INTEGRAL EQUATIONS FOR LINEAR ELASTO-VISCOUS SKELETONS

By

B. ROLLER

Department of Civil Engineering Mechanics, Technical University, Budapest

Received: November 15, 1980

1. Introduction

The altering state analysis of a structure involves relationships between the prescribed action of the structure (i.e. the load and the initial strain) and the response generated by the system (i.e. the displacement and the stresses) [9]. Similar rheology problems of elastic structures have been studied in detail since long, while the behaviour of structures, especially that of prestressed concrete is essentially influenced by creep and relaxation [5].

Survey of the special literature proves that a scientific team has developed in this country in the last fifteen years, concerned with the altering state analysis of perfectly elastic spatial structures. The methods suggested by this team have been widely spread in practice mainly in computing centers [7]. On the other hand, linear visco-elasticity has a well developed literature based ultimately on numerical methods [11] [12], that fit mainly the solution of continuum problems [4]. So it seems necessary to synthetize the results of both trends.

This paper is intended to generalize the basic equations of skeletons from *Hooke*an ones to structures consisting of bars each obeying a *Boltzmann—Volterra*-type [2] constitutive equation.

The validity conditions of *Picard*'s iteration [1] for the altering state equation are proved, as well. The constitutive law of concrete creep contained in Hungarian Codes [10] meets one condition.

The effect of permanent action on a gridwork of inhomogeneous construction will be illustrated by a numerical example. The problem has been solved by making use of numerical *Laplace*-retransform [3]. The procedure works well under the given particular circumstances, and leads to reasonable consequences [8].

2. Integral equations of the skeletons

2.1 *The basic equation*

The main altering state equation of the structure based on the validity of the first-order theory is

$$\begin{bmatrix} & \mathbb{G}^* \\ \mathbb{G} & \end{bmatrix} \begin{bmatrix} \mathbf{u}(t) \\ \mathbf{s}(t) \end{bmatrix} + \begin{bmatrix} \\ \Delta \mathbf{g}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(t) \\ \mathbf{d}(t) \end{bmatrix} = \mathbf{0} \tag{1}$$

[9], where

t — time
$\mathbf{q}(t)$ — load vector
$\mathbf{d}(t)$ — initial strain
$\mathbf{u}(t)$ — unknown displacement
$\mathbf{s}(t)$ — unknown stress
$\mathbb{G}^*$ — equilibrium matrix
$\Delta\mathbf{g}(t)$ — strain due to a stress.

It has to be mentioned that the rheological behaviour of the material makes the variables time-dependent even if forces of inertia are neglected (quasi-static effect). Besides, $\Delta\mathbf{g}(t)$ allows any proper constitutive relationship, finally (1) is assumed to fulfill certain scleronomic boundary conditions.

2.2 *The constitutive equations of uniaxial strain*

Creep and *relaxation* are the most important properties of visco-elastic behaviour. They are interdependent thus both of them suit to describe the criteria of linear visco-elasticity. Therefore the *Boltzmann—Volterra* constitutive law of uniaxial state of stress in the elementary strength of materials may be stated either as

$$\varepsilon(t) = Y_\varepsilon(t, t)\, \sigma(t) + \int_0^t K_\varepsilon(t, t')\, \sigma(t')\, dt' \tag{2}$$

or as

$$\sigma(t) = Y_\sigma(t, t)\, \varepsilon(t) + \int_0^t K_\sigma(t, t')\, \varepsilon(t')\, dt' \tag{3}$$

where $Y_\varepsilon(t, t')$ denotes the creep compliance effect at a time t due to a load acting since the instant t'; $Y_\sigma(t, t')$ is the relaxation modulus; finally

$$K_\varepsilon(t, t') = \frac{dY_\varepsilon(t, t')}{d(t - t')} \quad ; \quad K_\sigma(t, t') = \frac{dY_\sigma(t, t')}{dt - t')} \tag{4}$$

are the creep and the relaxation kernels, respectively. These formulae valid to normal stresses can be adopted to the case of pure shear, as well. For sake of simplicity, the quotient of the creep compliances and those of the relaxation moduli in shear and in normal stress, respectively, will be assumed to be constant.

$$Y_\gamma = \omega Y_\varepsilon \quad ; \quad Y_\tau = \frac{1}{\omega} Y_\sigma . \tag{5}$$

This is not an essential assumption.

2.3 *Stress resultants and displacements of a bar*

Flexibility equations of a straight-axed bar made of a homogeneous viscoelastic material are easy to develop taking relationships (2) to (5) and the principles of the elementary strength of materials into consideration.

With notations in Fig. 1 and considering axial stress:

$$\Delta w_{j,k,\xi}(t) = Y_{\varepsilon;j,k,}(t, t) \frac{l_{j,k}}{A_{j,k}} P_{j,k,\xi}(t) + \frac{l_{j,k}}{A_{j,k}} \int_0^t K_{\varepsilon;j,k}(t, t')\, P_{j,k,\xi}(t')\, dt' \tag{6}$$

and

$$P_{j,k,\xi}(t) = Y_{\sigma;j,k}(t, t) \frac{A_{j,k}}{l_{j,k}} \Delta w_{j,k,\xi}(t) + \frac{A_{j,k}}{l_{j,h}} \int_0^t K_{\sigma;j,k}(t, t')\, \Delta w_{j,k,\xi}(t')\, dt' \tag{7}$$

respectively. Furthermore, in case of bending,

$$-\frac{\partial^2 \Delta w_\zeta(\xi_{j,k}, t)}{\partial \xi_{j,k}^2} = Y_{\varepsilon;j,k}(t, t) \frac{1}{I_{j,k,\eta}} M_\eta(\xi_{j,k}, t) + \frac{1}{I_{j,k,\eta}} \int_0^t K_{\varepsilon;j,k}(t, t')\, M_\eta(\xi_{j,k}, t')\, dt' \tag{8}$$

