FREQUENCY-DEPENDENT GEOMETRICAL STIFFNESS MATRIX FOR THE VIBRATION ANALYSIS OF BEAM SYSTEMS

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1. Introduction

Frequency of beam vibrations is significantly affected by the static axial force. In analysing beam system vibrations by the method of finite elements, this fact is reckoned with by composing stiffness matrix $\mathbf{K}$ from the matrix differential equation

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = 0$$

from an elastic stiffness matrix and from a so-called geometrical one (containing normal force N) [1].

Geometrical stiffness matrix has been introduced in [2] so as to include all internal forces (N, M, T for in-plane beams). Deduction omits the effect of shear deformations.

In dynamic analyses, beam vibrations — especially at higher frequencies — are much affected by shear deformations. For mass and elastic stiffness matrices deduced reckoning with the effect of shear deformations we refer to [1]. In the following, determination of the geometrical stiffness matrix with respect to the effect of shear deformation will be presented. Analyses refer to plane structures composed from straight-axed bars of constant cross section, of homogeneous, isotropic, elastic material. The effect of the internal damping of the material on vibrations will be neglected.

2. Deduction of geometrical stiffness matrices taking shear deformation into account

According to the geometrical theory, equilibrium and compatibility equations of beam systems assuming zero kinematic load are [3]:

$$\begin{bmatrix} \mathbf{D} & \mathbf{G} \\ \mathbf{G} & \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{s} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ 0 \end{bmatrix} = 0 .$$

Let us determine geometrical stiffness matrix $\mathbf{D}$ according to procedures found in [2] and [4]. Matrix $\mathbf{D}$ will be obtained from

$$\mathbf{D}\mathbf{u} = \frac{d}{d\mathbf{u}} \Delta \mathbf{u}_2^* \mathbf{r} = \frac{dL_r}{d\mathbf{u}} \tag{1}$$

deduced from the principle of minimum potential energy, where $\Delta \mathbf{u}_2$ is a vector containing the quadratic term of displacement $\Delta \mathbf{u}$ due to load change, and $\mathbf{r}$ is the vector of developed internal forces.

Taking shear deformations into consideration, secondary energy L_r may be written as:

$$L_r = -\int_V \sigma_x \varepsilon_{x_2} dV - \int_V \tau_{xy} \gamma_{xy_2} dV \tag{2}$$

where ε_{x_2} and γ_{xy_2} are quadratic terms of the corresponding normal strains and shear strains, respectively:

$$\varepsilon_{x_2} = \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\} \tag{3}$$

$$\gamma_{xy_2} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}. \tag{4}$$

u and v being inner beam point displacements.

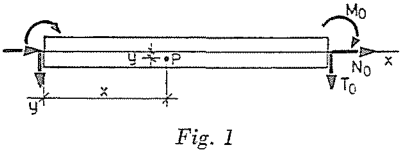

Fig. 1

These displacements, in terms of beam axis point displacements, are:

$$v(x, y) = v(x) \tag{5}$$

$$u(x, y) = u(x) - y\frac{\partial v'(x)}{\partial x} \tag{6}$$

where $v(x)$ and $u(x)$ are beam axis displacements along y and x, $v'(x)$ being the part due to the bending moment of the beam axis displacement along y. Substituted into (3) and (4):

$$\varepsilon_{x_2} = \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\} - y\frac{\partial u}{\partial x}\frac{\partial^2 v'}{\partial x^2} + \frac{y^2}{2}\left(\frac{\partial^2 v'}{\partial x^2}\right)^2 \tag{7}$$

$$\gamma_{xy_2} = -\frac{\partial u}{\partial x}\frac{\partial v'}{\partial x} + y\frac{\partial^2 v'}{\partial x^2}\frac{\partial v'}{\partial x}. \tag{8}$$

Substituting into (2) yields secondary energy L_r:

$$L_r = -\int_V \sigma_x \left\{\frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] + \frac{y^2}{2}\left(\frac{\partial^2 v'}{\partial x^2}\right)^2 - y\frac{\partial u}{\partial x}\frac{\partial^2 v'}{\partial x^2}\right\} dV -$$

$$-\int_V \tau_{xy}\left(y\frac{\partial^2 v'}{\partial x^2}\frac{\partial v'}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v'}{\partial x}\right) dV. \tag{9}$$

