SOME METHODS OF TENSOR REPRESENTATION AND CONSTRUCTION

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Four ways of representing second-order tensors will be presented, referring to plane stress state, plane strain state, second-order moments of plane configurations, in-plane forces of diaphragms and membrane shells, bending states of plates, this latter serving to illustrate construction methods.

The four representation possibilities are:

1. Mohr's circle, 2. tensor circle, 3. polar curve, 4. ellipse.

1. Four methods of representing moment tensors

Bending state at a point of a plate is described by the moment tensor M:

$$M = \begin{pmatrix} m_x & m_{xy} \\ m_{yx} & m_y \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}. \tag{1}$$

Principal moments are obtained from

$$m_{1,2} = \frac{m_x + m_y}{2} \pm \sqrt{\left(\frac{m_x - m_y}{2}\right)^2 + m_{xy}^2} \tag{2}$$

angle of principal direction being:

$$\operatorname{tg} 2\alpha_0 = \frac{2m_{xy}}{m_x - m_y}. \tag{3}$$

Moment components in arbitrary directions u, v are given by:

$$\begin{aligned} m_u &= \frac{m_x + m_y}{2} + \frac{m_x - m_y}{2}\cos 2\alpha - m_{xy}\sin 2\alpha \\ m_v &= \frac{m_x + m_y}{2} - \frac{m_x - m_y}{2}\cos 2\alpha + m_{xy}\sin 2\alpha \\ m_{uv} &= \qquad\qquad -\frac{m_x - m_y}{2}\sin 2\alpha + m_{xy}\cos 2\alpha. \end{aligned} \tag{4}$$

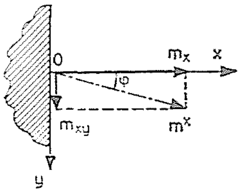
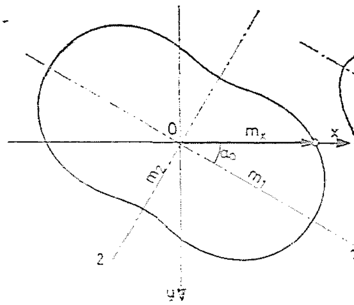


Fig. 1



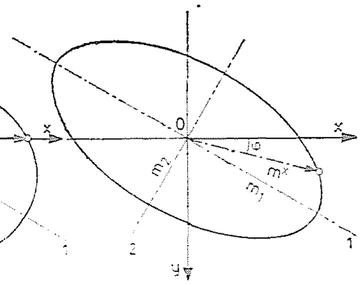
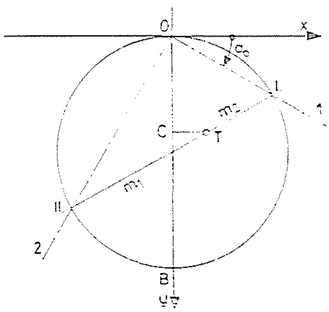


Fig. 7. Polar plot

Fig. 9. Ellipse



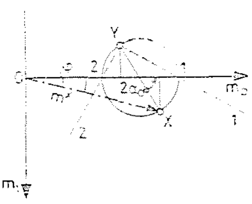


Fig. 4. Tensor circle $m_y =$ OC, $m_x =$ CB, $m_{xy} =$ CT

Fig. 2. Mohr's circle

A moment tensor may be considered as determined if moment components are given in two sections, e.g. moments m_x, m_{xy} in section x, and m_y, m_{xy} in section y. These are three data, namely $m_{xy} = -m_{yx}$. The well-known theorem of invariance:

$$m_x + m_y = m_u + m_v = m_1 + m_2. \tag{5}$$

Any section of a slab is generally acted upon by bending moment and twisting moment. Bending moment m_x, twisting moment m_{xy}, their resultant moment m^x developing in section x are seen in Fig. 1.

The bending state is said to be elliptic if principal moments have the same sign: $m_1 m_2 > 0$. Now the plate deformation surface will be elliptic around the point. The bending is hyperbolic if principal moments are of the opposite sign: $m_1 m_2 < 0$. The bending is parabolic if one principal moment is zero, a case termed that of uniaxial bending.

1.1 *Mohr's circle*

Mohr's representation of moment tensors is known to be feasible in a coordinate system m_b, m_t (bending moment — torsion moment) independent of the plate (Fig. 2).