and

$$M_\eta(\xi_{j,k}, t) = Y_{\sigma;j,k}(t, t)\, I_{j,k,\eta} \frac{\partial^2 \Delta w_\zeta(\xi_{j,k}, t)}{\partial \xi_{j,k}^2} + \\ + I_{j,k,\eta} \int_0^t K_{\sigma;j,k}(t, t') \frac{\partial^2 \Delta w_\zeta(\xi_{j,k}, t')}{\partial \xi_{j,k}^2}\, dt', \tag{9}$$

respectively. M_η is the bending moment varying along the bar axis, Δw_ξ is the variable relative displacement. $I_{j,k,\eta}$ is the inertia of the cross section re-

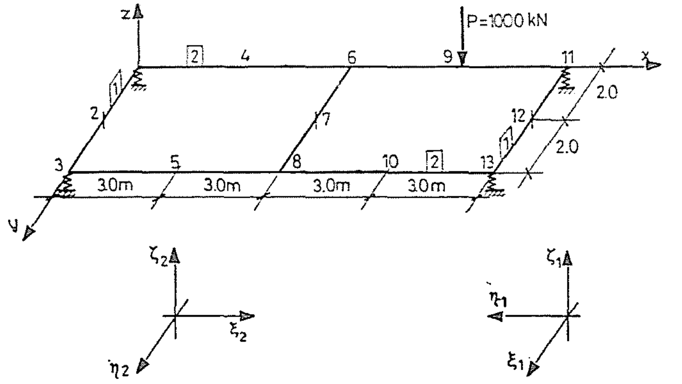


Fig. 1

ferred to the η-axis. Similar relationships are valid in torsion, too. Finally, force $P_{j,k}(t)$ acting at the bar end causes a relative displacement:

$$\Delta w_{j,k,\xi}(t) = Y_{\varepsilon;j,k}(t,t)\frac{l_{j,k}^3}{3I_{j,k,\eta}}P_{j,k,\zeta}(t) + \frac{l_{j,k}^3}{3I_{j,k,\eta}}\int_0^t K_{\varepsilon;j,k}(t,t')\,P_{j,k,\zeta}(t')\,dt'. \tag{10}$$

2.4 *The flexibility integral equation and the relationship of the structure stiffness*

The flexibility equation of the bar is a matrix relationship describing the vector of the relative displacements at the bar end by the vector of the stress history. On the other hand, the stiffness equation describes the vector of the respective stresses by the vector of the history of the relative displacement. Suppose the bar connecting nodes j and k not to be directly loaded. Let the vector of the relative displacement between the bar ends be

$$\Delta \mathbf{g}_{j,k}(t) = \begin{bmatrix} \Delta w_{j,k,\xi}(t) \\ \Delta w_{j,k,\eta}(t) \\ \Delta w_{j,k,\zeta}(t) \\ \Delta \vartheta_{j,k,\xi}(t) \\ \Delta \vartheta_{j,k,\eta}(t) \\ \Delta \vartheta_{j,k,\zeta}(t) \end{bmatrix}. \tag{11}$$

Be $\widetilde{\mathbf{F}}_{j,k}$ the flexibility matrix of the bar provided *Young*'s modulus $E = 1$ and let

$$\widetilde{\mathbf{S}}_{j,k} = \widetilde{\mathbf{F}}_{j,k}^{-1} \tag{12}$$

be a stiffness matrix. Furthermore, develop the variable flexibility and stiffness matrices as

$$\mathbf{F}_{j,k}(t,t) = Y_{\varepsilon;j,k}(t,t)\,\widetilde{\mathbf{F}}_{j,k} \tag{13}$$

$$\mathbf{S}_{j,k}(t,t) = Y_{\sigma;j,k}(t,t)\,\widetilde{\mathbf{S}}_{j,k}. \tag{14}$$

In addition, apply the matrix kernels

$$\begin{aligned} \mathbf{K}_{\varepsilon;j,k}(t,t') &= K_{\varepsilon;j,k}(t,t')\,\mathbf{I} \\ \mathbf{K}_{\sigma;j,k}(t,t') &= K_{\sigma,j,k}(t,t')\,\mathbf{I} \end{aligned} \tag{15}$$

as well ($\mathbf{I}$ stands for the unit matrix).

Thus the matrix integral equation of the bar flexibility is written as

$$\Delta \mathbf{g}_{j,k}(t) = \mathbf{F}_{j,k}(t,t)\,\mathbf{s}_{j,k}(t) + \widetilde{\mathbf{F}}_{j,k}\int_0^t \mathbf{K}_{\varepsilon;j,k}(t,t')\,\mathbf{s}_{j,k}(t')\,dt' \tag{16}$$

while the matrix integral equation of the stiffness states

$$\mathbf{s}_{j,k}(t) = \mathbf{S}_{j,k}(t, t)\,\Delta\mathbf{g}_{j,k}(t) + \tilde{\mathbf{S}}_{j,k}\int_0^t \mathbf{K}_{\sigma;j,k}(t, t')\,\Delta\mathbf{g}_{j,k}(t')\,dt'. \tag{17}$$

Eqs (16) and (17) are valid in turn to each bar in the structure separately. Making use of hyper-diagonals, they can be compiled into hypermatrix equations containing as many block equations as there are bars. Thus the flexibility and the stiffness integral equations of the complete structure may be written in forms (16) and (17), respectively, just subscripts j and k are to be omitted.