Beginning with integrations along the cross section, these are irrelevant to centroid displacement, permitting their functions to be factorized. For the remaining parts, the following relationships hold:

$$\int_F \sigma_x \, dF = N$$

$$\int_F y\,\sigma_x \, dF = M$$

$$\int_F \tau_{xy} \, dF = T.$$

Values of terms $\int_F \sigma_x y^2 \, dF$ and $\int_F \tau_{xy} y \, dF$ are also dependent on the cross-sectional form. If axes y and z, in, and normal to, the cross-sectional plane, respectively, are symmetry axes (e.g. circular and rectangular cross sections):

$$\int_F \sigma_x y^2 \, dF = i^2 N,$$

$$\int_F \tau_{xy} y \, dF = 0$$

(i being the cross section inertia radius). Now:

$$L_r = -\int_0^l N\frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\} dx - \int_0^l N\frac{1}{2} i^2 \left(\frac{\partial^2 v'}{\partial x^2}\right)^2 dx +$$

$$+\int_0^l M\frac{\partial u}{\partial x}\frac{\partial^2 v'}{\partial x^2} dx + \int_0^l T\frac{\partial u}{\partial x}\frac{\partial v'}{\partial x} dx, \tag{10}$$

and

$$L_r = -\int_0^l \Delta \mathbf{v}_2^* \tilde{\mathbf{r}} \, dx,$$

where

$$\tilde{\mathbf{r}}^* = [N\ M\ T]$$

$$\Delta \mathbf{v}_2 = \frac{1}{2} \begin{bmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + i^2 \left(\frac{\partial^2 v'}{\partial x^2}\right)^2 \\ -2 \frac{\partial u}{\partial x} \frac{\partial^2 v}{\partial x^2} \\ -2 \frac{\partial u}{\partial x} \frac{\partial v'}{\partial x} \end{bmatrix}.$$

Displacement functions of the beam axis can be written in terms of beam end displacements as:

$$u(x) = \mathbf{a}_u^* \mathbf{u}_k$$

$$v'(x) = \mathbf{a}_{v_M}^* \mathbf{u}_k = \mathbf{a}_{v'}^* \mathbf{u}_k$$

$$v(x) = (\mathbf{a}_{v_M}^* + \mathbf{a}_{v_T}^*) \mathbf{u}_k = \mathbf{a}_v^* \mathbf{u}_k$$

where

$\mathbf{a}_{v_M}^*$ and $\mathbf{a}_{v_T}^*$ are displacements obtained from functions for the bending moment, and the shear force, respectively.

Stresses in a beam cross section acted upon by beam end forces alone may be written as a function of these latter.

$$N(x) = N_0$$

$$M(x) = M_0 + (l - x) T_0$$

$$T(x) = T_0.$$

Substitutions according to (4) yield for the geometrical stiffness matrix:

$$\mathbf{D} = N_0 \int_0^l \mathbf{A}_N \, dx + M_0 \int_0^l \mathbf{A}_M \, dx + T_0 \int_0^l \mathbf{A}_T \, dx \tag{11}$$

where

$$\mathbf{A}_N = \frac{\partial \mathbf{a}_u}{\partial x} \frac{\partial \mathbf{a}_u^*}{\partial x} + \frac{\partial \mathbf{a}_v}{\partial x} \frac{\partial \mathbf{a}_v^*}{\partial x} + i^2 \frac{\partial^2 \mathbf{a}_{v'}}{\partial x^2} \frac{\partial^2 \mathbf{a}_{v'}^*}{\partial x^2} \tag{12}$$

$$\mathbf{A}_M = -\left\{ \frac{\partial \mathbf{a}_u}{\partial x} \frac{\partial^2 \mathbf{a}_{v}^*}{\partial x^2} + \frac{\partial^2 \mathbf{a}_{v}^*}{\partial x^2} \frac{\partial \mathbf{a}_u^*}{\partial x} \right\} \tag{13}$$

$$\mathbf{A}_T = (l - x) \mathbf{A}_M - \left\{ \frac{\partial \mathbf{a}_u}{\partial x} \frac{\partial \mathbf{a}_{v'}^*}{\partial x} + \frac{\partial \mathbf{a}_{v'}}{\partial x} \frac{\partial \mathbf{a}_u^*}{\partial x} \right\}. \tag{14}$$