To any section one point belongs on the circle which point has the bending and torsional moments as coordinates e.g. $X(m_x, m_{xy})$, $Y(m_y, m_{yx})$. In Mohr's circle, principal directions 1 and 2 can be constructed together with principal moments m_1 and m_2, and so can be moment components belonging to the coordinate system u, v rotated from x, y by angle α (Fig. 3). The point in this construction is that while axis x on the plate is rotated by α, in Mohr's

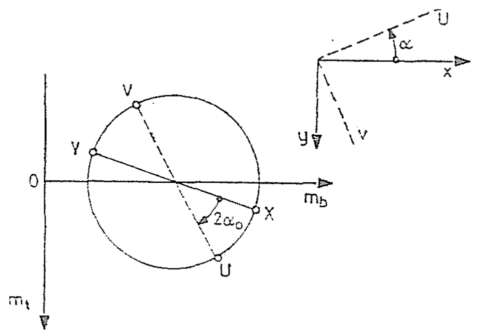


Fig. 3

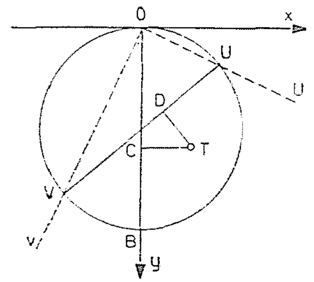


Fig. 6. m_u = VD; m_v = DU; m_{uv} = DT

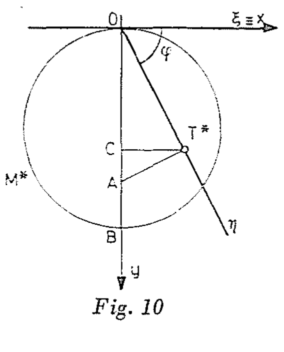


Fig. 10

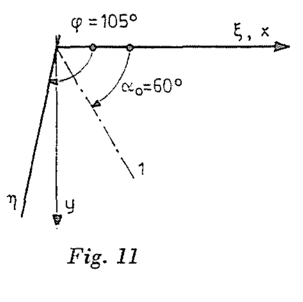


Fig. 11

circle the radius belonging to point X has to be rotated by α in the opposite direction to yield point U representing direction u. Its coordinates are moments m_u, m_{uv}.

1.2 *Tensor circle*

The moment tensor can be represented by a circle even in the plane coordinate system x, y in conformity with the theorem of invariance stating the sum of bending moments acting in perpendicular sections to be constant, and now, this will equal the diameter of a circle.

The tensor circle will be constructed as follows (Fig. 4). m_x, m_y and m_{xy} are given, the circle has to be put on axis y. Let us first admeasure distance $m_y =$ OC then $m_x =$ CB on axis y. Draw circle M with diameter $m_x + m_y$. Then admeasure torsion moment m_{xy} from point C parallel to axis x true to sign; if it is positive then in direction $+x$. The obtained principal point T and the circle M combine to the moment tensor circle.

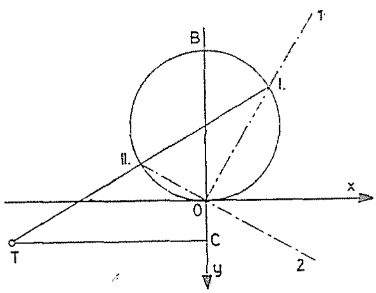


Fig. 5. $m_y =$ OC > 0; $m_x =$ CB < 0; $m_{xy} =$ CT < 0; $m_1 =$ II T > 0; $m_2 =$ T I < 0

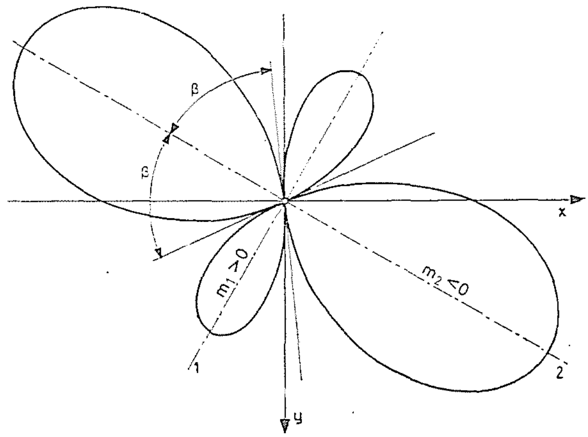


Fig. 8. β: negative bending sector $\operatorname{tg} \beta = \sqrt{-\frac{m_2}{m_1}}$