2.5 *The integral equations of the altering state*

The first integral equation of the altering state for a visco-elastic skeleton structure can be developed by combining the modified formula (16) and the basic relationship (1). Rearranged

$$\begin{bmatrix} & \mathbf{G}^* \\ \mathbf{G} & \mathbf{F}(t, t) \end{bmatrix}\begin{bmatrix} \mathbf{u}(t) \\ \mathbf{s}(t) \end{bmatrix} + \int_0^t \begin{bmatrix} & \\ & \mathbf{F}\tilde{\mathbf{K}}_{\varepsilon}(t, t') \end{bmatrix}\begin{bmatrix} \\ \mathbf{s}(t') \end{bmatrix} dt' + \begin{bmatrix} \mathbf{q}(t) \\ \mathbf{d}(t) \end{bmatrix} = \mathbf{0}. \tag{18}$$

This formula can be reduced in a manner quite similar to that used in the theory of perfectly elastic structures, where matrix equations of the equilibrium method and the compatibility method are developed by partitioning the altering state equation.

The procedure delivers the matrix integral equation of the equilibrium method:

$$\mathbf{M}(t, t)\,\mathbf{u}(t) + \int_0^t \mathbf{V}(t, t')\,\mathbf{u}(t')\,dt' = \mathbf{q}(t) - \mathbf{H}(t, t)\,\mathbf{d}(t) - \int_0^t \mathbf{I}(t, t')\,\mathbf{d}(t')\,dt' \tag{19}$$

where

$$\begin{aligned} \mathbf{G}^*\,\mathbf{S}(t, t)\,\mathbf{G} &= \mathbf{M}(t, t) \qquad & \mathbf{G}^*\,\tilde{\mathbf{S}}\mathbf{K}_{\sigma}(t, t')\,\mathbf{G} &= \mathbf{V}(t, t') \\ \mathbf{G}^*\,\mathbf{S}(t, t) &= \mathbf{H}(t, t) \qquad & \mathbf{G}^*\,\tilde{\mathbf{S}}\mathbf{K}_{\sigma}(t, t') &= \mathbf{I}(t, t'). \end{aligned} \tag{20}$$

Developing the integral equation of the compatibility method first, rows and columns may be suitably reversed, thus (1) holds in a partitioned form

$$\begin{bmatrix} & \mathbf{G}_1^* & \mathbf{G}_2^* \\ \mathbf{G}_1 & & \\ \mathbf{G}_2 & & \end{bmatrix}\begin{bmatrix} \mathbf{u}(t) \\ \mathbf{s}_1(t) \\ \mathbf{s}_2(t) \end{bmatrix} + \begin{bmatrix} \\ \Delta\mathbf{g}_1(t) \\ \Delta\mathbf{g}_2(t) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(t) \\ \mathbf{d}_1(t) \\ \mathbf{d}_2(t) \end{bmatrix} = \mathbf{0} \tag{21}$$

$\mathbf{s}_1(t)$ denotes the stress vector of the release system, $\mathbf{s}_2(t)$ stands for the redundant stresses while the remaining blocks correspond to those of the stresses.

The compatibility relationship will be written in a similar manner, that is, in partitioned form:

$$\Delta \mathbf{g}_1(t) = \mathbf{F}_1(t, t)\, \mathbf{s}_1(t) + \widetilde{\mathbf{F}}_1 \int_0^t \mathbf{K}_{\varepsilon,1}(t, t')\, \mathbf{s}_1(t')\, dt' \tag{22}$$

and

$$\Delta \mathbf{g}_2(t) = \mathbf{F}_2(t, t)\, \mathbf{s}_2(t) + \widetilde{\mathbf{F}}_2 \int_0^t K_{\varepsilon,2}(t, t')\, \mathbf{s}_2(t')\, dt'$$

respectively.

Introducing matrices

$$[-\mathbf{G}_2 \mathbf{G}_1^{-1} \quad \mathbf{I}] = \mathbf{D} \quad ; \quad \begin{bmatrix} \mathbf{G}_1^{*-1} \\ \ \end{bmatrix} = \mathbf{N}^* \tag{23}$$

results in the integral equation of the force method:

$$\mathbf{D}\mathbf{F}(t, t)\, \mathbf{D}^*\, \mathbf{s}_2(t) + \mathbf{D}\widetilde{\mathbf{F}} \int_0^t \mathbf{K}_\varepsilon(t, t')\, \mathbf{D}^*\, \mathbf{s}_2(t')\, dt' +$$

$$+ \mathbf{D}\mathbf{d}(t) - \mathbf{D}\mathbf{F}(t, t)\, \mathbf{N}^*\, \mathbf{q}(t) - \mathbf{D}\widetilde{\mathbf{F}} \int_0^t \mathbf{K}_\varepsilon(t, t')\, \mathbf{N}^*\, \mathbf{q}(t')\, dt' = \mathbf{0}. \tag{24}$$

$F(t,t)$, $\widetilde{\mathbf{F}}$ and $\mathbf{K}_\varepsilon(t, t')$ denote hyperdiagonals compiled from the corresponding blocks.