Displacement functions needed to obtain the matrices are:

$$\mathbf{a}_u^* = [1 - \xi \mid 0 \mid 0 \mid \xi \mid 0 \mid 0] \tag{15}$$

$$\mathbf{a}_{v'}^* = c_1 \left[0 \,|\, 1 - 3\xi^2 + 2\xi^3 \,|\, l \left(\xi - 2\xi^2 + \xi^3 - \frac{1}{2}(1 - \xi^2)\Phi \right) |\, 0 \,|\, 3\xi^2 - \right.$$

$$\left. - 2\xi^3 \,|\, l \left(\xi^3 - \xi^2 + \frac{1}{2}\xi^2 \Phi \right) \right] \quad (16)$$

$$\mathbf{a}_v^* = c_1 \left[0 \,|\, 1 - 3\xi^2 + 2\xi^3 + (1 - \xi)\Phi \,|\, l \left(\xi - 2\xi^2 + \xi^3 - \frac{1}{2}(\xi - \xi^2)\Phi \right) |\, 0 \,|\, 3\xi^2 - \right.$$

$$\left. - 2\xi^3 + \xi\Phi \,|\, \left(\xi^3 - \xi^2 - \frac{1}{2}(\xi - \xi^2)\Phi \right) l \right] \quad (17)$$

where $\xi = \frac{x}{l}$, $c_1 = \frac{1}{1 + \Phi}$, $\Phi = \frac{12EJ}{GF_s l}$ (F_s being the cross-sectional area modified by the form factor, G and E are moduli of elasticity, J is the cross-sectional moment of inertia).

After integrations, matrix $\mathbf{D}$ can be written as seen in Table 1. (The matrix being symmetrical, the table contains only the lower triangle.)

3. Deduction of the geometrical stiffness matrix from the approximate frequency-dependent function of displacement

Matrices $\mathbf{A}_N$, $\mathbf{A}_M$, $\mathbf{A}_T$ needed for the calculation of matrix $\mathbf{D}$ were seen in the previous Chapter to be determined using displacement functions (15) to (17), utilizing bar end displacements to yield the displacements of inner beam points determined e.g. by solving the beam differential equation. Displacement functions (15) to (17) concern static beam end displacements, and their application in dynamic analyses leads to approximations to be improved by densifying the nodes. Displacement functions obtained by solving the differential equation of vibrating beams are rather intricate and unfit to reduce the vibration problem to a linear eigenvalue problem. Analysis by means of a so-called approximate dynamic displacement function (1) leads to a double-size but linear eigenvalue problem, providing for a rapid convergence.

In this case:

$$\mathbf{a}_{u_0}^* = \mathbf{a}_{u_0}^* + \omega^2 \mathbf{a}_{u_2}^* \quad (18)$$

$$\mathbf{a}_v^* = \mathbf{a}_{v_0}^* + \omega^2 \mathbf{a}_{v_2}^* \quad (19)$$

$$\mathbf{a}_{v'}^* = \mathbf{a}_{v'_0}^* + \omega^2 \mathbf{a}_{v'_2}^* \quad (20)$$

where first terms are displacement functions (15) to (17), while displacement functions in the second terms have been presented in (1) and (5) omitting, and reckoning with, shear deformations, respectively. In this latter case:

Table 1

$\frac{N}{l}$					
$-c_1\frac{2T}{l}$	$\frac{N}{l}c_1^2\left(\frac{6}{5}+2\Phi+2\Phi^2+\frac{12i^2}{l^2}\right)$				
$-\frac{M}{l}-c_1T$	$Nc_1^2\left(\frac{1}{10}+\frac{6i^2}{l^2}\right)$	$Nc_1^2\left(\frac{2l}{15}+\frac{l\Phi}{6}+\frac{l\Phi^2}{12}+\frac{4i^2}{l}+\frac{2i^2\Phi}{l}+\frac{i^2\Phi^2}{l}\right)$			
$-\frac{N}{l}$	$c_1\frac{2T}{l}$	$\frac{M}{l}+c_1T$	$\frac{N}{l}$		
$c_1\frac{2T}{l}$	$-\frac{Nc_1^2}{l}\left(\frac{6}{5}+2\Phi+\Phi^2+\frac{12i^2}{l^2}\right)$	$-c_1^2N\left(\frac{1}{10}+\frac{6i^2}{l^2}\right)$	$-c_1\frac{2T}{l}$	$\frac{Nc_1^2}{l}\left(\frac{6}{5}+2\Phi+\Phi^2+\frac{12i^2}{l^2}\right)$	
$\frac{M}{l}+c_1T$	$Nc_1^2\left(\frac{1}{10}+\frac{6i^2}{l^2}\right)$	$Nc_1^2\left(-\frac{l}{30}-\frac{l\Phi^2}{12}-\frac{l\Phi}{6}-\frac{2i^2}{l}-\frac{2i^2\Phi}{l}-\frac{i^2\Phi^2}{l}\right)$	$-\frac{M}{l}+c_1T$	$-Nc_1^2\left(\frac{1}{10}+\frac{6i^2}{l^2}\right)$	$Nc_1^2\left(\frac{2l}{15}+\frac{l\Phi}{6}+\frac{l\Phi^2}{12}+\frac{4i^2}{l}+\frac{2i^2\Phi}{l}+\frac{i^2\Phi^2}{l}\right)$

$$\mathbf{a}^*_{u_2} = c_2\,[2\xi - 3\xi^2 + \xi^3 \mid 0 \mid 0 \mid \xi - \xi^3 \mid 0 \mid 0]$$

$$\mathbf{a}^*_{v_2'} = c_3 \Big[0 \mid 66\xi^2 - 156\xi^3 + 105\xi^4 - 21\xi^6 + 6\xi^7 \Big| \, l\Big(12\xi^2 - 22\xi^3 + 21\xi^5 -$$

$$- 14\xi^6 + \frac{1}{2}\Phi(-42\xi^2 + 112\xi^3 - 105\xi^4 + 42\xi^5 - 7\xi^6)\Big) \Big| \, 0 \,|\, 39\xi^2 - 54\xi^3 +$$

$$+ 21\xi^6 - 6\xi^7 \,|\, l\Big(-9\xi^2 + 13\xi^3 - 7\xi^6 + 3\xi^7 + \frac{1}{2}\Phi(21\xi^2 - 28\xi^3 + 7\xi^6)\Big)\Big]$$

$$\mathbf{a}^*_{v_2} = \mathbf{a}^*_{v_2'} + c_3\Big[0 \mid \Phi(63\xi^2 - 147\xi^3 + 105\xi^2 - 21\xi^5) \Big| \frac{1}{2} l\Phi(63\xi^2 - 147\xi^3 +$$

$$+ 105\xi^4 - 21\xi^5) \Big| \, 0 \,|\, \Phi(42\xi^2 - 63\xi^3 + 21\xi^5) \,|\, \frac{1}{2} l\Phi(-42\xi^2 + 63\xi^3 - 21\xi^5)\Big]$$

where $c_2 = \dfrac{\varrho l^2}{6E}$

$$c_3 = \frac{\varrho F l^4}{2520\,\mathrm{EJ}\,(1 + \Phi)} \quad \begin{array}{l}(\varrho - \text{density});\\ (F - \text{cross section area}).\end{array}$$

Substituting displacement functions (18) to (20) into (12) to (14) yields matrices $\mathbf{A}_N$, $\mathbf{A}_M$ and $\mathbf{A}_T$, in turn, yield matrix $\mathbf{D}$:

$$\mathbf{D} = \mathbf{D}_0 + \omega^2\,\mathbf{D}_2 + \omega^4\,\mathbf{D}_4 \tag{21}$$

In the matrix sum, $\mathbf{D}_0$ is matrix compiled in Table 1, matrices $\mathbf{D}_2$ and $\mathbf{D}_4$ being compiled in Tables 2 and 3.

