STATE CHANGE ANALYSIS OF STRUCTURES WITH GENERALIZED CONDITIONAL JOINTS

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In addition to plastic characteristics, the analysis of structures has to take uncertain displacements at in-situ joints into consideration. Both phenomena alter the stiffness of the structure and their computation in major structures requires much running time. Because of the physical and mathematical duality between both phenomena, it seemed advisable to develop a running-time-saving method for tracking the state change of structures, with so-called generalized conditional joints exhibiting both these phenomena, quite up to collapse. This method has been applied mainly for frameworks, but, relying on fundamental relationships in [2], it can be extended to any structure accessible to the finite element stiffness method. This procedure assumes a one-parameter load but it is also valid to multiparameter load processes, in section-wise one-parameter steps.

1. Generalized conditional joints

Recapitulation of physical and mathematical behaviour of generalized conditional joints relies on relationships in [1].

The elements of a structure whose forces or displacements or their combinations are limited by prescribed conditions are termed conditional joints. By nature of the condition, strength, geometry or generalized type conditional joints may be spoken of.

Figure 1 shows a generalized conditional joint with one degree of freedom, of a behaviour governed by strength and geometry conditions, such as:

if $-M_0 \leq M \leq M_0$, then $\varphi = 0$, and
if $|M| = M_0$, then $-\varphi_0 \leq \varphi \leq \varphi_0$ furthermore
if $|\varphi| = \varphi_0$, then $|M_0| \leq |M| \leq |M_0 + M_1|$, and
if $|M| = M_0 + M_1$, then φ is arbitrary.

In stress and displacement state with several degrees of freedom, strength and geometry conditions can be written as:

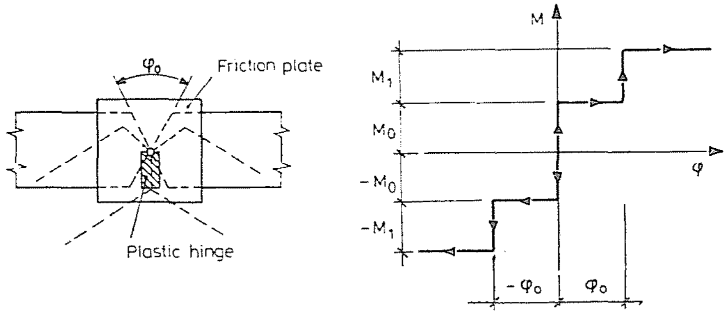


Fig. 1

$$F(\mathbf{s}) \leq 0 \quad \text{and} \quad f(\mathbf{t}) \leq 0$$

where $\mathbf{s}(s_1, s_2, \ldots, s_n)$ and $(t_1, t_2, \ldots, t_n)$ are vectors of generalized relative displacements at the same joint, respectively.

Plotting these conditions in coordinate systems $s_1, s_2, \ldots, s_n$ and $t_1, t_2, \ldots, t_n$ yields closed convex hypersurfaces each (Fig. 2). In course of the loading process, at the instant of each joint activation, end point of vector $\mathbf{s}$ or $\mathbf{t}$ corresponding to the joint nature lies on the respective hypersurface, and the corresponding increment vector d$\mathbf{t}$ or d$\mathbf{s}$ points to the outer normal of the hypersurface.

In the following, linearized conditions depictable by a convex polyhedron will be considered.

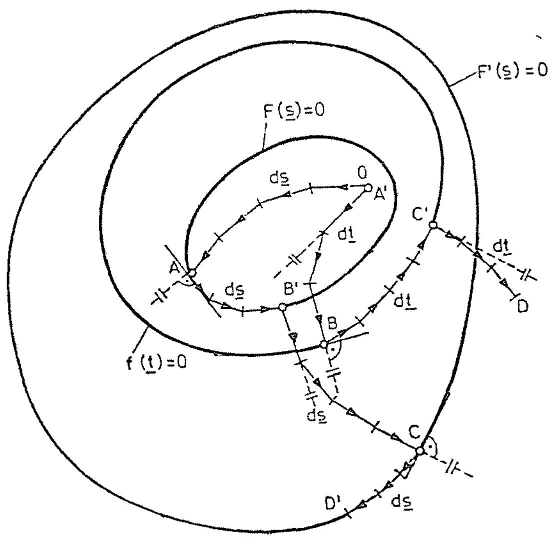


Fig. 2

2. Structures with generalized conditional joints

In the following, frameworks will be considered, making use of fundamental relationships and symbols in [2]. Development of the algorithm will rely on the fundamentals of plasticity described in [5].

The model of the analysis is a structure whose displacements of uncertain nature or yield stresses have been concentrated at the conditional joints, and in other parts of the structure continuous deformations and elastic behaviour have been assumed. Joints may include those of purely strength or purely geometry type beyond generalized conditional joints. Let the structure have r generalized joints, then the conditions are:

$$\left.\begin{array}{l} F_i(\mathbf{s}_m) \leq 0 \quad \text{and} \quad f_i(\mathbf{t}_m) \leq 0, \text{ that is:} \\ \text{if } F_i(\mathbf{s}_m) < 0, \text{ then } d\mathbf{t}_m = 0 \text{ and} \\ \text{if } f_i(\mathbf{t}_m) < 0, \text{ then } d\mathbf{s}_m = 0 \text{ but} \\ \left.\begin{array}{l} \text{if } F_i(\mathbf{s}_m) = 0, \text{ then } d\mathbf{t}_m \\ \text{if } f_i(\mathbf{t}_m) = 0, \text{ then } d\mathbf{s}_m \end{array}\right\} \text{depending on the above conditions, and their order} \end{array}\right\} (i = 1, 2, \ldots, r)$$

where m is the degree of freedom of the joint.

Considering the fundamental equation of frameworks from [2],

$$\begin{bmatrix} & \mathbf{G}^* \\ \mathbf{G} & \mathbf{F} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u} \\ \mathbf{s} \end{bmatrix} + \begin{bmatrix} \mathbf{q} \\ \mathbf{t} \end{bmatrix} = 0$$

relating given forces $\mathbf{q}$, kinematic loads $\mathbf{t}$, resulting displacements $\mathbf{u}$ and internal forces $\mathbf{s}$ of the structures, the above conditions may be written in the following form.

Let the strength condition for the i-th joint be the linearized plasticity condition [6]:

$$F_i(\mathbf{s}_i^j) = \mathbf{N}_i \mathbf{s}_i^j - \mathbf{k}_j \leq 0$$

where $\mathbf{s}_i^j$ refers to limited internal forces from among those $\mathbf{s}_i$ at the ith joint. Matrix $\mathbf{N}_i$ specifies their combinations by containing normal unit vectors belonging to each hyperplane of the convex polyhedron. Vector $\mathbf{k}_j$ refers to the distance of hyperplanes from the origin. If e. g. $\mathbf{N}_i$ equals the identity matrix, normals to the hyperplane are exactly the coordinate axes, thus, the condition involves only numerical comparison between developing and ultimate stresses.

Let the geometry condition for the i-th joint be the linearized form for relative displacements:

$$f_i(\mathbf{t}_i^j) = \mathbf{M}_i \mathbf{t}_i^j - \mathbf{l}_i \leq 0$$

$\mathbf{t}_i^j$ referring to limited relative displacements among those $\mathbf{t}_i$ at the i-th joint. $\mathbf{M}_i$ specifies their combinations related to given constants $\mathbf{l}_i$.

3. State change analysis of the structure using kinematic loads

State change of the structure is analyzed by tracking the loading process. Load increments in each step are limited by the conditions above, hence for

$$F_i(\mathbf{s}_i^j) < 0, \quad F_i(\mathbf{s}_i^j + \mathrm{d}\mathbf{s}_i^j) \leq 0,$$

and for

$$f_i(\mathbf{t}_i^j) < 0, \quad f_i(\mathbf{t}_i^j + \mathrm{d}\mathbf{t}_i^j) \leq 0.$$

These conditions determine the load increment causing another joint to be active. In each step, structural joints get rearranged, altering the stiffness matrix. While the activation of strength-type joints is known to be accompanied by a loss of structural stiffness, this latter increases during activation of geometry-type joints. In case of generalized conditional joints, stiffness may alternatively increase and decrease. Since, in addition, activation of conditional joints is a reversible process, alteration of the stiffness matrix may result from the reinactivation (unloading) of earlier active conditional joints.