3. Computation of the state variables

3.1 *The method of iteration and convergence conditions*

There are several methods for the solution of the hypermatrix integral equation of the altering state, here the successive iteration strategy will be discussed. Detailing the formulas of the procedure the conditions of its convergence are to be proved. Introducing notations

$$\mathbf{A}(t, t) = \begin{bmatrix} & \mathbf{G}^* \\ \mathbf{G} & \mathbf{F}(t, t) \end{bmatrix} \qquad \mathbf{W}(t\, t') = \begin{bmatrix} & \\ & \widetilde{\mathbf{F}}\mathbf{K}_\varepsilon(t, t') \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{u}(t) \\ \mathbf{s}(t) \end{bmatrix} \qquad \mathbf{b}(t) = \begin{bmatrix} \mathbf{q}(t) \\ \mathbf{d}(t) \end{bmatrix} \tag{25}$$

leads to the concise form

$$\mathbf{A}(t, t)\, \mathbf{x}(t) + \int_0^t \mathbf{W}(t, t')\, \mathbf{x}(t')\, dt' + \mathbf{b}(t) = \mathbf{0} \tag{26}$$

of the original relationship. The following iterative method is based on step-by-step fictitious elastic solutions obtained by applying the actual nonsingular value of matrix $\mathbf{A}(t, t)$. Every step of the procedure adopts the adequate form of the response obtained in the previous step as vector $\mathbf{x}(t')$ at each instant t. Thus we have the following algorithm:

$$\mathbf{x}_0(t) = \mathbf{0}; \quad \mathbf{x}_i(t) = -\mathbf{A}(t,t)^{-1}\mathbf{b}(t) - \int_0^t \mathbf{A}(t,t)^{-1}\mathbf{W}(t,t')\,\mathbf{x}_{i-1}(t')\,dt'$$

$$(i = 1, 2, \ldots, n;\ n \to \infty). \tag{27}$$

Provided it converges, the approximate solution of Eq. (26) is established. Consider the m-th iterative value of the solution, making use of (27):

$$\begin{aligned} \mathbf{x}_n(t) = &-\mathbf{A}(t,t)^{-1}\mathbf{b}(t) + \int_0^t \mathbf{A}(t,t)^{-1}\mathbf{W}(t,t')\,\mathbf{A}(t',t')^{-1}\mathbf{b}(t')\,dt' - \\ &- \ldots + (-1)^n \int_0^t \int_0^{t'} \int_0^{t''} \ldots \int_0^{t^{(n-2)}} \mathbf{A}(t,t)^{-1}\mathbf{W}(t,t')\,\mathbf{A}(t',t')^{-1}\mathbf{W}(t',t'')\ldots \\ &\ldots \mathbf{A}(t^{(n-2)}, t^{(n-2)})^{-1}\,\mathbf{W}(t^{(n-2)}, t^{(n-1)}) \times \\ &\times \mathbf{A}(t^{(n-1)}, t^{(n-1)})^{-1}\,\mathbf{b}(t^{(n-1)})\,dt'\,dt''\ldots dt^{(n-1)}. \end{aligned} \tag{28}$$

with parameters $t'', t''', \ldots, t^{(n-1)}$.

Convert the sequence $\mathbf{x}_n(t)$ into a series

$$\mathbf{x}_n(t) = \mathbf{x}_1(t) + \mathbf{x}_2(t) - \mathbf{x}_1(t) + \ldots + \mathbf{x}_n(t) - \mathbf{x}_{n-1}(t). \tag{29}$$

If $\|\mathbf{x}_n(t)\|$ is bounded, the procedure converges. The analysis is based on the norm of the difference vector $\mathbf{x}_n(t) - \mathbf{x}_{n-1}(t)$. Recalling (28):

$$\begin{aligned} \mathbf{x}_n(t) - \mathbf{x}_{n-1}(t) = (-1)^n \int_0^t \int_0^{t'} \int_0^{t''} \ldots \int_0^{t^{(n-2)}} &\mathbf{A}(t,t)^{-1}\mathbf{W}(t,t')\mathbf{A}(t',t')^{-1}\mathbf{W}(t',t'')\ldots \\ &\ldots \mathbf{A}(t^{(n-2)}, t^{(n-2)})^{-1}\,\mathbf{W}(t^{(n-2)}, t^{(n-1)}) \times \\ &\times \mathbf{A}(t^{(n-1)}, t^{(n-1)})^{-1}\,\mathbf{b}(t^{(n-1)})\,dt'\,dt''\ldots dt^{(n-1)}. \end{aligned} \tag{30}$$

Obviously, the inverse matrix of $\mathbf{A}$ occurring in the procedure has to be considered as a bounded function, for $\mathbf{A}$ must not be singular at any instant. Let the upper bound of the appropriate norm function be α i.e.:

$$\|\mathbf{A}(t,t)^{-1}\| \leq \alpha. \tag{31}$$

In addition, suppose that also all the creep compliance kernels $K_{\varepsilon;j;k}(t, t')$ of the bar elements in the structure are absolutely bounded, however variable t and parameter t' had been selected.

$$|K_{\varepsilon;j,k}(t,t')|,\ |K_{\varepsilon;j,k}(t',t'')|,\ \ldots,\ |K_{\varepsilon;j,k}(t^{(n-2)}, t^{(n-1)})| \leq \varkappa_{j,k}. \tag{32}$$

The maximum value of these bounds points to:

$$||\mathbf{W}(t, t')||, \ldots, ||\mathbf{W}(t^{(n-2)}, t^{(n-1)})|| \leq ||\tilde{\mathbf{F}}||\varkappa, \tag{33}$$

where $||\tilde{\mathbf{F}}||$ is obviously finite. Namely, for instance,

$$||\mathbf{W}(t, t')|| = ||\langle \tilde{\mathbf{F}}_1 K_{\varepsilon;1}(t, t'), \ \tilde{\mathbf{F}}_2 K_{\varepsilon;2}(t, t'), \ldots, \tilde{\mathbf{F}}_r K_{\varepsilon;r}(t, t')\rangle|| \tag{34}$$

subscripts referring to the bar number. Hence:

$$||\mathbf{W}(t, t')|| = \sqrt{\sum_{j=1}^{r} ||\tilde{\mathbf{F}}_j K_{\varepsilon;j}(t, t')||^2}, \tag{35}$$

and

$$||\tilde{\mathbf{F}}_j K_{\varepsilon;j}(t, t')|| = |K_{\varepsilon;j}(t, t')| \, ||\tilde{\mathbf{F}}_j|| \leq \varkappa \, ||\tilde{\mathbf{F}}_j||, \tag{36}$$

thus

$$||\mathbf{W}(t, t')|| \leq \varkappa \sqrt{\sum_{j=1}^{r} ||\tilde{\mathbf{F}}_j||^2} \, . \tag{37}$$