4. Statement of the matrix eigenvalue problem

Free vibrations of a moderately vibrating structure are described by matrix differential equation

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0\,. \tag{22}$$

Assuming $\mathbf{x} = \mathbf{v}\sin\omega t$ leads, after substitutions, to the eigenvalue problem

$$-\omega^2\mathbf{M}\mathbf{v} + \mathbf{K}\mathbf{v} = 0 \tag{23}$$

$$\mathbf{K}^{-1}\,\mathbf{M}\,\mathbf{v} = \frac{1}{\omega^2}\,\mathbf{v}$$

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

yielding, after solution, the natural frequencies and the first modes. Mass and stiffness matrices of the structure will be composed from mass and stiffness

Table 2

0					
$\frac{3M}{l^2}c_1c_2 + 1.6\frac{T}{l}c_1c_2$	$\frac{N}{l}c_1c_3(0.8 + 0.6\Phi)$				
$1.5\frac{M}{l}c_1c_2 + 0.8Tc_1c_2$	$Nc_1c_3(2 + 3.65\Phi + 1.75\Phi^2)$	$Nlc_1c_3(0.8 + 1.5\Phi + 0.75\Phi^2)$			
0	$\frac{3M}{l^2}c_1c_2 + 1.4\frac{T}{l}c_1c_2$	$1.5\frac{M}{l}c_1c_2 + 0.7Tc_1c_2$	0		
$-\frac{3M}{l^2}c_1c_2 - 1.6\frac{T}{l}c_1c_2$	$\frac{N}{l}c_1c_3(-0.8 - 0.6\Phi)$	$Nc_1c_3(1.5 + 3.35\Phi + 1.75\Phi^2)$	$-\frac{3M}{l^2}c_1c_2 - 1.4\frac{T}{l}c_1c_2$	$\frac{N}{l}c_1c_2(0.8 + 0.6\Phi)$	
$1.5\frac{M}{l}c_1c_2 + 0.8Tc_1c_2$	$Nc_1c_3(-1.5 - 3.35\Phi - 1.75\Phi^2)$	$Nlc_1c_3(-0.7 - 1.5\Phi + 9.75\Phi^2)$	$1.5\frac{M}{l}c_1c_2 + 0.7Tc_1c_2$	$Nc_1c_3(-2 - 3.65\Phi - 1.75\Phi^2)$	$Nlc_1c_3(0.8 + 1.5\Phi + 0.75\Phi^2)$

3

Table 3

$0.8\frac{N}{l}c_1^2$					
$-10.5\frac{M}{l^2}c_2c_3 - 21.8\frac{T}{l}c_2c_3$	$\frac{N}{l}c_3^2(53.765 + 106.888\Phi + 53.2\Phi^2 + \frac{i_0^2}{l^2}2317.091)$				
$-\frac{M}{l}c_2c_3(2.25 - 3\Phi) - Tc_2c_2(4.6 - 6.3\Phi)$	$Nc_3^2(11.537 + 22.87\Phi + 11.375\Phi^2 + (i_0^2/l^2)(486.545 - 672\Phi))$	$Nlc_3^2(2.487 + 4.932\Phi + 2.466\Phi^2 + (i_0^2/l^2)(103.273 - 280\Phi + 196\Phi^2))$			
$0.7\frac{N}{l}c_2^2$	$10.5\frac{M}{l^2}c_2c_3 - 9.7\frac{T}{l}c_2c_3$	$\frac{M}{l}c_2c_3(2.25 - 3\Phi) - Tc_2c_3(2.15 - 2.7\Phi)$	$0.8\frac{N}{l}c_2^2$		
$-10.5\frac{M}{l^2}c_2c_3 - 20.2\frac{T}{l}c_2c_3$	$\frac{N}{l}c_3^2(51.235 + 103.118\Phi + 51.8\Phi^2 + (i_0^2/l^2)2092.909)$	$Nc_3^2(11.213 + 22.63\Phi + 11.375\Phi^2 + (i_0^2/l^2)(458.545 - 588))$	$10.5\frac{M}{l^2}c_2c_3 - 11.3\frac{T}{l}c_2c_3$	$\frac{N}{l}c_3^2(53.765 + 106.888\Phi + 52.2\Phi^2 + (i_0^2/l^2)2317.091)$	
$\frac{M}{l}c_2c_3(2.25 - 3\Phi) - Tc_2c_3(-4.4 + 5.7\Phi)$	$Nc_3^2(-11.213 - 22.63\Phi - 11.375\Phi^2 + (i_0^2/l^2)(-458.545 + 588\Phi))$	$Nlc_3^2(-2.445 - 4.932\Phi - 2.466\Phi^2 + (i_0^2/l^2)(-99.727 + - 259\Phi - 164.5\Phi^2))$	$-\frac{M}{l}c_2c_3(2.25 - 3\Phi) - Tc_2c_3(-2.35 + 3.3\Phi)$	$Nc_3^2(-11.537 - 22.87\Phi - 11.375\Phi^2 + (i_0^2/l^2)(-486.545 + 672\Phi))$	$Nlc_3^2(2.487 + 4.932\Phi + 2.466\Phi^2 + (i_0^2/l^2)(103.273 - 280\Phi + 196\Phi^2))$