Step-wise alteration and repeated decomposition of the stiffness matrix is rather running-time-consuming, therefore a method has been developed for analyzing the structure of step-wise varying stiffness in each step relying on the stiffness matrix of the original structure. Joint activation is replaced by kinematic loads, reducing the operations to those on free vectors in each step. Although determination of the step-wise needed kinematic loads requires to solve a linear equation system in each step, still its step-wise changing size is by orders less than that of the stiffness matrix.

For the sake of simplicity, the method will be illustrated on a structure with *strength-type* conditional joints. In an intermediate step of the loading process, part $\mathbf{s}_k$ of internal forces $\mathbf{s}$ of the structure belongs to the already plasticized elements, the other part $\mathbf{s}_r$ to those still in elastic state (or originally inconditional ones). For a zero initial kinematic load $\mathbf{t}$, elements in the still elastic state are under a kinematic load $\mathbf{t}_r = 0$, while the maintenance of the state of the plastic element is assured by $\mathbf{t}_k$. The corresponding partition is:

$$\begin{bmatrix} & \mathbf{G}_r^* & \mathbf{G}_k^* \\ \mathbf{G}_r & \mathbf{F}_r & \\ \mathbf{G}_k & & \mathbf{F}_k \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u} \\ \mathbf{s}_r \\ \mathbf{s}_k \end{bmatrix} + \begin{bmatrix} \mathbf{q} \\ O \\ \mathbf{t}_k \end{bmatrix} = 0.$$

An increment $\mathrm{d}\mathbf{q}$ of load $\mathbf{q}$ increases forces $\mathbf{s}$ by $\mathrm{d}\mathbf{s}$. Assume — as mostly justified for one-parameter loads — forces $\mathbf{s}_k$ not to vary anymore, thus, condition $F(\mathbf{s}_k) = 0$ remains inaltered for the load increment $\mathrm{d}\mathbf{q}$. Assume for the force increment $\mathrm{d}\mathbf{s}_r$ the equality

$$F(\mathbf{s}_{ri} + \mathrm{d}\mathbf{s}_{ri}) = 0$$

to be met at the i-th joint, but the subsistence of this equality for another load increment $\Delta\mathbf{q}$ has to be provided for by a kinematic load $\Delta\mathbf{t}_{ki}$. The partition corresponding to this process is:

$$\begin{bmatrix} & \mathbf{G}_r^* & \mathbf{G}_i^* & \mathbf{G}_k^* \\ \mathbf{G}_r & \mathbf{F}_r & & \\ \mathbf{G}_i & & \mathbf{F}_i & \\ \mathbf{G}_k & & & \mathbf{F}_k \end{bmatrix} \cdot \left(\begin{bmatrix} \mathbf{u} \\ \mathbf{s}_r \\ \mathbf{s}_{ri} \\ \mathbf{s}_k \end{bmatrix} + \begin{bmatrix} d\mathbf{u} \\ d\mathbf{s}_r \\ d\mathbf{s}_{ri} \\ 0 \end{bmatrix} + \begin{bmatrix} \Delta\mathbf{u} \\ \Delta\mathbf{s}_r \\ 0 \\ 0 \end{bmatrix} \right) +$$

$$+ \left(\begin{bmatrix} \mathbf{q} \\ 0 \\ 0 \\ \mathbf{t}_k \end{bmatrix} + \begin{bmatrix} d\mathbf{q} \\ 0 \\ 0 \\ d\mathbf{t}_k \end{bmatrix} + \begin{bmatrix} \Delta\mathbf{q} \\ 0 \\ \Delta\mathbf{t}_{ki} \\ \Delta\mathbf{t}_k \end{bmatrix} \right) = 0.$$

Let us consider the third and fourth matrix equations at the instant of activation of the i-th element:

$$\mathbf{G}_i(\mathbf{u} + d\mathbf{u}) + \mathbf{F}_i(\mathbf{s}_{ri} + d\mathbf{s}_{ri}) = 0$$

$$\mathbf{G}_k(\mathbf{u} + d\mathbf{u}) + \mathbf{F}_k \cdot \mathbf{s}_k + (\mathbf{t}_k + d\mathbf{t}_k) = 0$$

and for a further load increment $\Delta\mathbf{q}$:

$$\mathbf{G}_i(\mathbf{u} + d\mathbf{u} + \Delta\mathbf{u}) + \mathbf{F}_i(\mathbf{s}_{ri} + d\mathbf{s}_{ri}) + \Delta\mathbf{t}_{ki} = 0$$

$$\mathbf{G}_k(\mathbf{u} + d\mathbf{u} + \Delta\mathbf{u}) + \mathbf{F}_k\mathbf{s}_k + (\mathbf{t}_k + d\mathbf{t}_k + \Delta\mathbf{t}_k) = 0$$

consequently,

$$\Delta\mathbf{t}_{ki} = -\mathbf{G}_i \cdot \Delta\mathbf{u}$$

$$\Delta\mathbf{t}_k = -\mathbf{G}_k \cdot \Delta\mathbf{u}$$

hence, $\Delta\mathbf{t}_k$ and $\Delta\mathbf{t}_{ki}$ are obtained from $\Delta\mathbf{u}$ relying on the original stiffness matrix. Stress increment $d\mathbf{s}_{ri}$ can be determined from the yield condition, making the load parameter for that step to be known. Kinematic load $d\mathbf{t}_k$ is obtained from $\mathbf{t}_k$ by taking the load parameter into consideration.

Further steps of the computation are similar. In each step, however, relative displacements at active joints have to be checked since an eventual sign reversal hints to unloading of the joint to be considered inactive again.

For the sake of illustrativeness, the process has been represented in a diagram.

In Fig. 3, linear transformations describing relationships between external and internal forces and displacements have been plotted, with norms of the corresponding vectors indicated on coordinate axes.