And since

$$||\tilde{\mathbf{F}}|| = ||\langle \tilde{\mathbf{F}}_1, \tilde{\mathbf{F}}_2, \ldots, \tilde{\mathbf{F}}_r\rangle|| \tag{38}$$

(33) is proven. Finally, suppose also the norm of the prescribed vector $\mathbf{b}(t)$ to be bounded:

$$||\mathbf{b}(t)|| \leq \beta. \tag{39}$$

Now, since (30) contains $\mathbf{A}^{-1}$ n times while $\mathbf{W}$ only $(n - 1)$ times, we obtain

$$||\mathbf{x}_n(t) - \mathbf{x}_{n-1}(t)|| \leq \int_0^t \int_0^{t'} \int_0^{t''} \ldots \int_0^{t^{(n-2)}} \alpha^n \varkappa^{n-1} ||\tilde{\mathbf{F}}||^{n-1} \beta \, dt' dt'' \ldots dt^{(n-1)}. \tag{40}$$

The integrand in the right-hand side of (40) is constant, besides

$$\int_0^t \int_0^{t'} \int_0^{t''} \ldots \int_0^{t^{(n-2)}} dt' \, dt'' \ldots dt^{(n-1)} = \frac{t^{n-1}}{(n-1)!}, \tag{41}$$

thus

$$||\mathbf{x}_n(t) - \mathbf{x}_{n-1}(t)|| \leq \alpha^n \beta \frac{(\varkappa ||\tilde{\mathbf{F}}|| \, t)^{n-1}}{(n-1)!} \, . \tag{42}$$

Let

$$\alpha\varkappa ||\tilde{\mathbf{F}}|| = \lambda. \tag{43}$$

then also

$$||\mathbf{x}_n(t) - \mathbf{x}_{n-1}(t)|| \leq \alpha\beta \frac{(\lambda t)^{n-1}}{(n-1)!} \, . \tag{44}$$

Furthermore using (42), (29) delivers

$$||\mathbf{x}_n(t)|| \leq \alpha\beta\left[1 + \lambda\frac{t}{1!} + \lambda^2\frac{t^2}{2!} + \ldots + \lambda^{n-1}\frac{t^{n-1}}{(n-1)!}\right] \leq \alpha\beta\, e^{\lambda t}. \quad (45)$$

Recalling the intermediary stipulations, conditions of the absolute and uniform convergence can be stated to be obviously sufficient.

If

a) *the fictitious instantaneous values of all the elastic displacements and stresses can be determined at any instant throughout the analysis hence*

a^1) *the skeleton is other than hypostatic,*

a^2) *it contains no completely compliant part thus the creep compliance function of each bar is bounded,*

a^3) *provided the structure itself is hyperstatic it contains at least one primary system which is not completely stiff,*

furthermore *if*

b) *all the creep compliance kernels of each bar material are absolutely bounded,*

c) *norms of both the load vector and the initial strain vector are bounded up to the end of the analysis,*

then algorithm (27) describes a convergent procedure.

3.2 *Concrete in ageing*

Dischinger's theory describing the creep of ageing concrete fulfills those among the conditions above that concern the creep compliance function. *Dischinger*'s theory is founded basically on three assumptions:

a) In case of permanent stresses, that is, in the proper case of creep, the material obeys an exponential law.

b) Storage of the material increases the initial stiffness (also describable by an exponential function of time).

c) Besides, ageing reduces the visco-elastic after-effect to be taken into consideration by depressing the compliance curves along the time axis.

Hence the rheological function is of the form:

$$Y_\varepsilon(t, t') = \frac{1}{E_0(0)}\left\{\frac{E_0(0)}{E_0(t')} + \psi(t) - \psi(t')\right\} \quad (46)$$

the creep function being:

$$\psi(t) = A(1 - e^{-t}). \quad (47)$$

The initial modulus of elasticity is expressed as:

$$E_0(t') = [E_0(\infty) - E_0(0)]\,(1 - e^{-Bt'}) + E_0(0) \quad (48)$$

with

$$E_0(0) > 0 \quad E_0(\infty) - E_0(0) > 0 \quad A > 0 \quad B > 0. \tag{49}$$

Substituting (47) and (48) into (46) we obtain

$$Y_\varepsilon(t, t') = \frac{1}{E_0(0)} \left\{ \frac{1}{1 + \frac{E_0(\infty) - E_0(0)}{E_0(0)} (1 - e^{-Bt'})} + A(e^{-t'} - e^{-t}) \right\} \tag{50}$$

and considering (49), (50) appears to be a bounded nonzero quantity in the interval $(0, \infty)$.

The creep compliance kernel belonging to (50) is

$$K_\varepsilon(t, t') = \frac{1}{E_0(0)} \left\{ \frac{E_0(\infty) - E_0(0)}{E_0(0)} \frac{Be^{-Bt'}}{\left[1 + \frac{E_0(\infty) - E_0(0)}{E_0(0)} (1 - e^{-Bt'})\right]^2} + Ae^{-t'} \right\} \tag{51}$$

again bounded in $(0, \infty)$ and vanishing for t'. Thus, iteration suits in case of the concrete of r.c. skeletons.

