and trial runs were made on the computer ODRA-1204 of the Faculty of Civil Engineering, Technical University, Budapest.

First, the program has been tested, and the convergence examined on an example detailed, with its numerical results, on p. 75 in [1]. Starting data:

$$m = 7;\ n = 9;\ a = b = 2.0 \text{ m}$$

$$\mathbf{H}_x = 20\,\langle 1;1;1;1;1;1;1;1;1\rangle$$

$$\mathbf{H}_y = 50\,\langle 0.712;\ 0.872;\ 0.968;\ 1.0;\ 0.968;\ 0.872;\ 0.712\rangle.$$

Other data can be read off Fig. 2. The first numerical observations are rather favourable and justify the efficiency of convergency acceleration. Fig. 3 is a semilog. plot of α *vs.* iteration steps. In the first-type iteration (Eqs 7), the required accuracy was obtained in 18 iteration steps, as against 7 steps in the accelerated iteration. In both cases, one iteration step took 11.7 sec.

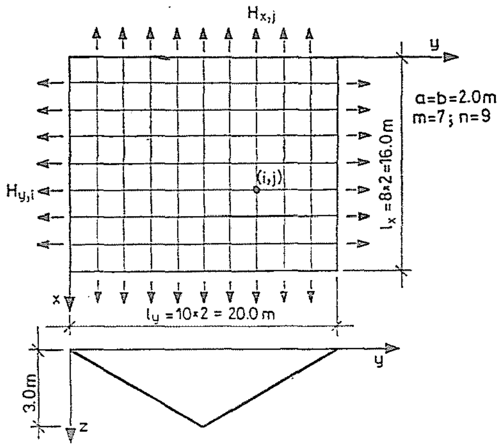

Fig. 2

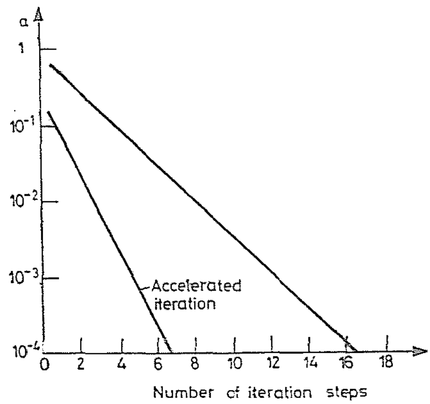

Fig. 3

Final outcome of accelerated iteration:

$$\mathbf{Z} = \begin{bmatrix} 0.548386 & 1.044140 & 1.479323 & 1.822911 & 1.999452 \dots \\ 0.470439 & 0.876091 & 1.201680 & 1.423154 & 1.507366 \dots \\ 0.413764 & 0.762695 & 1.031175 & 1.202809 & 1.262615 \dots \\ 0.393942 & 0.723986 & 0.975021 & 1.133072 & 1.187261 \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

(Dotted part refers to double symmetry values. Remark that outputs obtained by either of both iterations are perfectly identical up to four digits in compliance with bounds for α.)

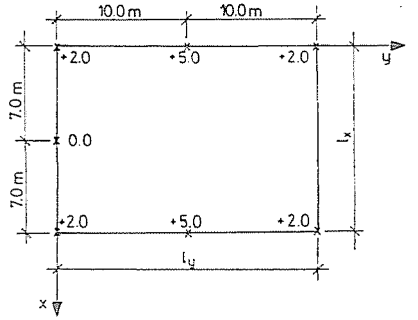

Fig. 4

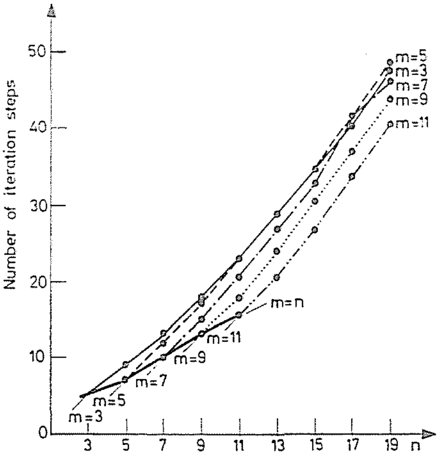

Fig. 5

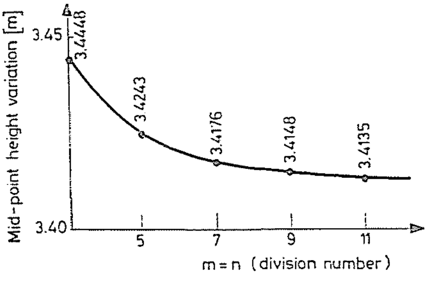

Fig. 6

After having successfully tried the program, the effect of division number has been tested on a problem with input data keeping the load to stretching force relation constant.

Basic data are:

$$l_x = 14 \text{ m}; \qquad l_y = 20 \text{ m}; \quad m, n \text{ vary.}$$

$$a = \frac{l_x}{m+1}; \qquad b = \frac{l_y}{n+1}$$

$$k = \text{entier}\left(\frac{n}{2}\right) + 1$$

$$H_{x,j} = H_{x,n+1-j} = 5\, b \sin\frac{j\pi}{2k}; \qquad (j = 1, 2, \ldots, n)$$

$$l = \text{entier}\left(\frac{m}{2}\right) + 1$$

$$H_{y,i} = H_{y,m+1-i} = 10\, a \sin\frac{i\pi}{2l}; \qquad (i = 1, 2, \ldots, m).$$

Elevation data of edge break points are seen in Fig. 4 (in a projection with data), intermediate reaches being straight, the program computed them automatically, corresponding to the division numbers. The vertical load was 2.0 kN/m^2.

The number of iteration steps vs. n has been plotted in Fig. 5.

Fig. 6 presents the variation of the mid-point elevation *vs.* division number, for $m = n$.

Summary

An iteration method is presented for determining the equilibrium form of cable nets stretched over rigid frames, over rectangular ground plane, for the case of cable-wise varying stretching forces. The problem may be considered as the finite model of an inextensible membrane stressed by varying forces, or as the numerical solution of Poisson's equation describing the equilibrium form of this membrane.

References

1. Szabó, J.—Kollár, L.: Analysis of Suspension Roofs. (In Hungarian). Műszaki Könyvkiadó, Budapest, 1974.
2. Szabó, J.: Ein neues Verfahren zur unmittelbaren numerischen Lösung der Dirichletschen Randwertaufgaben. ZAMM, **38.** (1—4).

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