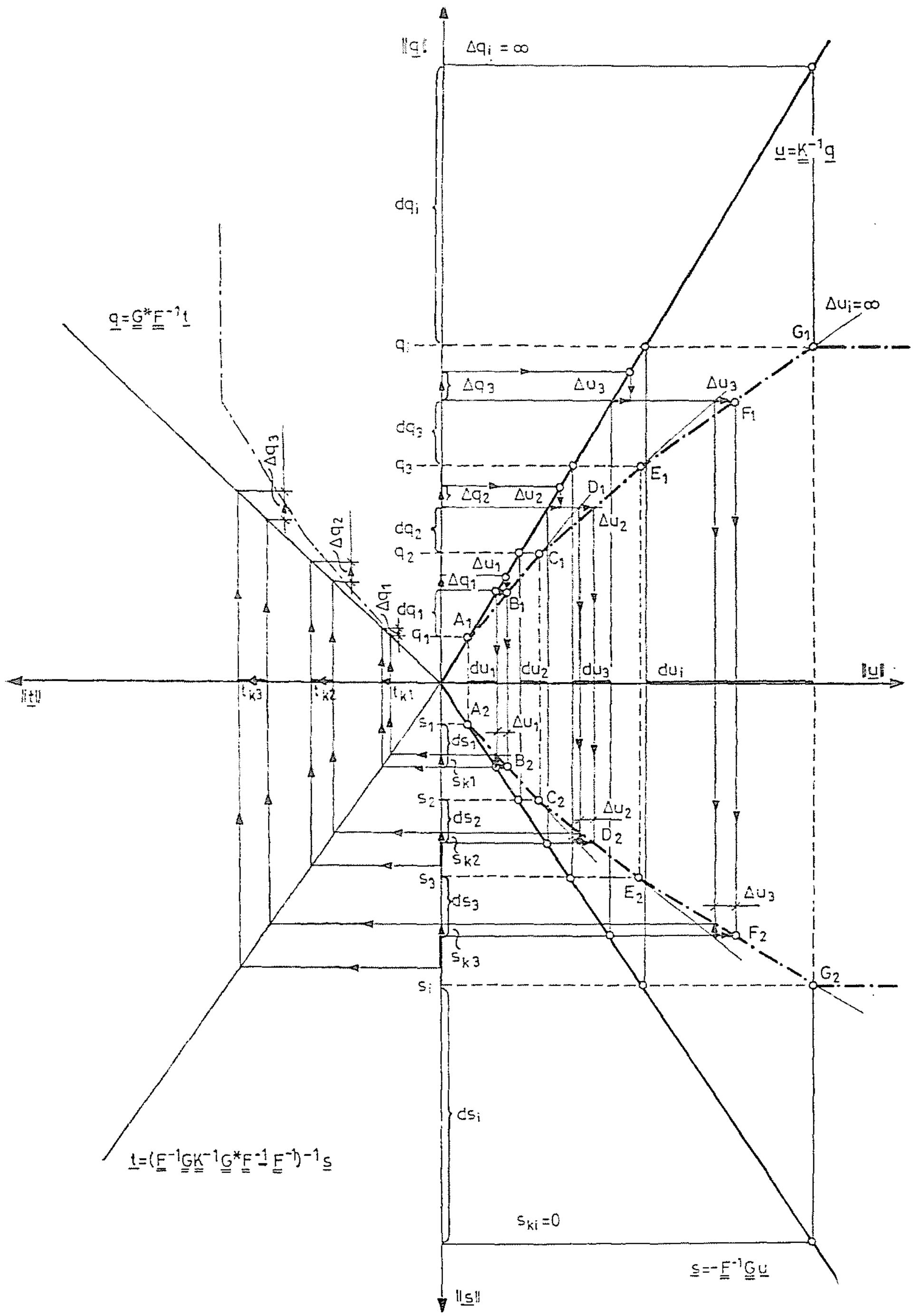

‖q‖
Δq_i = ∞
u=K^-1q
dq_i
Δu_i=∞
q=G*F^-1t
q_i
G_1
Δq_3
Δu_3
Δu_3
F_1
dq_3
q_3
E_1
Δq_2
Δu_2
D_1
Δu_2
Δq_2
dq_2
q_2
C_1
Δu_1
Δq_1
B_1
Δq_1
dq_1
A_1
q_1
du_1
du_2
du_3
du_i
‖u‖
‖t‖
t_k3
t_k2
t_k1
A_2
s_1
Δu_1
ds_1
B_2
s_k1
s_2
C_2
Δu_2
ds_2
D_2
s_k2
s_3
E_2
Δu_3
ds_3
F_2
s_k3
G_2
s_i
ds_i
t=(F^-1GK^-1G*F^-1F^-1)^-1s
s_ki=0
s=-F^-1Gu
‖s‖

Fig. 3

First step of the loading process leads to load value q_1, to that an increment dq_1 causes one (or simultaneously several) strength joints to become active (points A_1, A_2). Load $q_1 + dq_1$ involves displacements $u_1 + du_1$ and internal forces $s_1 + ds_1$. But the internal forces s_{k1} belonging to the joints becoming just active will take a share in the further load bearing, in compliance with the respective conditions, consequently the residual forces $ds_1 - s_{k1}$ will be rearranged. Condition for forces s_{k1} is reckoned with as kinematic load t_{k1}, causing, in turn, a displacement Δu_1 to be added to $u_1 + du_1$ to obtain displacement belonging to $q_1 + dq_1$ of the structure of changed stiffness (point B_1). Transformation corresponding to the new stiffness matrix appears from line $\overline{A_1B_1}$. Displacement increment Δu_1 hints to a rearrangement in ds_1 (point B_2), marking transformation change corresponding to line $\overline{A_2B_2}$.

A further load increase to q_2 activates another joint, and part s_{k2} of internal forces ds_2 belonging to load increment dq_2 becomes limited. Conditions prescribed for s_{k2} or still prevailing for joints activated in the previous step are provided for by a kinematic load t_{k2} resulting in a displacement increment Δu_2 (points D_1, D_2). Thus, the change of transformations is indicated by lines $\overline{C_1D_1}$, $\overline{C_2D_2}$.

The procedure is continued in similar steps until the structure or a part of it becomes unstable, appearing from zeroing of the internal forces s_{ki} belonging to the next load increment dq_i i.e. $s_{ki} = 0$, and for Δq_i, $\Delta u_i = \infty$. Namely then no further joint is activated, the structure is unable to take further loads and internal forces, and performs arbitrary displacements.

Gradual decrease of the structure stiffness along the loading process clearly appears in the figure, nevertheless analysis of the structure of varying stiffness relies throughout on the original stiffness matrix. The linear equation system for determining the kinematic load in each step is of a size equal to the number of already active joints, much less than the order of the stiffness matrix of the complete structure.

Analysis of structures with *geometry-type* joints may follow similar lines. Now, computation relies on the structure stiffness matrix belonging to the active state of all geometry joints (perfect closure), while the real initial state where all joints are still inactive is provided for by kinematic loads. Let us consider an intermediate step of the loading process where inactive geometry joints involve internal forces $\mathbf{s}_g = 0$, and joint inactivity is provided for by kinematic loads $\mathbf{t}_g$. At activated geometry joints, the corresponding element of $\mathbf{s}_r$ may be arbitrary. Let us notice that $\mathbf{s}_r$ comprises forces both at already activated joints and at originally conditioned joints. Since the initial load on the structure did not involve kinematic loads, the corresponding elements $\mathbf{t}'_r$, of $\mathbf{t}_r$ are under condition $f(\mathbf{t}'_r) = 0$.

$$\begin{bmatrix} & \mathbf{G}_r^* & \mathbf{G}_g^* \\ \hline \mathbf{G}_r & \mathbf{F}_r & \\ \mathbf{G}_g & & \mathbf{F}_g \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u} \\ \hline \mathbf{s}_r \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{q} \\ \hline \mathbf{t}_r \\ \mathbf{t}_g \end{bmatrix} = 0.$$

Increment $d\mathbf{q}$ of load $\mathbf{q}$ reduces the kinematic loads $\mathbf{t}_g$ by $d\mathbf{t}_g$. Assume kinematic load $\mathbf{t}_r'$ not to vary any more, condition $f(\mathbf{t}_r') = 0$ being unchanged for a load increment $d\mathbf{q}$, and assume relative displacement change $d\mathbf{t}_g$ to meet equality

$$f(\mathbf{t}_{gi} + d\mathbf{t}_{gi}) = 0$$

at joint i, subsistence of which for another load increment $\Delta\mathbf{q}$ means rearrangement of the remaining kinematic loads. This involves the following partitioning:

$$\begin{bmatrix} & \mathbf{G}_r^* & \mathbf{G}_i^* & \mathbf{G}_g^* \\ \hline \mathbf{G}_r & \mathbf{F}_r & & \\ \mathbf{G}_i & & \mathbf{F}_i & \\ \mathbf{G}_g & & & \mathbf{F}_g \end{bmatrix} \cdot \left(\begin{bmatrix} \mathbf{u} \\ \hline \mathbf{s}_r \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} d\mathbf{u} \\ \hline d\mathbf{s}_r \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \Delta\mathbf{u} \\ \hline \Delta\mathbf{s}_r \\ \Delta\mathbf{s}_i \\ 0 \end{bmatrix} \right) +$$

$$+ \left(\begin{bmatrix} \mathbf{q} \\ \hline \mathbf{t}_r \\ \mathbf{t}_{gi} \\ \mathbf{t}_g \end{bmatrix} + \begin{bmatrix} d\mathbf{q} \\ \hline 0 \\ d\mathbf{t}_{gi} \\ d\mathbf{t}_g \end{bmatrix} + \begin{bmatrix} \Delta\mathbf{q} \\ \hline 0 \\ 0 \\ \Delta\mathbf{t}_g \end{bmatrix} \right) = 0.$$