3.3 *Analysis of a simple gridwork*

The constitutive law of *Dischinger*'s theory can be reduced to that of the three-parameter solid subject to permanent load. Now the matrix integral equations of the altering state can be turned via *Laplace*-transforms into the equations of the matrix displacement method and of the matrix force method [6], these consist, however, of variable vectors and matrices. In case the structure has many degrees of freedom, analytic retransformation is cumbersome, next to impossible. Therefore one of the numerical methods of retransformation has to be applied, essentially solving the transformed equation starting from suitably selected values of the independent variable. Results will undergo *Lagrangian* interpolation leading to approximate polynomials similar to the *Laurent*-series, easy to retransform.

The procedure was tested on the problem of an elastically supported rectangular gridwork with significant stiffness to twisting. The calculations were carried out on the computer CDC 3300 of the Hungarian Academy of Sciences [8].

The arrangement of the gridwork is shown in Fig. 2. The structure consists of two main girders of rectangular cross section and three cross-beams. The main girders have a cross section of 0.5×1.0 m and the cross-beams 0.4×0.6 m.

The gridwork supported at the four corner nodes models a bridge structure. In addition, 7 other fictitious mid-bar nodes were selected.

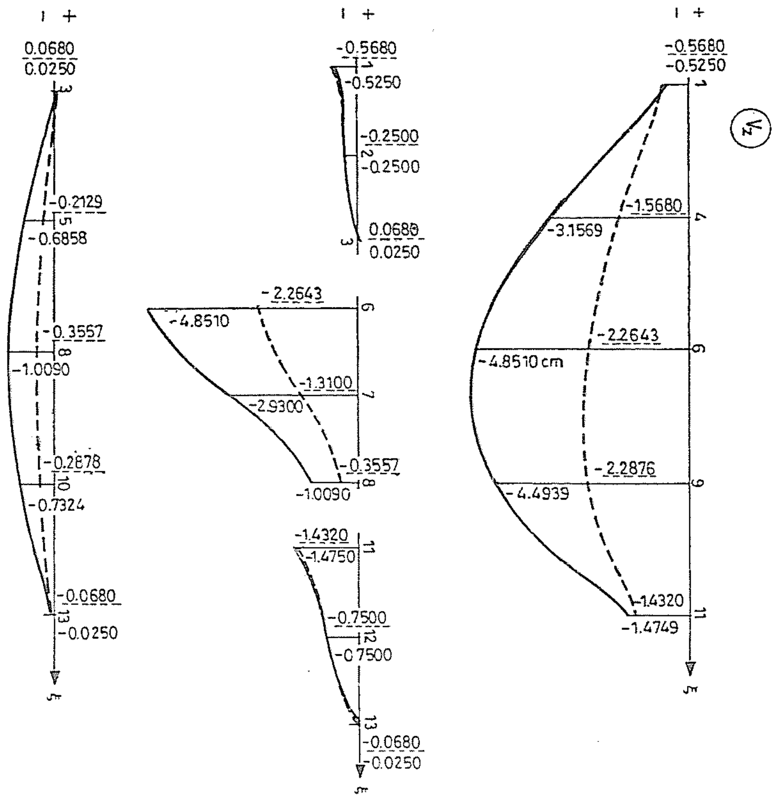


Fig. 2

The supports were supposed ideally elastic with a uniform spring constant of 50000 kN/m. The constitutive laws of the material of the longitudinal girders differed from those of the cross-beams, so the structure was considered as a heterogeneous system of three-parameter solids. The constitutive equations of the bars conform to the particular case of concrete creep under permanent stress involved in the Hungarian Code for Highway Bridges.

The initial modulus of elasticity of concrete was calculated from the cube strength as:

$$E_0 = 5500 \frac{K}{K + 20} \qquad [E_0] = \frac{kN}{\text{cm}^2}. \tag{52}$$

The compliance function

$$Y_\varepsilon(t) = \frac{1}{E_0} \{1 + \varphi_\infty (1 - e^{-\lambda t})\} \tag{53}$$

is applied, as well. φ_∞ is a constant depending on the concrete age and connected with the phenomenon of ageing, but irrelevant to *Poynting—Thomson* materials. The value of the delay factor amounts to 0.12 $\frac{1}{\text{month}}$. Longitudinal main girders and cross-beams are made of concrete grades B40 and B28 involving $\varphi_\infty = 2.0$ and $\varphi_\infty = 2.55$ corresponding to storage times of 28 days and 7 days, respectively. The modulus of elasticity G_0 was calculated using a *Poisson*'s ratio of 1/6.

Table 1

Node	$10^3 \cdot \varphi_x$	$10^3 \cdot \varphi_y$	v_z
1	2.09094	3.45217	−0.568019
2	1.33967	2.25757	−0.250000
3	2.09094	1.06297	0.068019
4	3.10991	3.02080	−1.568014
5	3.10991	0.75798	−0.212895
6	4.12887	1.42757	−2.264310
7	5.09290	0.78485	−1.309999
8	4.12887	0.14217	−0.355688
9	3.84031	−1.57515	−2.287612
10	3.84031	−0.53697	−0.287842
11	3.55174	−3.45021	−1.431981
12	3.33899	−2.16060	−0.750000
13	3.55174	−0.87100	−0.068019

Table 2

Node	$10^3 \cdot \varphi_x$	$10^3 \cdot \varphi_y$	v_z
1	2.73204	9.17970	−0.525008
2	0.69654	5.93939	−0.250000
3	2.73204	2.69908	0.025008
4	5.06509	7.78819	−3.156912
5	5.06509	1.88150	−0.685814
6	7.39815	2.92791	−4.851046
7	10.70878	1.52121	−2.929998
8	7.39815	0.11452	−1.008951
9	5.93660	−6.20409	−4.493926
10	5.93660	−1.79893	−0.732434
11	4.47505	−11.87208	−1.474992
12	3.19991	−7.31515	−0.750000
13	4.47505	−2.75822	−0.025008

The load is due to a single concentrated permanent force acting at the fictitious node No. 9. Let us examine the nodal displacements and the stress resultants.