matrices of each beam. Unit mass and stiffness matrices taking also shear deformations into consideration are found in [1].

If also initial static stresses of the structure have to be reckoned with, then differential equation

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{K} + \mathbf{D})\mathbf{x} = 0$$

will be started from, with $\mathbf{D}$ taken from Table 1. The problem will be solved as above.

In the case of approximate dynamic displacement functions, the mass matrix becomes

$$\mathbf{M} = \mathbf{M}_0 + \omega^2 \mathbf{M}_2$$

omitting the term multiplied by ω^4; and the stiffness matrix:

$$\mathbf{K} = \mathbf{K}_0 + \omega^4 \mathbf{K}_4,$$

$\mathbf{M}_0$ and $\mathbf{K}_0$ being matrices obtained from the static displacement function, identical to those in the preceding problem.

Matrices $\mathbf{M}_2$ and $\mathbf{K}_4$ taking shear displacement into consideration are found in [5].

Solution of the homogeneous equation

$$(-\omega^2 \mathbf{M}_0 - \omega^4 \mathbf{M}_2)\mathbf{v} + (\mathbf{K}_0 + \omega^4 \mathbf{K}_4)\mathbf{v} = 0$$

$$[\mathbf{K}_0 - \omega^2 \mathbf{M}_0 - \omega^4(\mathbf{M}_2 - \mathbf{K}_4)]\mathbf{v} = 0 \qquad (25)$$

may be obtained by iteration [1], using results of problem (23), or by solving a double-size eigenvalue problem [1], [5].

Provided the analysis by an approximate dynamic displacement function is wanted to take the initial static stresses into account, stiffness matrix has to be increased by matrix $\mathbf{D}$ under (21).

$$\mathbf{K} = \mathbf{K}_0 + \mathbf{D}_0 + \omega^2 \mathbf{D}_2 + \omega^4(\mathbf{K}_4 + \mathbf{D}_4).$$

Now, the homogeneous equation delivering natural frequencies is:

$$[\mathbf{K}_0 + \mathbf{D}_0 - \omega^2(\mathbf{M}_0 - \mathbf{D}_2) - \omega^4(\mathbf{M}_2 - \mathbf{K}_4 - \mathbf{D}_4)]\mathbf{v} = 0$$

$$[\mathbf{A} - \omega^2 \mathbf{B} - \omega^4 \mathbf{C}]\mathbf{v} = 0 \qquad (26)$$

and the double-size matrix eigenvalue problem:

$$\begin{bmatrix} \mathbf{O} & \mathbf{E} \\ \mathbf{C}^{-1}\mathbf{A} & -\mathbf{C}^{-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \end{bmatrix} = \omega^2 \begin{bmatrix} \mathbf{v} \\ \mathbf{z} \end{bmatrix}. \qquad (27)$$

5. Evaluation of numerical results

The beam seen in Fig. 2, with the given cross-sectional dimensions, has been divided into four parts. In each beam, internal forces seen in the stress diagrams have been assumed. First, the case of normal force alone has been investigated. Analyses involved matrices determined by the static displacement function. ω^2 values belonging to the first three bending vibrations due to increasing normal forces, taking shear deformations into consideration, have been