Let us consider the third and the fourth matrix equation at the instant of the i-th joint activation:

$$\mathbf{G}_i(\mathbf{u} + d\mathbf{u}) + \mathbf{t}_{gi} + d\mathbf{t}_{gi} = 0$$

$$\mathbf{G}_g(\mathbf{u} + d\mathbf{u}) + \mathbf{t}_g + d\mathbf{t}_g = 0$$

and for a further load increment $\Delta\mathbf{q}$:

$$\mathbf{G}_i(\mathbf{u} + d\mathbf{u} + \Delta\mathbf{u}) + \mathbf{F}_i\Delta\mathbf{s}_i + \mathbf{t}_{gi} + d\mathbf{t}_{gi} = 0$$

$$\mathbf{G}_g(\mathbf{u} + d\mathbf{u} + \Delta\mathbf{u}) + \mathbf{t}_g + d\mathbf{t}_g + \mathbf{t}_g = 0$$

thus:

$$\mathbf{F}_i\,\Delta\mathbf{s}_i = -\mathbf{G}_i\,\Delta\mathbf{u}$$

$$\Delta\mathbf{t}_g = -\mathbf{G}_g\,\Delta\mathbf{u}$$

hence, $\Delta\mathbf{s}_i$ and $\Delta\mathbf{t}_g$ can be produced from $\Delta\mathbf{u}$ on the basis of the original stiffness matrix. Relative displacement increment $d\mathbf{t}_{gi}$ can be determined from the

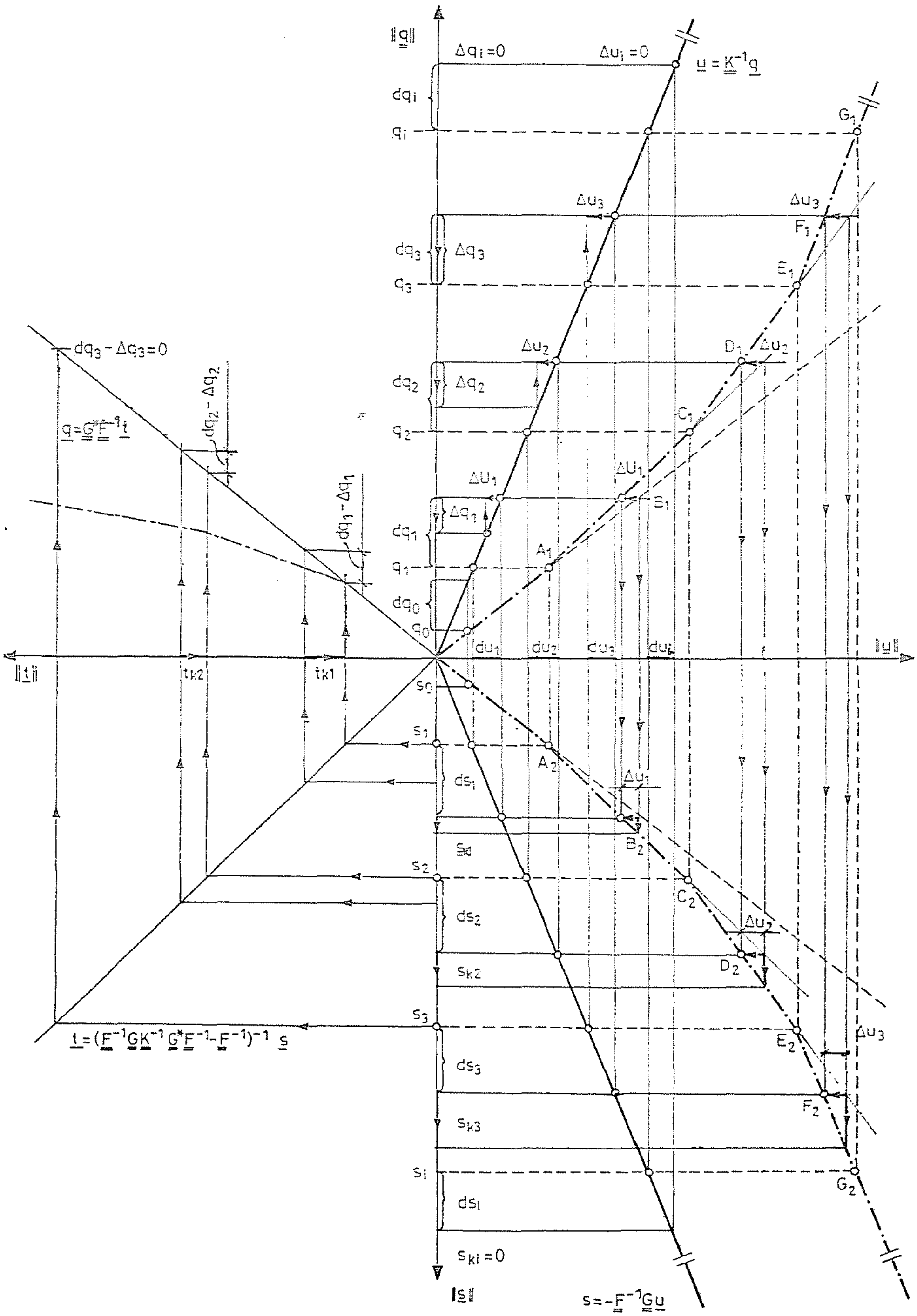

‖q‖
Δqi=0
Δui=0
u = K⁻¹q
dqi
qi
G1
Δu3
Δu3
F1
dq3
Δq3
q3
E1
dq3 - Δq3=0
Δu2
D1
Δu2
dq2
Δq2
dq2 - Δq2
q=G*F⁻¹t
q2
C1
ΔU1
ΔU1
B1
Δq1
dq1 - Δq1
dq1
A1
q1
dq0
q0
du1
du2
du3
dui
‖u‖
‖t‖
tk2
tk1
s0
s1
A2
ds1
Δu1
B2
s2
C2
ds2
Δu2
D2
sk2
s3
t=(F⁻¹GK⁻¹G*F⁻¹-F⁻¹)⁻¹ s
E2
Δu3
ds3
F2
sk3
si
G2
dsi
ski=0
‖s‖
s=-F⁻¹Gu

Fig. 4

condition prescribed for the joint making the load parameter pertaining to the step to be known, hence $d\mathbf{t}_g$ can be obtained from $\mathbf{t}_g$ with respect to the parameter.

Further computation steps are similar but in each step, stresses in earlier activated elements have to be checked, since an eventual sign reversal hints to the unloading of the joint to be considered as inactive again.

The procedure has graphically been represented in Fig. 4, where load q_1 comprises kinematic loads providing for the inactivity of all geometry joints. Assume a geometry joint to become active under a load increment dq_1 to load q_1, hence at the activated elements new forces s_{ki} to rise besides the internal force increment ds_1 corresponding to dq_1. Thereby the number of inactive joints decreases, and so does the kinematic load t_0 to t_{k1}. Thereby also displacements decrease by Δu_1 corresponding to $t_0 - t_{k1}$. Now, $q_1 + dq_1$ has $u_1 + du_1 - \Delta u_1$ as counterpart (point B_1), involving the change of the stiffness transformation (line $\overline{A_1B_1}$). Displacement change Δu_1 yields transformation change for the forces (line $\overline{A_2B_2}$).

Upon further load increase to q_2, another joint gets activated, and beyond forces ds_2 belonging to load increment dq_2, further forces s_{k2} arise at newly activated joints. Thereby the number of forces limited by the conditions decreases, and so does the kinematic load replacing inactive joints, causing a displacement reduction Δu_2 equal to the transformation changes (lines $\overline{C_1D_1}$, $\overline{C_2D_2}$).

The procedure continues along these lines until every joint becomes active if it ever can. In that event the next load increment dq_1 causes no internal force increment any more, i.e. $s_{ki} = 0$, all joints have become activated so that the kinematic load providing for inactivity is zeroed. Now $\Delta q_i = \Delta u_i = 0$ and the structure stiffness equals that of the substituting structure, both stiffness transformations run parallel. Actually the structure, with a stiffness corresponding to the last state, has an arbitrary load capacity — to a given limit.

Analysis of the structure of step-wise increasing stiffness relies in each step on the original stiffness matrix. The size of the linear equation system for determining the kinematic load for each step equals only the number of the still inactive joints.

Analysis of the structures with *generalized* conditional joints is the combination of both former methods.

The stiffness matrix underlying the method is that of a structure that would arise if all its conditional joints were inactive for strength but active for geometry. Therefore first the kinematic loads providing for geometric inactivity in a state of strength inactivity will be determined.

The resulting structure of identically inactive joints is the starting step of the loading process.

