SINGULAR VALUE DECOMPOSITION OF MATRICES AND ITS APPLICATION IN NUMERICAL ANALYSIS

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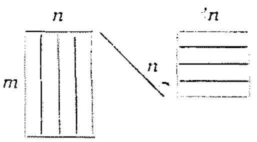
Singular Value Decomposition (SVD)

A real $m \times n$ matrix $\mathbf{A}$ with $m \geq n$ can always be written in the form

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T =$$ (1)



where

$$\mathbf{U}^T\mathbf{U} = \mathbf{V}^T\mathbf{V} = \mathbf{V}\mathbf{V}^T = \mathbf{I}_n,$$

and $\mathbf{D} = \langle \sigma_1, \ldots, \sigma_n \rangle$ denotes a diagonal matrix. The matrix $\mathbf{U}$ consists of n orthonormalized eigenvectors associated with the n largest eigenvalues of $\mathbf{A}\mathbf{A}^T$, and the matrix $\mathbf{V}$ consists of the orthonormalized eigenvectors of $\mathbf{A}^T\mathbf{A}$. The diagonal elements of $\mathbf{D}$ are the non-negative square roots of the eigenvalues of $\mathbf{A}^T\mathbf{A}$; they are called *singular values* of $\mathbf{A}$. We shall assume the ordering $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$. If $\mathbf{A}$ is a matrix of rank r, then $\sigma_r > 0$ and either $r = n$ or $\sigma_{r+1} = \sigma_{r+2} = \ldots = \sigma_n = 0$, (i.e. the number of positive singular values is exactly the rank of $\mathbf{A}$).

A constructive proof of SVD is given by the following.

The product $\mathbf{A}^T\mathbf{A}$ is a real symmetric matrix of order n; moreover it is positive semidefinite, since with any vector $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} = (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) = ||\mathbf{A}\mathbf{x}||_2^2 \geq 0$. Consequently, all eigenvalues of $\mathbf{A}^T\mathbf{A}$ are non-negative and hence they may be denoted by $\sigma_1^2, \ldots, \sigma_n^2$, where $\sigma_1 \geq \ldots \geq \sigma_n \geq 0$. Let r be the subscript for which $\sigma_r > 0$ and either $r = n$ or $\sigma_{r+1} = \ldots = \sigma_n = 0$.

A real symmetric matrix is known to be of simple structure and hence there exists a real orthogonal matrix $\mathbf{V}$ such that

$$\mathbf{V}^T\mathbf{A}^T\mathbf{A}\mathbf{V} = \mathbf{D}^2 \tag{2}$$

where $\mathbf{D}^2 = \langle \sigma_1^2, \ldots, \sigma_n^2 \rangle$ is diagonal and the columns of $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ are the orthonormalized eigenvectors of $\mathbf{A}^T\mathbf{A}$. Eq. (2) may be written in a more detailed form as:

$$\begin{bmatrix} (\mathbf{A}\mathbf{v}_1)^T \\ \vdots \\ (\mathbf{A}\mathbf{v}_n)^T \end{bmatrix} [\mathbf{A}\mathbf{v}_1, \ldots, \mathbf{A}\mathbf{v}_n] = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix}. \tag{2a}$$

By equating the main diagonal elements in both sides of Eq. (2a) we have $(\mathbf{A}\mathbf{v}_i)^T (\mathbf{A}\mathbf{v}_i) = \sigma_i^2$ or

$$||\mathbf{A}\mathbf{v}_i||_2 = \sigma_i, \qquad i = 1, \ldots, n \tag{3}$$

and by equating non-diagonal elements, $\mathbf{A}\mathbf{v}_1, \ldots, \mathbf{A}\mathbf{v}_n$ are seen to be mutually orthogonal vectors. According to our previous assumption $\sigma_1 \geqq \ldots \geqq \sigma_r > 0$, $\sigma_{r+1} = \ldots = \sigma_n = 0$ it follows from Eq. (3) that

$$\mathbf{A}\mathbf{v}_i \neq \mathbf{0} \quad \text{for} \quad i = 1, \ldots, r \tag{4a}$$

and

$$\mathbf{A}\mathbf{v}_i = \mathbf{0} \quad \text{for} \quad i = r+1, \ldots, n. \tag{4b}$$