Remind that also bending moments have to be admeasured true to sign; if they are positive, then along $+y$, and if negative, then in the opposite direction. All three values in Fig. 4 are positive, the bending is elliptic. Figure 5 shows a tensor circle where $m_y > 0$, $m_x < 0$ and $m_{xy} < 0$, this is a case of hyperbolic bending.

In case of hyperbolic bending, principal point T is outside the circle, in elliptic bending it is inside the circle, while in parabolic bending, principal point T lies exactly on the circle.

Moments belonging to some axis u in the tensor circle have to be constructed (Fig. 6) by drawing diameter UV belonging to axis u, and projecting principal point T normally to it, resulting in moments $m_u = \mathrm{VD}$, $m_v = \mathrm{DU}$ and $m_{uv} = \mathrm{TD}$.

The sign of the twisting moment depends on what side of the diameter the principal point T is; in our figure it is on side $+u$, thus, $m_{uv} > 0$.

Construction of principal moments starts by drawing a diameter crossing principal point T (Figs 4 and 5), then $m_1 = \mathrm{II, T}$ and $m_2 = \mathrm{I, T}$. Obtained axes 1 and 2 are principal moment directions in their original positions.

1.3 *Polar curve of bending moments*

Plotting bending moments in any direction yields polar curve in Fig. 7. Principal moments m_1 and m_2 are admeasured in principal moment directions 1 and 2, respectively. In this figure, both principal moments are positive and the bending is an elliptic one. Fig. 8 shows a moment tensor with principal moments of opposite sign, the bending is a hyperbolic one, the polar curve is a "cloverleaf".

The polar curve is a good illustration of the bending moment value in any direction but does not suit direct construction. It can be drawn by plotting bending moments constructed in the tensor circle (or in Mohr's circle).

1.4 *Moment ellipse*

Plotting resultant moment rather than bending moment in every direction yields the moment ellipse (Fig. 9). For instance, resultant moment m^x is admeasured to direction x, at angle φ indicated also in Fig. 1. Final points of resultant moments are aligned on an ellipse. Principal axes of the ellipse are principal moments m_1 and m_2 in principal directions 1 and 2, respectively.

2. Tensor circle of the ultimate moment of a reinforced concrete slab

As an example of application, let us consider the tensor circle presenting ultimate moments of a r.c. slab in the general case of skew reinforcement.

Assume reinforcements in directions ξ and η, including an angle φ to be given. Also design moments $\overline{m}_\xi$ and $\overline{m}_\eta$ are given that are ultimate moments if steel bars exist only in one direction (ξ or η). In the coordinate system of Fig. 10 where x coincides with one reinforcement direction, $x \equiv \xi$, components of the tensor of ultimate moments are:

$$\begin{aligned} m_x^* &= \overline{m}_\xi + \overline{m}_\eta \cos^2 \varphi \\ m_y^* &= \overline{m}_\eta \sin^2 \varphi \\ m_{xy}^* &= -m_{yx}^* = \overline{m}_\eta \cos \varphi \sin \varphi . \end{aligned} \tag{6}$$

Tensor circle of ultimate moments will be drawn by consecutively admeasuring design moments $\overline{m}_\eta = \mathrm{OA}$ and $\overline{m}_\xi = \mathrm{AB}$ on axis y. A circle will be drawn with $\overline{m}_\eta + \overline{m}_\xi = \mathrm{OB}$ as diameter. This will be tensor circle M^*, with principal point T* obtained by projecting point A normally on line η. Normal projection of point T* on axis y yields point C, yielding, in turn, normal components of the tensor: $m_x^* = \mathrm{CB}$, $m_y^* = \mathrm{OC}$, $m_{xy}^* = \mathrm{CT}^*$. This could be easily verified by applying Eq. (6) on Fig. 10.

In this representation, principal point T* of the tensor circle of ultimate moments always lies on reinforcement axis η.