Some of the results of the computation are presented below.

Elements of the nodal displacement vector at times $t = 0$ and $t \to \infty$ have been compiled in Tables 1 and 2, respectively, in radian and cm units. The displacements have been determined by making use of appropriate moduli, that is,

$E(0) = 3666.666$ kN/cm^2 and $E(\infty) = 1222.222$ kN/cm^2 for the main girders;
$E(0) = 3208.333$ kN/cm^2 and $E(\infty) = 903.756$ kN/cm^2 for the cross-beams.

The displacements at $t = 5$ years obtained by 10-point interpolation are shown in Table 3.

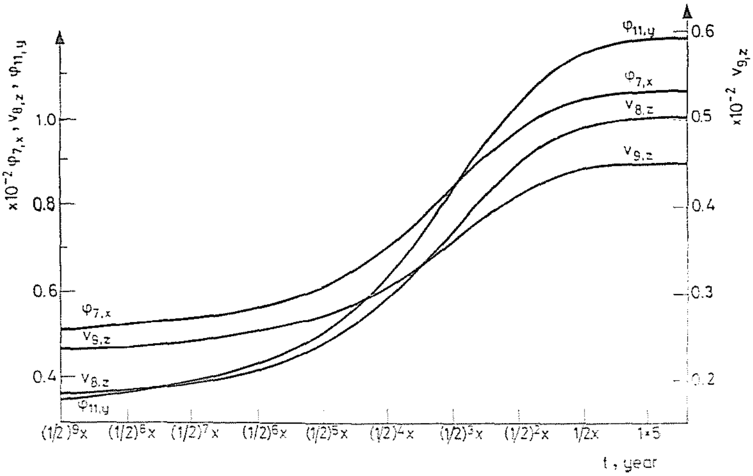
Table 3

Node	$10^3 \cdot \varphi_x$	$10^3 \cdot \varphi_y$	v_z
1	2.73807	9.16176	−0.52743
2	0.71225	5.93099	−0.24993
3	2.73802	2.70021	0.02755
4	5.06382	7.77279	−3.15418
5	5.06383	1.88358	−0.68371
6	7.38958	2.92051	−4.84475
7	10.69397	1.51969	−2.92626
8	7.38965	0.11883	−1.00775
9	5.92681	−6.19709	−4.48737
10	5.92684	−1.79235	−0.73289
11	4.46400	−11.85629	−1.47230
12	3.18549	−7.30351	−0.74995
13	4.46410	−2.75070	−0.27597

The results have been plotted in graphs: Fig. 3 contains the elements of the vertical displacements along the bar axis in both the initial and the final states, drawn in broken and in continuous line, respectively. Figure 4 shows the histories of the displacement components $\varphi_{7,x}$, $\varphi_{11,y}$ $v_{8,z}$ and $v_{9,z}$ to semilog scale. In Fig. 5, bending moments on the bar axes have been plotted in initial and final states.

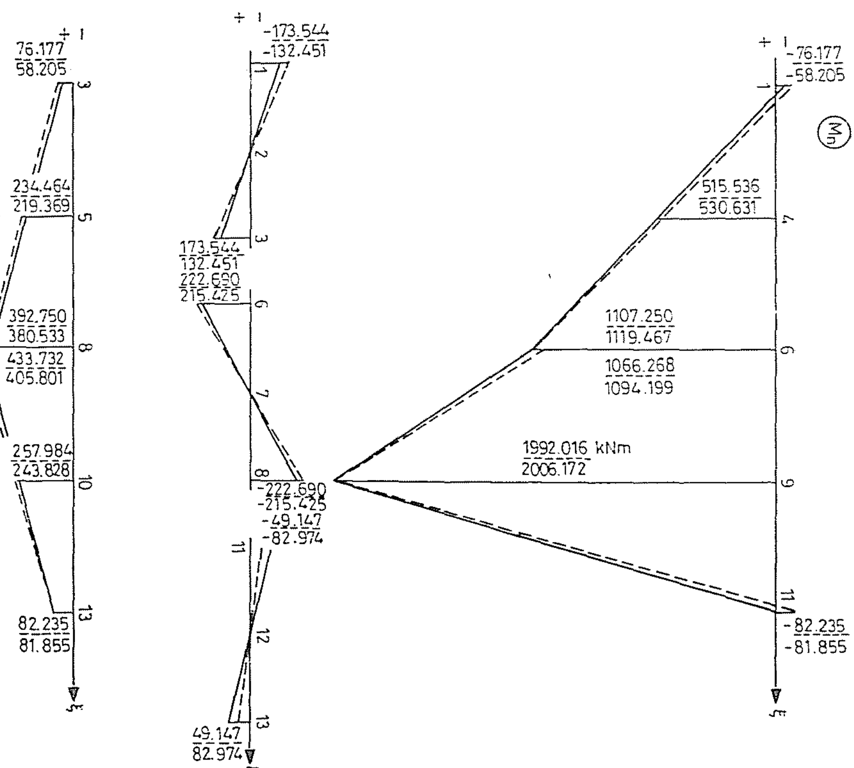
The following conclusions have been drawn:

a) Each displacement developed similarly as the creep of *Poynting—Thomson*-type materials. The values at the first and the last selected instant calculated by the approximate method are in fair agreement with those obtained