Premultiplying the definition equations of the eigenvalue problem

$$\mathbf{A}^T\mathbf{A}\mathbf{v}_i = \sigma_i^2 \mathbf{v}_i, \qquad i = 1, \ldots, n \tag{5}$$

from the left by $\mathbf{A}$, we have

$$\mathbf{A}\mathbf{A}^T(\mathbf{A}\mathbf{v}_i) = \sigma_i^2(\mathbf{A}\mathbf{v}_i), \qquad i = 1, \ldots, n. \tag{6}$$

Because of (4a), from Eq. (6) it follows that eigenvalues $\sigma_1^2, \ldots, \sigma_r^2$ of $\mathbf{A}^T\mathbf{A}$ are also eigenvalues of $\mathbf{A}\mathbf{A}^T$, and the mutually orthogonal vectors $\mathbf{A}\mathbf{v}_i$, $i = 1, \ldots, r$ are eigenvectors of $\mathbf{A}\mathbf{A}^T$. Hence vectors

$$\mathbf{u}_i = \frac{\mathbf{A}\mathbf{v}_i}{\sigma_i}, \qquad i = 1, \ldots, r \tag{7}$$

are orthonormalized eigenvectors of $\mathbf{A}\mathbf{A}^T$, since Eq. (3) implies $||\mathbf{u}_i||_2 = 1$, $i = 1, \ldots, r$.

Next we shall use the following *lemma*:
If $\mathbf{A}$ is a $m \times n$ matrix of rank r, then $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ are also of rank r.

Proof: Since $\mathbf{A}$ is of rank r, there exist precisely $n-r$ linearly independent solutions of the equation

$$\mathbf{A}\mathbf{x} = \mathbf{0}. \tag{8}$$

These solutions are at the same time non-trivial solutions of equation

$$\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{0} \tag{9}$$

hence they are eigenvectors of A^TA corresponding to its zero eigenvalue. Moreover zero is A^TA's eigenvalue of multiplicity not smaller than $n-r$; hence the rank of A^TA is at most r. To show that the rank of A^TA cannot be even smaller than r, suppose that zero is A^TA's eigenvalue of multiplicity $n-r+1$. Then there exists a vector $y \neq 0$ for which

$$A^TAy = 0 \tag{10}$$

and $Ay \neq 0$ (else y would be one among the eigenvectors obtained by solving Eq. (8)). Since $Ay \neq 0$ and $y \neq 0$, we have $||Ay||_2^2 = y^TA^TAy > 0$, which contradicts Eq. (10). Therefore the zero is an eigenvalue of multiplicity not higher than $n-r$, hence the rank of A^TA is at least r. So we have proved that the rank of A^TA is exactly r.

Let $B = A^T$, then $B^TB = AA^T$ hence our proof for A^TA holds also for AA^T.

From the lemma it follows that matrix A^TA of order n has the zero as its eigenvalue exactly of multiplicity $n-r$ and matrix AA^T of order m has the zero eigenvalue with multiplicity $m-r$. Consequently $\sigma_1^2, \ldots, \sigma_r^2$ are the non-zero eigenvalues of both matrices A^TA and AA^T.

Similarly, AA^T is a real symmetric matrix hence of simple structure. Thus AA^T has an orthogonalized eigensystem $u_1, \ldots, u_m$. For eigenvectors u_i, $i = 1, \ldots, r$ corresponding to positive eigenvalues holds (7). Let U be the $m \times n$ matrix with columns $u_1, \ldots, u_n$, orthonormalized eigenvectors of AA^T corresponding to eigenvalues $\sigma_1^2, \ldots, \sigma_n^2$. Thus $U^TU = I_n$.

From (7) and our previous considerations it follows that

$$Av_i = \sigma_i u_i, \qquad i = 1, \ldots, n$$

or in form of matrix equation:

$$AV = UD.$$

Finally, pre-multiplying this equation from the right by the transpose of orthogonal matrix V results in Eq. (1).

A variant of the SVD theorem for square matrices has been proved by FORSYTHE and MOLER in [2].

ALGOL procedures for computation of the singular values and complete orthogonal decomposition of a real rectangular matrix based on very effective numerical methods have been given by GOLUB and REINSCH in [1]. FORTRAN variants of the mentioned procedures have been developed at the Department of Civil Engineering Mechanics, Technical University, Budapest.