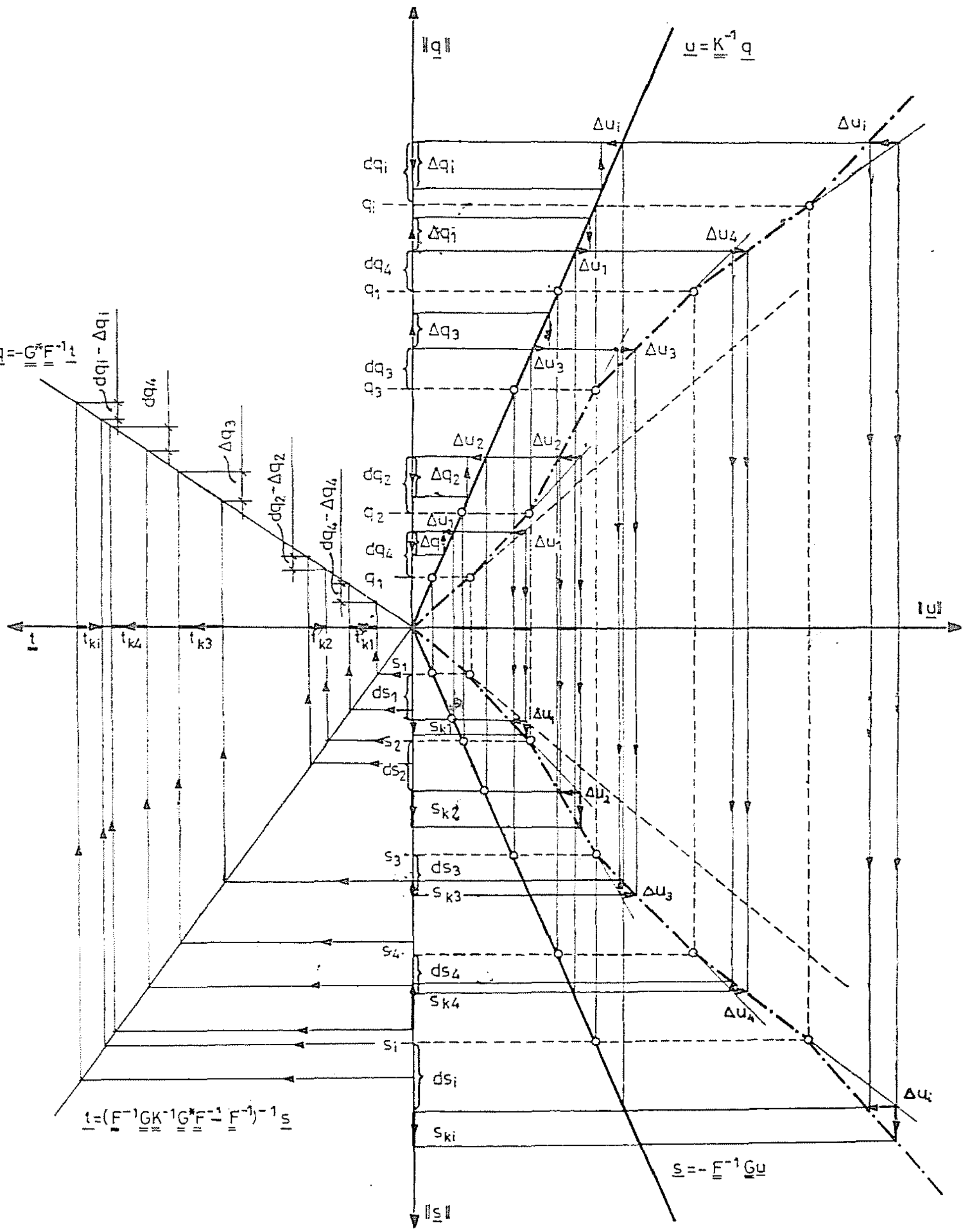


Fig. 5

The type of the conditional joint first becoming active in course of the load increase, and the relevant load value, depend on the relation between the stress or displacement state developing in each joint and the condition prescribed for the given joint type.

Behaviour of a structure with generalized conditional joints in course of the loading process has been plotted in Fig. 5. It is irrelevant for the analysis whether strength or geometry-type joints occur together in the same cross section of the structure (generalized joint) or separately, in different cross sections. Comprehensive survey of Figs 3 and 4 yields a clue to Fig. 5 without further comments.

Joint activation in each step goes on in ideal cases up to the specified load value or up to the total or partial instability of the structure. Instability may result from the development of plastic joints in itself, but also from the coincidence of already plastic and still not closed joints. Also an unloading at certain spots, due to step-wise stress rearrangement, may be realized, causing a strength-type joint to be elastic again, and a geometry-type joint previously closed to reopen. This occurrence has to be checked step by step.

4. Structural state change analysis with mathematical programming

Approximation of state change analysis by mathematical programming relies on fundamentals in [8].

Increment vectors arising in joint activations (Fig. 2) are:

$$\mathrm{ds} = \mathrm{d}\lambda \frac{\partial f}{\partial \mathbf{t}} \quad \text{and} \quad \mathrm{d}\mathbf{t} = \mathrm{d}\Lambda \frac{\partial \mathcal{F}}{\partial \mathbf{s}}$$

where λ and Λ are so-called strength and geometry multipliers, resp., for the combinations of the arising internal force and relative displacement components. For joints of one degree of freedom they equal the increment itself.

In course of the loading process, relationship between velocities (increments) of the state characteristics $\dot{\mathbf{q}}$, $\dot{\mathbf{t}}$, $\dot{\mathbf{u}}$, $\dot{\mathbf{s}}$, $\dot{\boldsymbol{\lambda}}$ and $\dot{\boldsymbol{\Lambda}}$ in the process of *strength-type* activation of generalized conditional joints are:

$$\begin{bmatrix} \mathbf{0} & \mathbf{G}^* & \mathbf{0} \\ \mathbf{G} & \mathbf{F} & \mathbf{N}_K^* \\ \mathbf{0} & \mathbf{N}_K & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{s}} \\ \dot{\boldsymbol{\Lambda}}_K \end{bmatrix} + \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{t}} \\ -\dot{\mathcal{F}}_K \end{bmatrix} = 0 \qquad \text{(a)}$$

where K is the set of subscripts where $\mathcal{F}_K = 0$. Furthermore:

$$\dot{\boldsymbol{\Lambda}}_K \geq \mathit{0},\ \dot{\mathcal{F}}_K \leq \mathbf{0} \quad \text{and} \quad \dot{\boldsymbol{\Lambda}}_K^* \dot{\mathcal{F}}_K = \mathit{0}$$

and for elements in state $\mathcal{F}_{\overline{K}} < 0$

$$\dot{\boldsymbol{\Lambda}}_{\overline{K}} = \mathit{0}$$

where $\overline{K}$ is the complementary set of K for the set of all subscripts affected by the conditions.

Relationships (a) comprise equations for equilibrium, compatibility and joint strength conditions. Alongside with the strength activation of joints, the compatibility equation is seen to be completed by the relative displacement velocities at active joints.

Relationships between state change velocities in the process of *geometry-type* activation of generalized conditional joints are:

$$\begin{bmatrix} \mathbf{0} & \mathbf{G}^* & \mathbf{G}^*\mathbf{M}_J^* \\ \mathbf{G} & \mathbf{F} & \mathbf{0} \\ \mathbf{M}_J\mathbf{G} & \mathbf{0} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{s}} \\ \dot{\lambda}_J \end{bmatrix} + \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{t}} \\ -\dot{\mathbf{f}}_J \end{bmatrix} = 0 \tag{b}$$

furthermore:

$$\dot{\lambda}_J \geq O, \quad \dot{\mathbf{f}}_J \leq O \quad \text{and} \quad \dot{\lambda}_J^* \cdot \dot{\mathbf{f}}_J = O$$

and in elements in state $f_J^- < 0$:

$$\dot{\lambda}_J^- = O.$$

Relationships (b) comprise equilibrium, compatibility and joint geometry condition equations. In course of the geometry-type activation of joints, the equilibrium equation is seen to be completed by the velocity of internal forces arising at active joints.