Invariant of the ultimate moment tensor: $m_1^* + m_2^* = m_x^* + m_y^* = \overline{m}_\xi + \overline{m}_\eta$ equals the sum of design moments, it does not depend on the angle φ of the reinforcing bars.

3. Optimum design of reinforced concrete slabs

The tensor circle lends itself to complexer constructions, such as the following optimization problem. Let the bending state in a point of the r.c. slab be given: $m_1 = +80$ kNm/m, $m_2 = -40$ kNm/m and $\alpha_0 = +60°$. It is a hyperbolic bending state. Both in top and in bottom of the r.c. slab a mesh of steel bars each is needed. Mark out directions ξ and η of the reinforcements including an angle $\varphi = 105°$, as seen in Fig. 11.

Let us determine now for what particular moments the top and bottom reinforcements in directions ξ and η are to be designed to meet both the ultimate condition and the optimum condition $\overline{m}_\xi + \overline{m}_\eta = \min$!

This problem can analytically be solved by calculating the following formula pairs referring to the four cases:

case a):

$$\overline{m}_\xi = m_x - m_y \frac{\cos \varphi}{1 + \cos \varphi} + m_{xy} \frac{1 - 2 \cos \varphi}{\sin \varphi} = +89.2$$

$$\overline{m}_\eta = m_y \frac{1}{1 + \cos \varphi} + m_{xy} \frac{1}{\sin \varphi} = +121.3$$