These procedures may be applied for the numerical solution of a high number of problems; some of them will be presented in the following:

Computation of the pseudoinverse of A

Let A be a real $m \times n$ matrix. An $n \times m$ matrix X is said to be the pseudoinverse of A if X satisfies the following four properties:

$$\mathbf{AXA} = \mathbf{A}; \qquad \mathbf{XAX} = \mathbf{X};$$

$$(\mathbf{AX})^T = \mathbf{AX}; \qquad (\mathbf{XA})^T = \mathbf{XA}.$$

The unique solution is denoted by $\mathbf{A}^+$. It is easy to verify that if $\mathbf{A} = \mathbf{UDV}_E^T$, then $\mathbf{A}^+ = \mathbf{VD}^+\mathbf{U}^T$ where $\mathbf{D}^+ = \langle \sigma_1^+, \ldots, \sigma_n^+ \rangle$ and

$$\sigma_i^+ = \begin{cases} \dfrac{1}{\sigma_i} & \text{for} \quad \sigma_i > 0 \\ 0 & \text{for} \quad \sigma_i = 0. \end{cases}$$

Thus $\mathbf{A}^+$, the pseudoinverse of A, is easy to compute if the SVD of A has been accomplished i.e. if matrices U, D and V are known.

Solution of homogeneous equations

Let A be a matrix of rank r, and suppose we wish to solve

$$\mathbf{Ax}_i = \mathbf{0} \quad \text{for} \quad i = r+1, \ldots, n.$$

Let

$$\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_n] \quad \text{and} \quad \mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_n].$$

Then, since $\mathbf{Av}_i = \sigma_i \mathbf{u}_i$ $(i = 1, \ldots, n)$, and $\sigma_{r+1} = \ldots = \sigma_n = 0$, we have

$$\mathbf{Av}_i = \mathbf{0} \quad \text{for} \quad i = r+1, \ldots, n$$

and $\mathbf{x}_i = \mathbf{v}_i$.

If the rank of A is known, then the system of linear homogeneous equations may be solved by a simpler algorithm.

Solution of linear least squares problems

Let A be a real $m \times n$ matrix with $m > n$ and let b be a given vector with m elements. A vector x with n elements has to be determined so that

$$||\mathbf{b} - \mathbf{Ax}||_2 = \min. \tag{11}$$

If rank r of A is less than n, then there is no unique solution. Thus we require amongst all x which satisfy (11) that

$$||\hat{x}||_2 = \min$$

and this solution is unique; further

$$\hat{\mathbf{x}} = \mathbf{A}^{+}\mathbf{b} = \mathbf{V}\mathbf{D}^{+}\mathbf{U}^{T}\mathbf{b}.$$

This statement will be proven by the following. Since $||\mathbf{b} - \mathbf{A}\mathbf{x}||_2^2 = (\mathbf{b} - \mathbf{A}\mathbf{x})^T(\mathbf{b} - \mathbf{A}\mathbf{x})$, the condition for minimum $\frac{d}{d\mathbf{x}}||\mathbf{b} - \mathbf{A}\mathbf{x}||_2^2 = 0$ leads to

$$\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}. \tag{12}$$

Thus our problem is equivalent to solving Eq. (12).

Substituting $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$ into Eq. (12) we obtain

$$\mathbf{V}\mathbf{D}\underbrace{\mathbf{U}^T\mathbf{U}}_{\mathbf{I}_n}\mathbf{D}\mathbf{V}^T\mathbf{x} = \mathbf{V}\mathbf{D}\mathbf{U}^T\mathbf{b}$$

and since $\mathbf{V}$ is orthogonal we have

$$\mathbf{D}^2\mathbf{V}^T\mathbf{x} = \mathbf{D}\mathbf{U}^T\mathbf{b}. \tag{13}$$

If $\mathbf{A}$ has a rank $r = n$, then $\mathbf{D}$ is non-singular and so

$$\mathbf{x} = \mathbf{V}\mathbf{D}^{-1}\mathbf{U}^T\mathbf{b} = \mathbf{V}\mathbf{D}^{+}\mathbf{U}^T\mathbf{b} = \mathbf{A}^{+}\mathbf{b}$$

is the unique solution of our problem. For $r < n$, Eq. (13) may be rewritten in partitioned form:

$$\left[\begin{array}{c|c} \mathbf{D}_r^2 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right]\begin{bmatrix} \mathbf{y}_r \\ \mathbf{y}_s \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{D}_r & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right]\begin{bmatrix} \mathbf{c}_r \\ \mathbf{c}_s \end{bmatrix}$$

where

$$\mathbf{V}^T\mathbf{x} = \mathbf{y} = \begin{bmatrix} \mathbf{y}_r \\ \mathbf{y}_s \end{bmatrix} \quad \text{and} \quad \mathbf{U}^T\mathbf{b} = \mathbf{c} = \begin{bmatrix} \mathbf{c}_r \\ \mathbf{c}_s \end{bmatrix}.$$