Let us consider now the relationship between state characteristic velocities for the case where both *strength* and *geometry* activations arise in the structure:

$$\begin{bmatrix} \mathbf{0} & \mathbf{G}^* & \mathbf{0} & \mathbf{G}^*\mathbf{M}_J^* \\ \mathbf{G} & \mathbf{F} & \mathbf{N}_K^* & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_K & \mathbf{0} & \mathbf{0} \\ \mathbf{M}_J\mathbf{G} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{s}} \\ \dot{\Lambda}_K \\ \dot{\lambda}_J \end{bmatrix} + \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{t}} \\ -\dot{\mathfrak{F}}_K \\ -\dot{\mathbf{f}}_J \end{bmatrix} = 0 \tag{c}$$

furthermore:

$$\begin{bmatrix} \dot{\Lambda}_K \\ \dot{\lambda}_J \end{bmatrix} \geq \mathbf{0} \begin{bmatrix} \dot{\mathfrak{F}}_K \\ \dot{\mathbf{f}}_J \end{bmatrix} \leq O \text{ and } [\dot{\Lambda}_K \; \dot{\lambda}] \begin{bmatrix} \dot{\mathfrak{F}}_K \\ \dot{\mathbf{f}}_J \end{bmatrix} = O.$$

In this case both equilibrium and compatibility equations are seen to be completed by the corresponding stress and relative displacement velocities. In the following, subscripts J and K will be omitted.

Eliminating the non-sign-dependent unknowns from Eqs (c) yields:

$$\left[\begin{array}{c|c} \mathbf{N}\mathbf{F}^{-1}\mathbf{G}\mathbf{K}^{-1}\mathbf{G}^*\mathbf{F}^{-1}\mathbf{N}^* - \mathbf{N}\mathbf{F}^{-1}\mathbf{N}^* & \\ \hline & \mathbf{M}\mathbf{G}\mathbf{K}^{-1}\mathbf{G}^*\mathbf{M}^* \end{array}\right] \cdot \left[\begin{array}{c} \dot{\Lambda} \\ \hline \dot{\lambda} \end{array}\right] +$$

$$+ \left[\begin{array}{c} -\mathbf{N}[\mathbf{F}^{-1}\mathbf{G}\mathbf{K}^{-1}(\dot{\mathbf{q}} - \mathbf{G}^*\mathbf{F}^{-1}\dot{\mathbf{t}}) + \mathbf{F}^{-1}\dot{\mathbf{t}}] - \dot{\mathfrak{F}} \\ \hline \mathbf{M}[\mathbf{K}^{-1}(\dot{\mathbf{q}} - \mathbf{G}^*\mathbf{F}^{-1}\dot{\mathbf{t}})] - \dot{\mathbf{f}} \end{array}\right] = 0 \tag{d}$$

or, in a simpler form:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^* \\ \mathbf{B} & \mathbf{C} \end{bmatrix} \cdot \begin{bmatrix} \dot{\Lambda} \\ \dot{\lambda} \end{bmatrix} + \begin{bmatrix} \dot{\mathbf{a}} \\ \dot{\mathbf{b}} \end{bmatrix} - \begin{bmatrix} \dot{\mathfrak{F}} \\ \dot{\mathbf{f}} \end{bmatrix} = 0$$

still simpler:

$$\mathbf{D}\dot{\mathbf{x}} + \dot{\mathbf{d}} - \dot{\mathbf{y}} = 0$$

where

$$\dot{\mathbf{x}} \geq O \quad \dot{\mathbf{y}} \leq O \quad \text{and} \quad \dot{\mathbf{x}}^* \dot{\mathbf{y}} = O.$$

This problem corresponds to a linear complementary problem:

$$\text{LK}: \{\mathbf{D}\dot{\mathbf{x}} - \dot{\mathbf{y}} + \dot{\mathbf{d}} = O,\ \dot{\mathbf{x}} \geq O,\ \ \dot{\mathbf{x}} \leq O,\ \dot{\mathbf{x}}^* \dot{\mathbf{y}} = O\}$$

equivalent to the primal-dual problem couple of the quadratic programming problem:

$$\text{K1}: \min \left\{ \frac{1}{2} \dot{\mathbf{x}}^* \mathbf{D}\dot{\mathbf{x}} + \dot{\mathbf{d}}^* \dot{\mathbf{x}} \mid \dot{\mathbf{x}} \geq O \right\}$$

$$\text{K2}: \min \left\{ \frac{1}{2} \dot{\mathbf{x}}^* \mathbf{D}\dot{\mathbf{x}} \mid \mathbf{D}\dot{\mathbf{x}} + \dot{\mathbf{d}} \leq O,\ \dot{\mathbf{x}} \geq O \right\}.$$

The linear programming problem, equivalent to the linear complementary problem, has been solved relying on a procedure equivalent to the simplex algorithm, with a physical purport corresponding to the solution by kinematic loads described in the previous chapter.

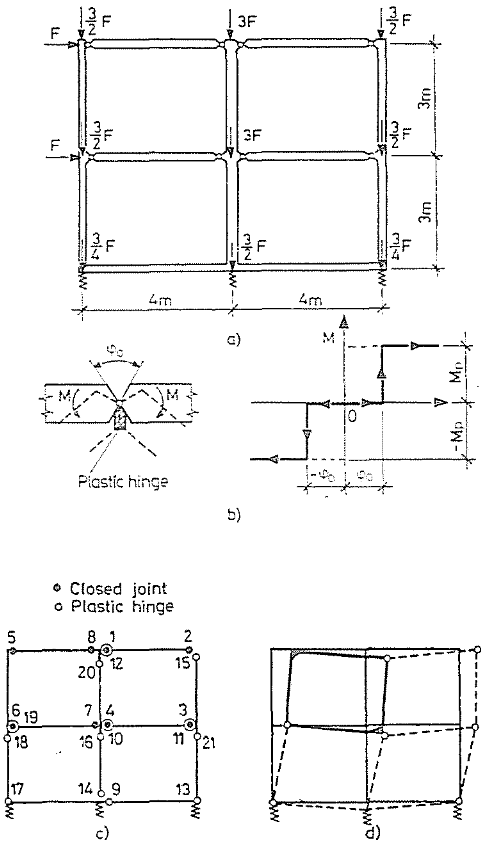
5. Applications

Computer programs have been established for the application of the presented method.

The program reckoning with strength-type joints has been applied for the analysis of plastic load capacity of frameworks. The program handled big-size problems, leading to numerical comparisons concerning the running time saving due to this method [3].

The program reckoning with geometry-type joints has been applied for the analysis of in-situ joints in precast frameworks and panel buildings [4].

The program reckoning with generalized conditional joints has been applied for the analysis of structures bedded on elastic soil, modelled by frameworks. Some numerical examples will be presented.



Steps i	Load par. α_i
1	0.0093
2	0.0095
3	0.0127
4	0.0139
5	0.0377
6	0.0470
7	0.0955
8	0.197
9	0.453
10	0.527
11	0.568
12	0.661
13	0.669
14	0.711
15	0.745
16	0.750
17	0.954
18	0.964
19	0.973
20	0.996
21	0.999

Fig. 6

Example 1

Framework seen in Fig. 6a models a structure composed of precast beams with data:

	Cross section area (m^2)	Moment of inertia (m^4)
Sole beam	1.2	0.06
Higher beams	0.6	0.03
Internal columns	0.4	0.02
Outer columns	0.2	0.01

Taking elastic properties by *Ohde*'s method into consideration, the characteristics are:

$$E_{str} = 3 \cdot 10^7 \text{ kN/m}^2;\ E_{soil} = 10^3 \text{ kN/m}^2;\ \nu_{str} = \nu_{soil} = 0.15.$$

Generalized conditional joints have been assumed at final cross sections of middle and upper horizontal beams uniformly prescribing relative rotation limit $|\varphi_0| = 2 \cdot 10^{-5}$ and plastic moment $|M_p| = 100$ kNm. Thereby the generalized joint of one degree of freedom has the following characteristics (Fig. 6b): if $-2 \cdot 10^{-5} \leq \varphi \leq 2 \cdot 10^{-5}$ then $M = 0$ and if $|\varphi| = 2 \cdot 10^{-5}$, then -100 kNm $\leq M \leq 100$ kNm, furthermore if $|M| = 100$ kNm, then φ is arbitrary.