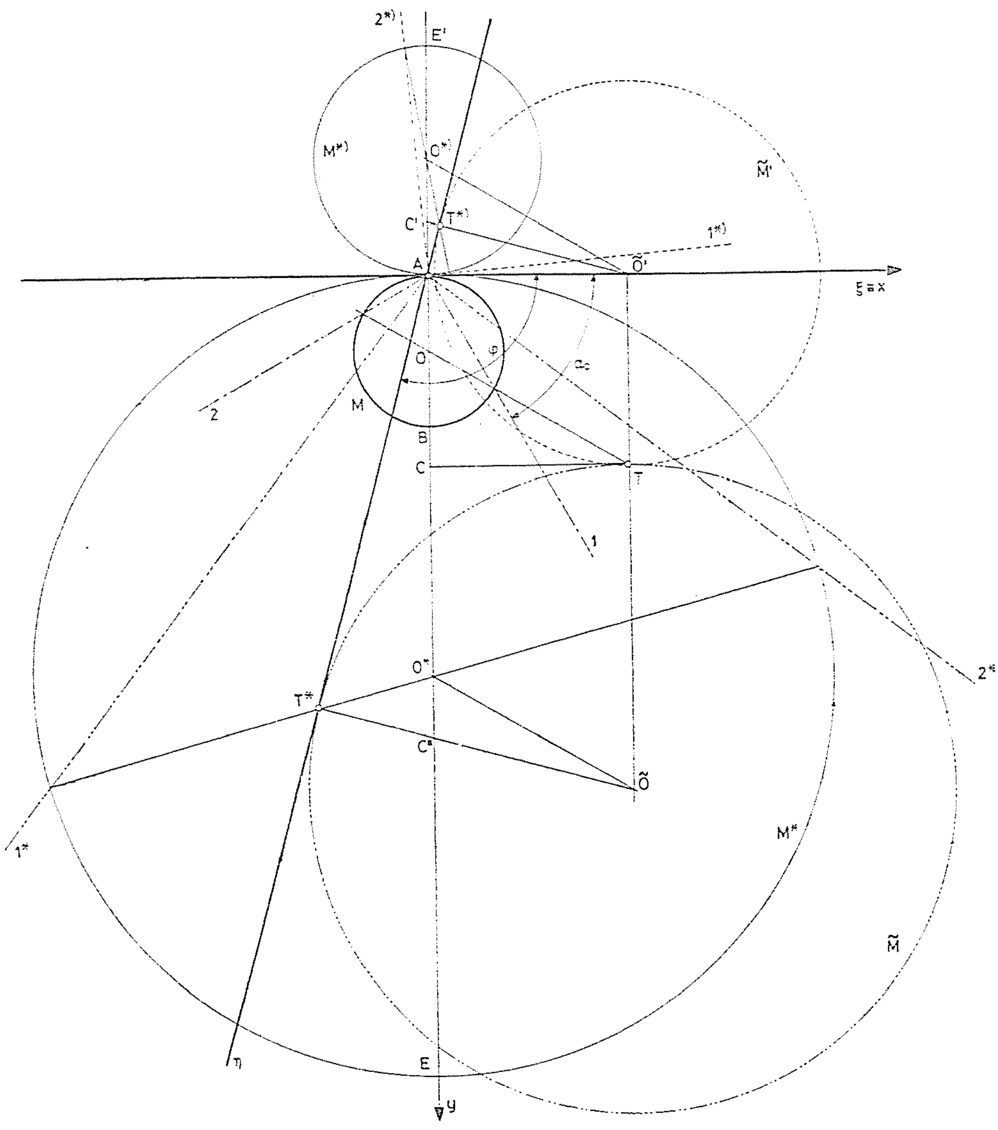


Fig. 12. M: applied moment tensor. $m_y = \text{AC} = +50$, $m_x = \text{CB} = -10$; $m_{xy} = \text{CT} = +52$; M^*: bottom resisting moment tensor, $\overline{m}_\xi = \text{CE} = +89.2$; $\overline{m}_\eta = \text{AC} = 121.3$; $\widetilde{M}$: bottom reserve moment tensor; $M^{*)}$: top resisting moment tensor $\overline{m}_\xi = \text{C'E'} = -46.2$; $\overline{m}_\eta = \text{AC'} = -14.1$; $\widetilde{M}'$: top reserve moment tensor

case b):

$$\overline{m}_\xi = m_x + m_y \frac{\cos\varphi}{1-\cos\varphi} - m_{xy}\frac{1+2\cos\varphi}{\sin\varphi} = -46.2$$

$$\overline{m}_\eta = m_y \frac{1}{1-\cos\varphi} - m_{xy}\frac{1}{\sin\varphi} = -14.1$$

case ξ):

$$\overline{m}_\xi = m_x - \frac{m_{xy}^2}{m_y} = -64.1$$

$$\overline{m}_\eta = 0.$$

case η):

$$\overline{m}_\xi = 0$$

$$\overline{m}_\eta = \frac{m_x m_y - m_{xy}^2}{m_x \sin^2\varphi + m_y \cos^2\varphi - m_{xy}\sin 2\varphi} = -160.0.$$

Case a) refers to the design of the bottom reinforcement ($\overline{m}_\xi + \overline{m}_\eta = 210.5$). Case b) yields optimum design moments of the top reinforcement ($\overline{m}_\xi + \overline{m}_\eta = -60.3$). Both cases ξ) and η) refer to the top reinforcement (design moments are negative) assumed to comprise bars in direction ξ or η alone. These solutions are, however, other than optima, case b) being the more favourable ($60.3 < 64.1$ and $60.3 < 160.0$).

Graphic solution of the same problem is seen in Fig. 12. Circle M with principal point T represents the given applied moment tensor. Resisting moment tensor circles M^* and $M^{*\prime}$ corresponding to cases a) and b), resp., have been constructed as follows:

A line parallel to axis y is drawn from point T, along that the centre of a circle passing through point T and contacting line η is to be found. There are two such circles, $\tilde{M}$ and $\tilde{M}'$, with centres $\tilde{O}$ and $\tilde{O}'$. Contact points T^* and $T^{*\prime}$ will be principal points of optimum resisting moment tensors M^* and $M^{*\prime}$, respectively. Centres O^* and $O^{*\prime}$ are obtained by drawing lines through points $\tilde{O}$ and $\tilde{O}'$ parallel to OT, intersecting axis y at the centres. Now, normals to η are drawn from $\tilde{O}$ and $\tilde{O}'$ cutting out points C and C′. Then, design moments are:

$$\text{case a)}\quad \overline{m}_\xi = \mathrm{CE} = +89.2,\ \overline{m}_\eta = \mathrm{AC} = +121.3$$

$$\text{case b)}\quad \overline{m}_\xi = \mathrm{C'E'} = -46.2,\ \overline{m}_\eta = \mathrm{AC'} = -14.1.$$

Figure 12 shows construction of applied moment tensor circle M, resisting moment tensor circles M^* and $M^{*\prime}$ for cases a) and b), and in addition, circles $\tilde{M}$ and $\tilde{M}'$. These are tensor circles of reserve moments: $\tilde{M} = M^* - M$ and $\tilde{M}' = M^{*\prime} = M$, differences of ultimate and applied moments.