From the partitioned form of Eq. (13) it follows that $\mathbf{y}_r$ may be obtained as the unique solution of equation $\mathbf{D}_r^2\mathbf{y}_r = \mathbf{D}_r\mathbf{c}_r$ and $\mathbf{y}_s$ may be chosen arbitrarily. If $\mathbf{y}_s = \mathbf{0}$ then obviously $||\hat{\mathbf{y}}||_2 = \min$. Then also $||\hat{\mathbf{x}}||_2 = \min$ since $||\hat{\mathbf{x}}||_2 = ||\mathbf{V}\hat{\mathbf{y}}||_2 = (\hat{\mathbf{y}}^T\mathbf{V}^T\mathbf{V}\hat{\mathbf{y}})^{1/2} = ||\hat{\mathbf{y}}||_2$.

Writing equations

$$\mathbf{y}_r = \mathbf{D}_r^{-1}\mathbf{c}_r$$

$$\mathbf{y}_s = \mathbf{0}$$

in the form

$$\begin{bmatrix} \mathbf{y}_r \\ \mathbf{y}_s \end{bmatrix} = \begin{bmatrix} \mathbf{D}_r^{-1}, & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{c}_r \\ \mathbf{c}_s \end{bmatrix}$$

or

$$\mathbf{V}^T\hat{\mathbf{x}} = \mathbf{D}^{+}\mathbf{U}^T\mathbf{b}$$

leads to

$$\hat{\mathbf{x}} = \mathbf{V}\mathbf{D}^{+}\mathbf{U}^{T}\mathbf{b} = \mathbf{A}^{+}\mathbf{b}$$

and the proof is complete.

Note that if matrix $\mathbf{U}$ is not needed, it would appear that one could apply the usual diagonalization algorithms to symmetric matrix $\mathbf{A}^T\mathbf{A}$ which has to be formed explicitly. However, the computation of $\mathbf{A}^T\mathbf{A}$ involves unnecessary numerical inaccuracy. For example, let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \quad \text{then} \quad \mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1+\alpha^2, & 1 \\ 1 & 1+\alpha^2 \end{bmatrix}$$

so that $\sigma_1(\mathbf{A}) = \sqrt{2+\alpha^2}$, $\sigma_2(\mathbf{A}) = |\alpha|$. If $\alpha^2 < \varepsilon$, the computer precision, the computed $\mathbf{A}^T\mathbf{A}$ has the form $\begin{bmatrix} 1, & 1 \\ 1, & 1 \end{bmatrix}$ and the best one obtained by diagonalization is $\tilde{\sigma}_1(\mathbf{A}) = \sqrt{2}$, $\tilde{\sigma}(\mathbf{A})_2 = 0$.

Some properties of square matrices

Consider a linear mapping $A : X \to Y$ between two n-dimensional spaces X and Y, where A is represented by a square matrix $\mathbf{A}$ of order n. This means that to every $\mathbf{x} \in X$ there is a $\mathbf{y} = \mathbf{A}\mathbf{x} \in Y$. Using the singular value decomposition of $\mathbf{A}$ we have

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \mathbf{U}\mathbf{D}\mathbf{V}^T\mathbf{x}$$

or

$$\underbrace{\mathbf{U}^T\mathbf{y}}_{\mathbf{y}'} = \mathbf{D}\underbrace{\mathbf{V}^T\mathbf{x}}_{\mathbf{x}'}.$$

By $\mathbf{x}' = \mathbf{V}^T\mathbf{x}$ and $\mathbf{y}' = \mathbf{U}^T\mathbf{y}$ we have only introduced orthogonal change of variables in spaces X and Y. Then $||\mathbf{x}'||_2 = ||\mathbf{V}^T\mathbf{x}||_2 = \sqrt{\mathbf{x}^T\mathbf{V}\mathbf{V}^T\mathbf{x}} = \sqrt{\mathbf{x}^T\mathbf{x}} = ||\mathbf{x}||_2$ and analogously $||\mathbf{y}'||_2 = ||\mathbf{y}||_2$; this means that orthogonal transformations have left the Euclidean norm of vectors unchanged.