Besides, final cross sections of columns and sole beams were assumed to have pure strength joints, with plastic moment $|M_p| = 100$ kNm. Load was assumed at $F = 100$ kN.

The structure was examined by tracking the loading process. In course of the first eight steps, generalized joints got activated geometrically, then in further thirteen steps — alternately with pure strength joints, — they were also activated from strength aspect. Activation order is seen in Fig. 6c, while Fig. 6d shows the produced yield mechanism and load parameters belonging to activation steps.

Figure 7 tracks state changes of end cross sections with generalized joints of beams A—B and C—D during loading. In load increments, closure rate is linear increasing. Before closure, cross-sectional bending moments are zero, after closure they increase section-wise linearly for each load increment until formation of a plastic hinge at the plastic moment. In case of further load increase, subsistence of the yield condition is provided for by kinematic loads, in the actual case, by relative rotation. In course of the loading process, this step-wise changing kinematic load has proven to be of the same sign as the relative rotation in the former closure process, excluding unloading of the already closed joint. The introduced kinematic load is, by physical purport, simply a relative rotation at a plastic hinge.

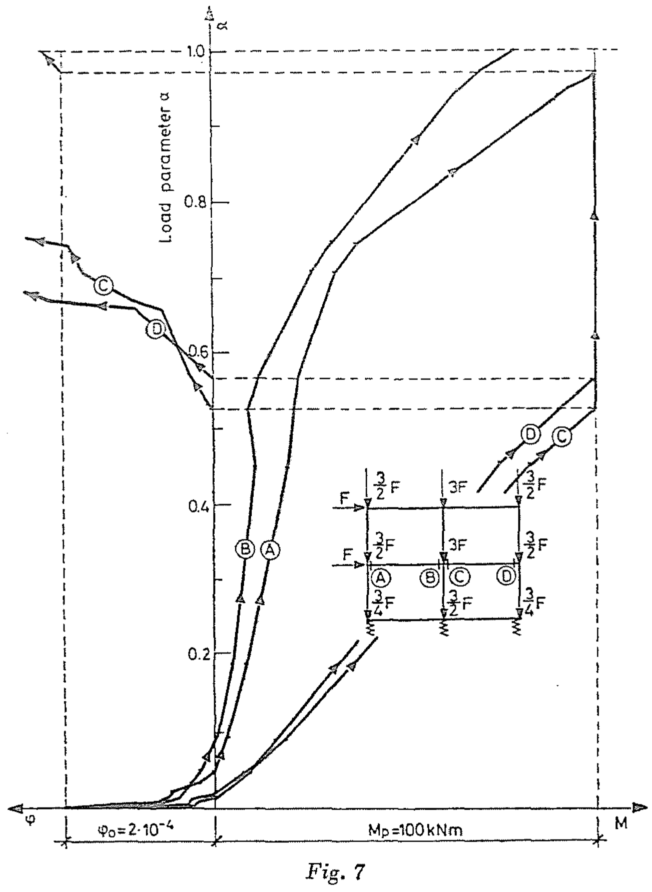
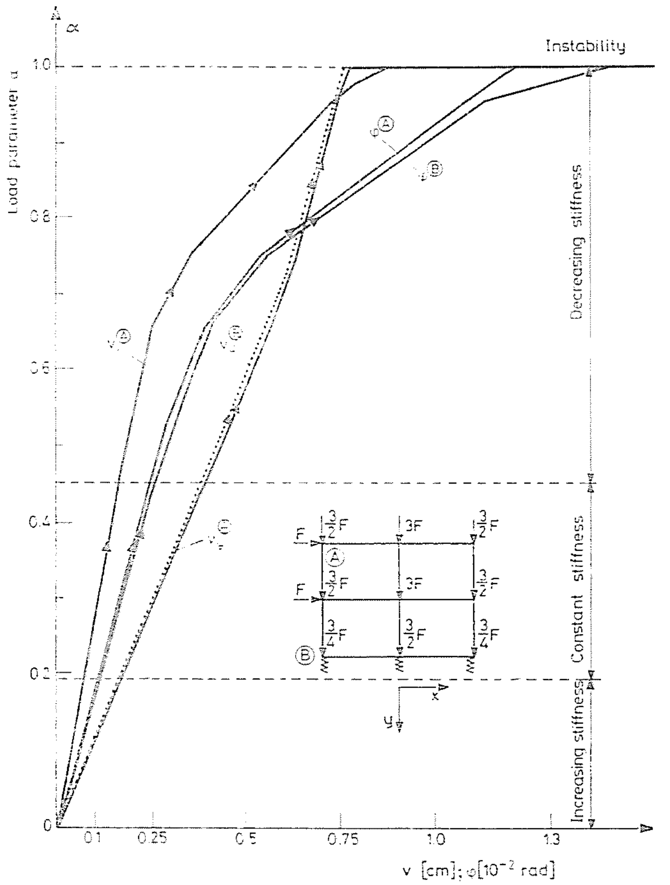
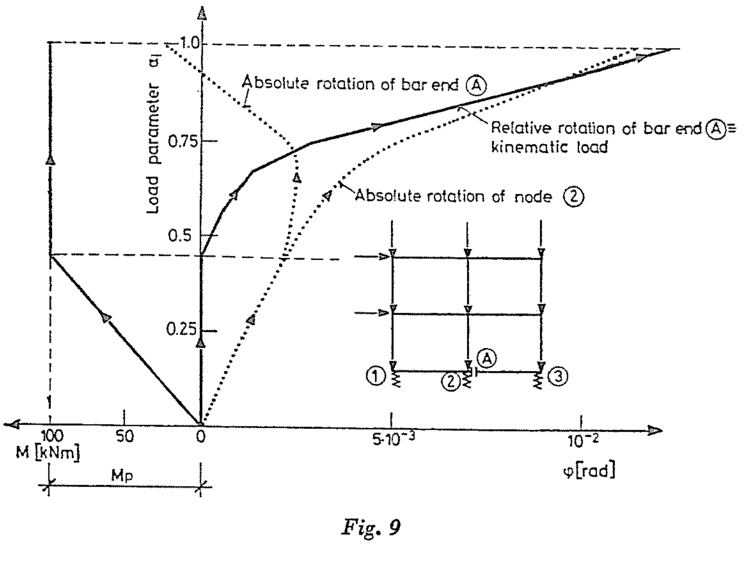


Fig. 7



Instability
Load parameter α
Decreasing stiffness
Constant stiffness
Increasing stiffness
3F
v [cm]; φ[10⁻² rad]

Fig. 8



Load parameter α
Absolute rotation of bar end Ⓐ
Relative rotation of bar end Ⓐ≡ kinematic load
Absolute rotation of node ②
M [kNm]
Mp
φ[rad]

Fig. 9

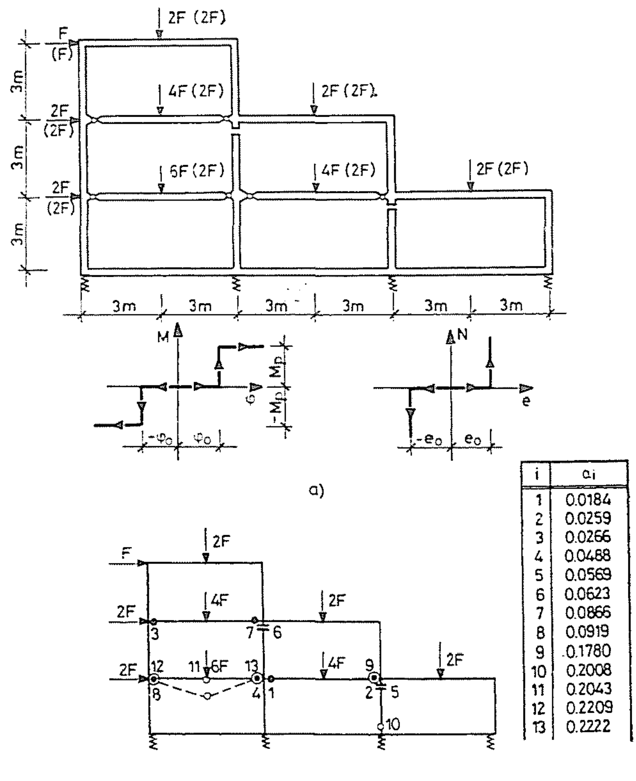
State change of the complete structure has been illustrated in Fig. 8 by tracking nodal displacements all along the loading process. During activation of the geometry joints, structural stiffness can be read off to increase, then, with increasing development of plastic joints, to decrease. This is apparent from the variation of displacement components φ^A, φ^B, v_x^A, while variation of v_y^B hints to the increase of stiffness even in the plastic range. Namely, vertical displacements mainly depend on soil rigidity, relatively increasing in the decreasing stage of structural stiffness.