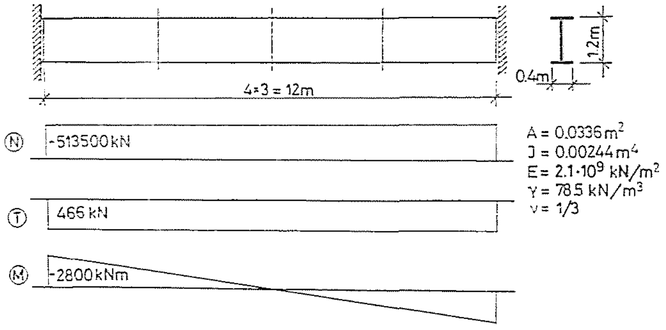

Fig. 2

compiled in rows 1 to 3 of Table 4. For frequencies in rows 4 to 6, the effect of shear deformation has been neglected in matrix $\mathbb{D}$, it being fully neglected in rows 7 to 9. Tables point out the significance of taking shear deformations into consideration, it being of course greater for stout, and less for flexible, beams.

The table contains squares of natural circular frequencies multiplied by 10^{-5}.

Table 4

$\bar{N}$	0	0.1 N	0.5 N	N
1	1.3476	1.1941	0.5701	−0.2491
2	8.5159	7.9501	5.6841	2.8449
3	28.117	26.8270	21.658	15.195
4	1.3476	1.1897	0.5496	−0.2902
5	8.5159	7.9009	5.4381	2.3514
6	28.117	26.020	20.5210	12.887
7	1.6481	1.4849	0.8252	−0.0201
8	11.986	11.3960	9.0326	6.0684
9	43.037	41.8060	36.8890	30.755

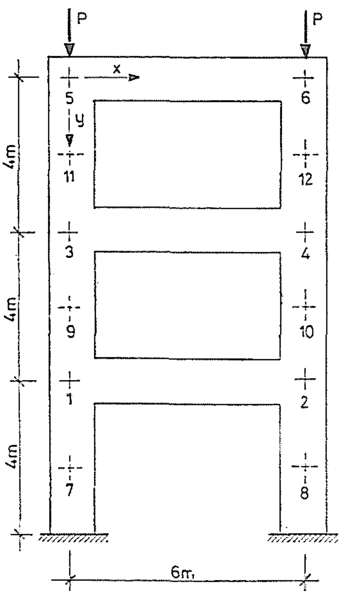

Fig. 3

The tabulated natural circular frequencies may be considered as approximations refinable by increasing the number of nodes. For instance, five nodes yield 1.6446, 11.819 and 42.099 in first columns of rows 7—8—9. Negative values in the last column of the table mean that the normal force exceeds the first critical value.

Examination of the effects of bending moment and shear force showed them to be rather irrelevant, except for moments several times the ultimate one for the given structure.

A framework made from beams with cross-sectional characteristics indicated in the former example is seen in Fig. 3. Columns are assumed to develop normal forces P. Analysis involved first matrices obtained from the

Table 5

N	a)	b)	c)
1	−0.01822	−0.01857	−0.01875
2	0.5983	0.5865	0.5802
3	2.5838	2.5689	2.5319
4	3.5929	3.5067	3.4360
5	6.3939	6.2577	6.1758

static displacement function, assuming nodes a) at framework nodes (6 nodes) and b) at framework nodes and at mid-columns (12 nodes). The analysis was remade by applying the matrix determined by approximate dynamic displacement functions (case c) with framework nodes (6 nodes).

Table 5 contains the squares of the first five natural circular frequencies multiplied by 10^{-5} under a compressive force $P = 25 \cdot 10^4$ kN. Tabulated values of the natural circular frequencies are approximations from above. Analysis by the dynamic displacement function is seen to be the exacter.

Summary

Frequency-dependent geometrical stiffness matrices of beams have been deduced by taking shear deformations into consideration. Numerical analyses showed shear deformations to significantly affect the natural frequencies of the tested beams. At the same time, the effect of initial static bending moments and shear forces on the natural frequency is negligible. Accuracy of the vibration analysis of structures is improved by stiffness and mass matrices obtained from dynamic displacement functions. Geometrical stiffness matrices have been deduced using dynamic displacement functions. Analyses involving the deduced matrices demonstrated the purposefulness of the method.

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