The orthogonal change of variables in both spaces X and Y caused the original transformation A to take a new diagonal matrix form:

$$\mathbf{y}' = \mathbf{D}\mathbf{x}' \tag{14}$$

or written more detailed:

$$\left.\begin{aligned} y_1' &= \sigma_1 x_1' \\ &\vdots \\ y_r' &= \sigma_r x_r' \\ y_{r+1}' &= 0 \\ &\vdots \\ y_n' &= 0 \end{aligned}\right\} \tag{15}$$

where r denotes the rank of $\mathbf{A}$.

Using (15) it is easy to see that the unit hypersphere $\{\mathbf{x}' : ||\mathbf{x}'||_2 = 1\}$ i.e. $x_1'^2 + \ldots + x_n'^2 = 1$ after transformation by $\mathbf{D}$ will take the form of an r-dimensional hyperellipse

$$\frac{y_1'^2}{\sigma_1^2} + \ldots + \frac{y_n'^2}{\sigma_n^2} = 1 \quad \text{if} \quad r = n$$

or

$$\frac{y_1'^2}{\sigma_1^2} + \ldots + \frac{y_r'^2}{\sigma_r^2} \leq 1 \quad \text{and} \quad y_{r+1}' = \ldots = y_n' = 0,$$

$$\text{if} \quad r < n.$$

One of the farthest points of the hyperellipse from the origin is that with coordinates $(\sigma_1, 0, \ldots, 0)$. If $r < n$, then the origin is a point of the hyperellipse. If $r = n$, the origin is not a point of the hyperellipse and one of its points nearest to the origin is that with coordinates $(0, \ldots, 0, \sigma_n)$.

If $r < n$ then both $\mathbf{D}$ and $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$ are singular matrices. If $r = n$ then $\mathbf{D} = \langle \sigma_1, \ldots, \sigma_n \rangle$ and $\mathbf{A}$ are non-singular and it follows from (14) or (15) that $\mathbf{D}^{-1} = \langle \sigma_1^{-1}, \ldots, \sigma_n^{-1} \rangle$. Since $\mathbf{A}^{-1} = (\mathbf{U}\mathbf{D}\mathbf{V}^T)^{-1} = (\mathbf{V}^T)^{-1}\mathbf{D}^{-1}\mathbf{U}^{-1} = \mathbf{V}\mathbf{D}^{-1}\mathbf{U}^T$, hence the singular values of $\mathbf{A}^{-1}$ are values $\sigma_1^{-1}, \ldots, \sigma_n^{-1}$.

Corresponding to any vector norm $||.||$ for any real matrix $\mathbf{A}$ of order n the *matrix-norm* (Hilbert-norm) may be defined by the quantity

$$||\mathbf{A}|| = \max_{\mathbf{x} \neq 0} \frac{||\mathbf{A}\mathbf{x}||}{||\mathbf{x}||} = \max_{||\mathbf{x}||=1} ||\mathbf{A}\mathbf{x}||. \tag{16}$$

Note that (16) implies

$$||\mathbf{A}\mathbf{x}|| \leq ||\mathbf{A}||\,||\mathbf{x}||. \tag{17}$$

The geometric interpretation of matrix norm (16) is that $||\mathbf{A}||$ is the maximum length of a unit vector after transformation by $\mathbf{A}$. But, using the Euclidean vector norm $||.||_2$ the unit hypersphere $\{\mathbf{x} : ||\mathbf{x}||_2 = 1\}$ after transformation by $\mathbf{A}$ is seen to become a hyperellipse, where the length of the major half-axis is equal to σ_1, hence

$$||\mathbf{A}||_2 = \max_{||\mathbf{x}||_2=1} ||\mathbf{A}\mathbf{x}||_2 = ||\mathbf{D}||_2 = \sigma_1. \tag{18}$$

$||\mathbf{A}||_2$ is also called the *spectral norm* of $\mathbf{A}$.

In conformity with the above, the singular values of $\mathbf{A}^{-1}$ are $\sigma_1^{-1}, \ldots, \sigma_n^{-1}$. Since at the same time $\max \{\sigma_1^{-1}, \ldots, \sigma_n^{-1}\} = \sigma_n^{-1}$, for $r = n$ we have

$$||\mathbf{A}^{-1}||_2 = ||\mathbf{D}^{-1}||_2 = \sigma_n^{-1}. \tag{19}$$

From these facts and their geometric interpretations if follows that the maximum possible deformation on the unit hypersphere after transformation by $\mathbf{A}$ may be expressed as ratio σ_1/σ_n.