Finally, behaviour of a conditional pure strength joint, cross section A in Fig. 9, has been plotted. Up to plastic moment, beam end rigidly joined to the node performs the same absolute rotation. Relative rotation at the formation of a plastic hinge corresponds to the introduced kinematic load, and the beam end undergoes an absolute rotation independent of the node.

Example 2

Let us consider a framework with conditional joints in Fig. 10, with further data:

	Cross section area [m²]	Moment of inertia [m⁴]
Sole beam	0.6	0.03
Upper beams	0.6	0.03
Columns	1.2	0.06



i	a_i
1	0.0184
2	0.0259
3	0.0266
4	0.0488
5	0.0569
6	0.0623
7	0.0866
8	0.0919
9	0.1780
10	0.2008
11	0.2043
12	0.2209
13	0.2222

Fig. 10

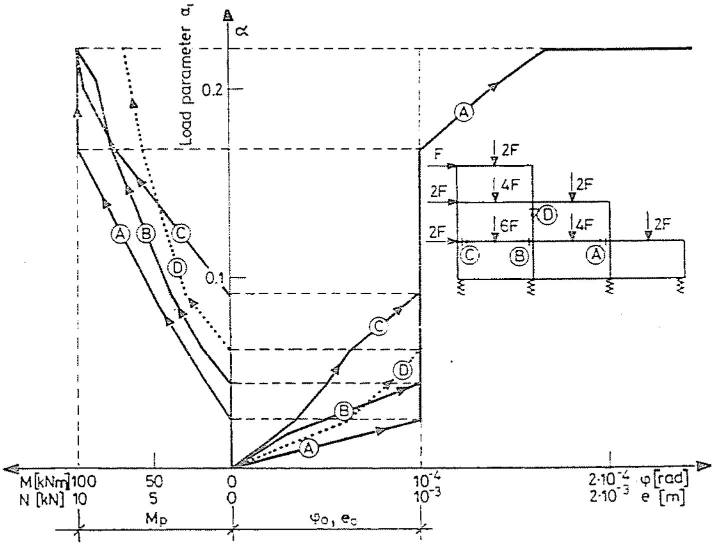

Load parameter α_i
α
0.2
0.1
F
2F
4F
6F
A
B
C
D
M[kNm] 100 50 0
N [kN] 10 5 0
10^{-4}
10^{-3}
$2\cdot10^{-4}$ φ[rad]
$2\cdot10^{-3}$ e [m]
M_p
φ_0, e_c

Fig. 11

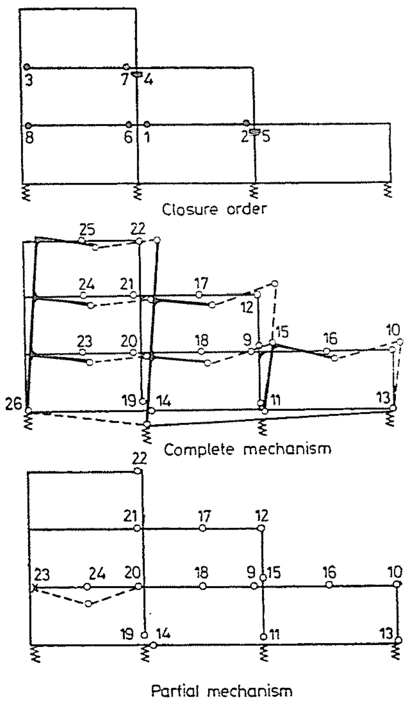

Closure order
Complete mechanism
Partial mechanism

Fig. 12

Furthermore:

$$E_{str} = 3 \cdot 10^7 \text{ kN/m}^2;\ E_{soil} = 10^3 \text{ kN/m}^2;\ \nu_{str} = \nu_{soil} = 0.15$$

and

$$|\varphi_0| = 10^{-5} \text{ radians};\ |e_0| = 10^{-4} \text{ m};\ |M_p| = 100 \text{ kNm};\ F = 100 \text{ kN}.$$

During the gradual load increase, in the first eight steps, generalized conditional joints got geometrically activated. Thereafter, in mere five steps, the structure got into ultimate plastic condition, or better, a partial yield mechanism has developed for a load parameter $\alpha = 0.222$ as seen in Fig. 10b.

Behaviour of some joints *vs.* load parameter is seen in Fig. 11. As soon as respective displacement components of joints are at closure value, the to then zero stress tends to increase. Beyond eventual prescribed stress limits again displacements occur. Assuming load values in parentheses in Fig. 10 causes the structure to get much slower to the ultimate condition for load parameter $\alpha = 0.5$, as seen from Fig. 12 together with the order of closure and the alternative mechanisms with and without taking geometrical unloading into consideration.

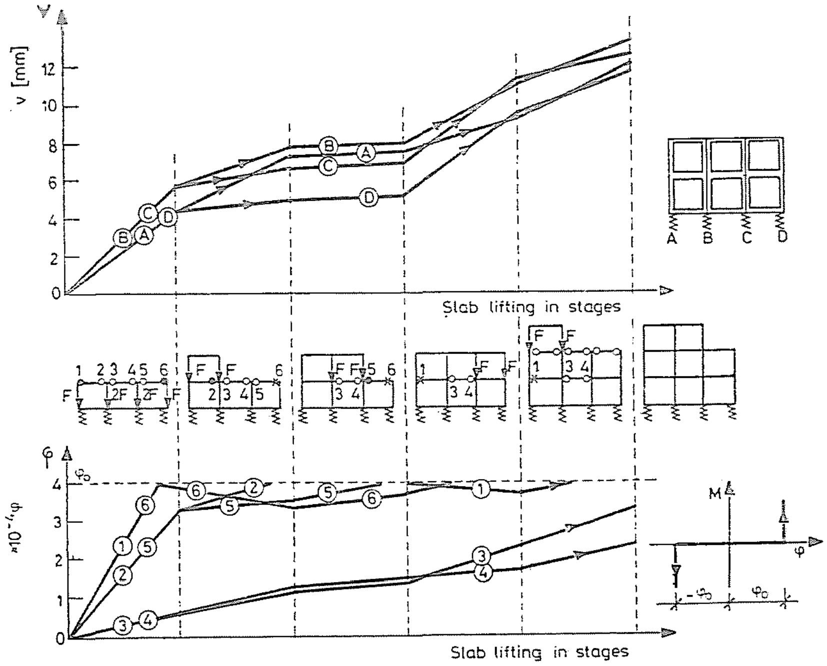


Fig. 13

Example 3

The described computation method permits to analyse multiparameter-type loading processes even if they are section-wise one-parameter ones. Thereby state change of a building under multiparameter loads due to the consecutive lifting in of panels in course of the loading process can be tracked. For example, let us consider a wall assembled from panels, modelled by the framework in Fig. 6. Characteristics of the structural material and of soil elasticity are the same as those in the figure.

Figure 13 shows the lift-in process of panels, testing in each step the change due to the newly lifted-in panel in the connection state developed in conformity with the loading on the building erected to then. This is a set of analyses of one-parameter loads separately analysing each load increment. The first few panel lifting-in steps of the loading process have been tracked in the figure, indicating connection states, and support settlement changes.

Summary

In the analysis of load-bearing structures, beside plastic characteristics, reckoning with uncertain displacements at in-situ joints is justified by the increasing use of prefabrication. Both phenomena affect stiffness of the structure and to reckon with them makes the problem rather running time consuming. The physical and mathematical duality between both phenomena suggested to develop a running time saving method for tracking the state change of structures with generalized conditional joints comprising both phenomena above, quite up to collapse.

This method, primarily devised for frameworks, can be extended to any structure accessible to the finite element stiffness method.

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* **In Hungarian**