$\times 10^{-2}$ $\varphi_{7,x}$, $v_{8,z}$, $\varphi_{11,y}$
$\times 10^{-2}$ $v_{9,z}$
1.0
0.8
0.6
0.4
0.6
0.5
0.4
0.3
0.2
$\varphi_{11,y}$
$\varphi_{7,x}$
$v_{8,z}$
$v_{9,z}$
$(1/2)^9x$ $(1/2)^8x$ $(1/2)^7x$ $(1/2)^6x$ $(1/2)^5x$ $(1/2)^4x$ $(1/2)^3x$ $(1/2)^2x$ $1/2x$ $1 \cdot 5$
t, year

Fig. 3



76.177
58.205
234.464
219.369
392.750
380.533
433.732
405.801
257.984
243.828
82.235
81.855
-173.544
-132.451
173.544
132.451
222.690
215.425
-222.690
-215.425
-49.147
-82.974
49.147
82.974
-76.177
-58.205
515.536
530.631
1107.250
1119.467
1066.268
1094.199
1992.016 kNm
2006.172
-82.235
-81.855
M_n

Fig. 4

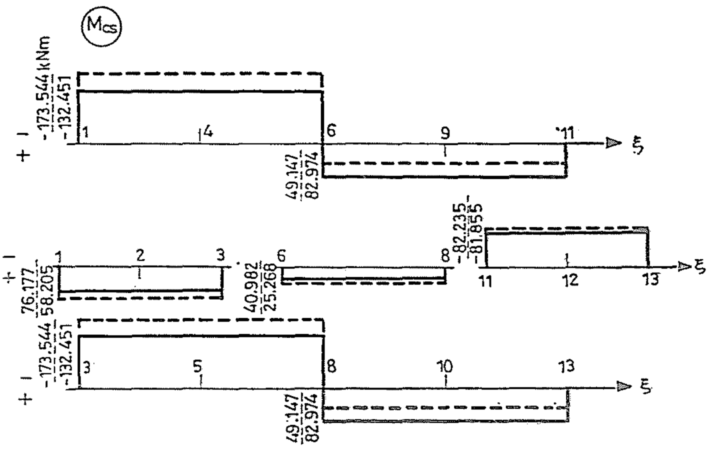


Fig. 5

in a quite other way for the initial and the permanent stages, respectively. Also they remain within the bounds assigned by these latter (compare Tables 2 and 3). So the approximate computation delivers fairly realistic results and can be considered as suitable. Also the interval of observation has been correct and it proves really typical.

b) There is no significant difference between the initial and the permanent stress values. The bending moment diagram of the main girders — of outstanding importance — develops in such a way that the loaded main girder carries a somewhat greater part of the load in the steady than in the initial state. Both circumstances point to the fact that the transverse load distribution of medium degree is little influenced by either the heterogeneity of the system or the creep process itself. Or else, interaction between the longitudinal girders decreases since the cross-beams are more prone to "yield" than are the longitudinal ones. Mind that the supports are rather stiff, besides the difference between the district creep parameters φ_∞ is negligible, the results obtained are considered reasonable.

Summary

Basic equations of structures of visco-elastic bars described by a *Boltzmann-Volterra*-type constitutive equation each can be generalized, provided the convergence conditions of Picard's iteration applied to the solution of the altering state equation are met.

The constitutive law of concrete creep involved in the Hungarian code usually means fulfillment of these conditions.

The theory has been applied on the analysis of a gridwork subject to a permanent load. The problem has been solved by means of a numerical variant of *Laplace*-retransform. The procedure works well under certain particular circumstances as well and it yields reasonable conclusions.

References

1. Fenyő, I.—Frey, T.: Mathematics for Electrical Engineers.* Vol. II. Műszaki Könyvkiadó, Budapest 1965.
2. Flügge, W.: Viscoelasticity. 2. rev. ed., Springer, Berlin—Heidelberg—New York, 1975
3. Krylow—Skoblya: A Handbook of Numerical Laplace-Retransforms. Publ. "Mir", Moscow 1974.
4. Nowacki, W.: Theorie des Kriechens—Lineare Visko-Elastizität, F. Deuticke, Wien 1965.
5. Palotás, L.: Statics of Reinforced Concrete Structures under Consideration of Residual Deformation.* Anyagvizsgálók Közlönye, **XIII**, 3 (1940).
6. Roller, B.: Altering State Analysis of Viscoelastic Structures.* Építés- Építészettudomány Vol. **XI**, No. 3—4. (1979).
7. Roller, B.: A Survey on Spatial Skeletons With Particular Respect to Up-to-date Computation Methods,* Műszaki Tudomány, **55**, 3—4, Akadémiai Kiadó, Budapest (1979).
8. Roller, B.: Altering State Analysis of Viscoelastic Skeletons.* D. Sci. Thesis, Budapest, 1980.
9. Szabó, J.—Roller, B.: Anwendungen der Matrizenrechnung auf Stabwerke. Akadémiai Kiadó, Budapest, 1978.
10. Szalai, J.: Inconsistencies in the Linear Theory of Creep of Concrete. Acta Technica Acad. Sci. Hung., Vol. **79**, 1—2 (1974).
11. Webber, J. O. H.: Stress Analysis in Viscoelastic Bodies Using Finite Elements and a Correspondence Rule with Elasticity. Journal of Strain Analysis, 4 (1969).
12. Zienkiewicz, O. C.—Watson, M.—King, I. P.: A Numerical Method of Viscoelastic Stress Analysis. Intern. Journal of Mech. Sci., **10** (1968).

Associate Prof. Dr. Béla Roller D. Techn. Sci. Budapest, H-1521

* In Hungarian