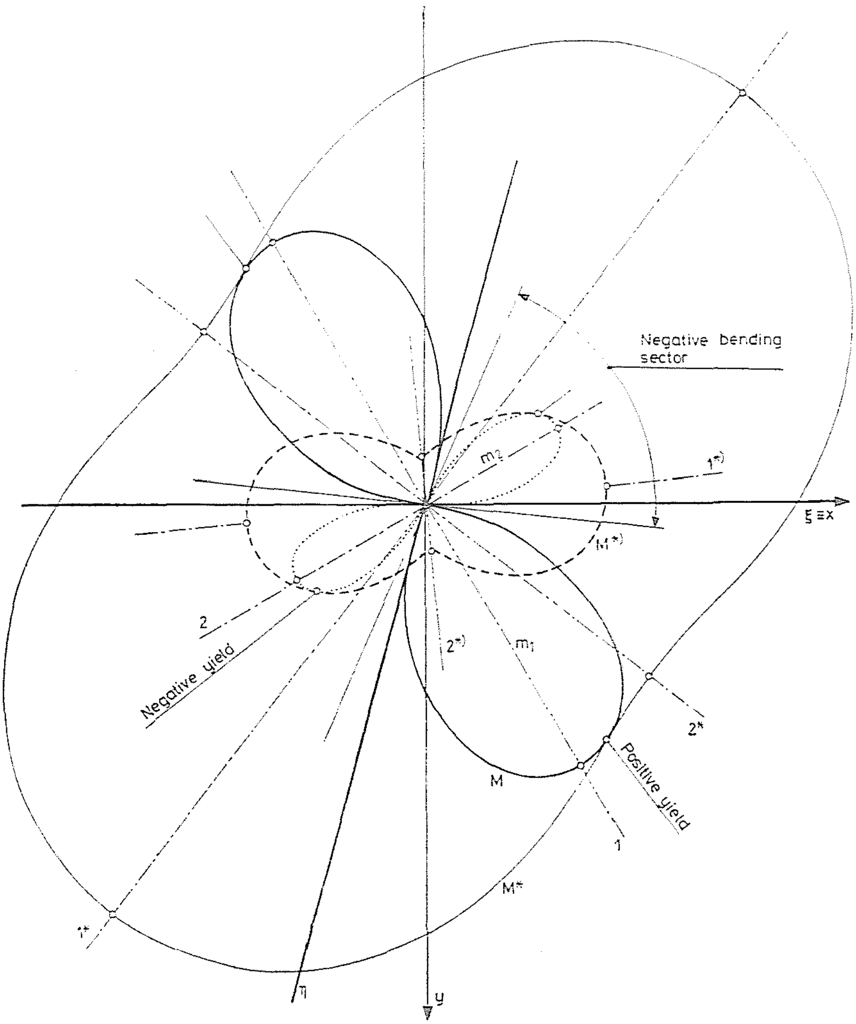


Fig. 13. M: applied moment tensor. $m_1 = +80$; $m_2 = -40$; M^*: bottom resisting moment tensor; $M^{*)}$: top resisting moment tensor

Figure 13 shows polar curves of the same problem. A bigger positive moment acts in principal direction 1 of applied moments M, and a lower negative moment in principal direction 2. Also the segment with negative bending is seen.

Polar plot M^* is curve of positive ultimate moments for the bottom reinforcement (case a). Where it contacts the curve of the applied moment tensor, hence where the ultimate moment equals the applied bending moment, there the positive (bottom) yield line develops. Polar plot $M^{*\prime}$ is the curve of negative ultimate moments for the top reinforcement (case b). In the direction of its contact with the negative limb of applied moments, the negative (top) yield line develops.

Summary

Four methods of representing second-order tensors are illustrated on the moment tensor of a slab. Beside the well-known Mohr's circle, the well constructible tensor circle, the polar curve of bending moments clearly illustrating the bending state, and the ellipse of resultant moments are involved. Application examples include the ultimate moment tensor circle of a r.c. slab with skew reinforcement, as well as the optimum solution of a r.c. slab problem.

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