On the other hand, for any square matrix A, the condition of A with respect to inversion and to the particular norm used by the *condition number* may be defined as:

$$\text{cond}\,(\mathbf{A}) = \begin{cases} \|\mathbf{A}\|\,\|\mathbf{A}^{-1}\| & \text{if } \mathbf{A} \text{ is non-singular} \\ +\infty & \text{if } \mathbf{A} \text{ is singular.} \end{cases}$$

For the spectral norm it follows from (18) and (19) that

$$\text{cond}_2\,(\mathbf{A}) = \|\mathbf{A}\|_2\,\|\mathbf{A}^{-1}\|_2 = \frac{\sigma_1}{\sigma_n} \geq 1, \tag{20}$$

where σ_1 and σ_n are maximum and minimum singular values of A, respectively. Hence $\text{cond}_2(\mathbf{A})$ is the measure of the maximum possible deformation of the unit hypersphere after transformation by A. For the sake of completeness let us note that for any non-singular symmetric matrix A, (20) implies $\text{cond}_2(\mathbf{A}) = |\lambda|_{\max}/|\lambda|_{\min}$, where $|\lambda|_{\max}$ and $|\lambda|_{\min}$ are maximum and minimum eigenvalues in modulus of A, respectively.

It is easy to see why cond(A) plays a dominant role as a reliable measure of the conditioning (stability) of the solution of the system of linear equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{21}$$

Assume A to be non-singular hence (21) to have a unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. Let us see how small changes in the data of Eq. (21) affect its solution. Perturbation of the elements of vector b alone leads to:

$$\mathbf{A}(\mathbf{x} + \delta x) = \mathbf{b} + \delta\mathbf{b}$$

and subtracting Eq. (21) from the last equation, we have $\mathbf{A}\delta\mathbf{x} = \delta\mathbf{b}$ or

$$\delta\mathbf{x} = \mathbf{A}^{-1}\delta\mathbf{b}.$$

Applying inequality (17) to the last equation and to Eq. (21) yields:

$$\|\delta\mathbf{x}\| \leq \|\mathbf{A}^{-1}\|\,\|\delta\mathbf{b}\| \text{ and } \|\mathbf{b}\| \leq \|\mathbf{A}\|\,\|\mathbf{x}\|.$$

The product of the two inequalities above is:

$$\|\delta\mathbf{x}\|\,\|\mathbf{b}\| \leq \|\mathbf{A}\|\,\|\mathbf{A}^{-1}\|\,\|\delta\mathbf{b}\|\,\|\mathbf{x}\|$$

so that

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \text{cond}\,(\mathbf{A})\left|\frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}\right..$$

This result shows the possibility of a large relative error in x even for a small relative error in b if the condition number of A is high.

The following simple example will illustrate the dangers inherent in solving ill-conditioned systems. Consider the system

$$\begin{bmatrix} 2, & 6 \\ 2, & 6.00001 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 8.00001 \end{bmatrix}$$

with exact solution $x_1 = x_2 = 1$, and the perturbed system

$$\begin{bmatrix} 2, & 6 \\ 2, & 5.99999 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 8.00002 \end{bmatrix}$$

which has the solution $x_1 = 10$, $x_2 = -2$. Here changes of 2×10^{-5} in a_{22} and 10^{-5} in b_2 have caused a gross change in the solution.

The coefficient matrices described above are seen to be almost singular. But let us underline that in general the ill-condition of a matrix is independent of the smallness of its determinant. In fact, if, for instance $\sigma_1 = \ldots = \sigma_n = 10^{-30}$, then the singular value decomposition of A and the fact that determinants of orthogonal matrices are equal to ± 1 imply

$$|\det(\mathbf{A})| = \det(\mathbf{U}) \det(\mathbf{D}) \det(\mathbf{V}^T) = \sigma_1, \ldots, \sigma_n = 10^{-30n},$$

which is a very small number. Nevertheless $\operatorname{cond}_2(\mathbf{A}) = \sigma_1/\sigma_n = 1$ hence A is perfectly conditioned.

Summary

A constructive proof of the singular value decomposition theorem and its application to the numerical solution and analysis of some linear algebraic problems have been presented.

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