INHOMOGENEOUS BOUNDARY CONDITIONS FOR FINITE STRIPS

By

Gy. SZILÁGYI

Department of Civil Engineering Mechanics, Technical University, Budapest

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Presented by Prof. Dr. S. KALISZKY

1. Introduction

The finite strip method serves for the analysis of thin-walled, prismatic folded plates made of a linear elastic, orthotropic material subject to small displacements. The structure is divided into strips parallel to its generatrices by means of the so-called nodal lines (Fig. 1). For each strip, shape functions, products of two functions with one variable, are assumed, one comprising unknown generalized displacements of the nodal lines, the other consisting of interpolation polynomials. Nodal line displacements are determined from the equilibrium equation system of line forces, to be deduced according to the principles either of the virtual displacements or of the minimum potential energy. The shape function should be kinematically admissible. Thus, this procedure is a generalized displacement method.

In conformity with the shape function, the finite strip method may be considered as combined from an analytic method and the method of finite elements. The analytic method is decided by the form of the nodal line displacement function. Two fundamental cases may be distinguished [1], [2]:

a) Orthogonal function series, e.g. *Fourier*-series;
b) Non-orthogonal function series.

In the *Fourier*-series alternative of the finite strip method, orthogonality of shape functions causes the linear algebraic equation system to decompose to small equation systems, main advantage of the finite strip method.

Originally, the method has been developed by applying *Fourier*-series satisfying homogeneous boundary conditions for strip ends. In the following, the possibility of satisfying the most frequent strip-end inhomogeneous boundary conditions is investigated, keeping the *Fourier*-series shape functions, hence without renouncing of the main advantage of this method.

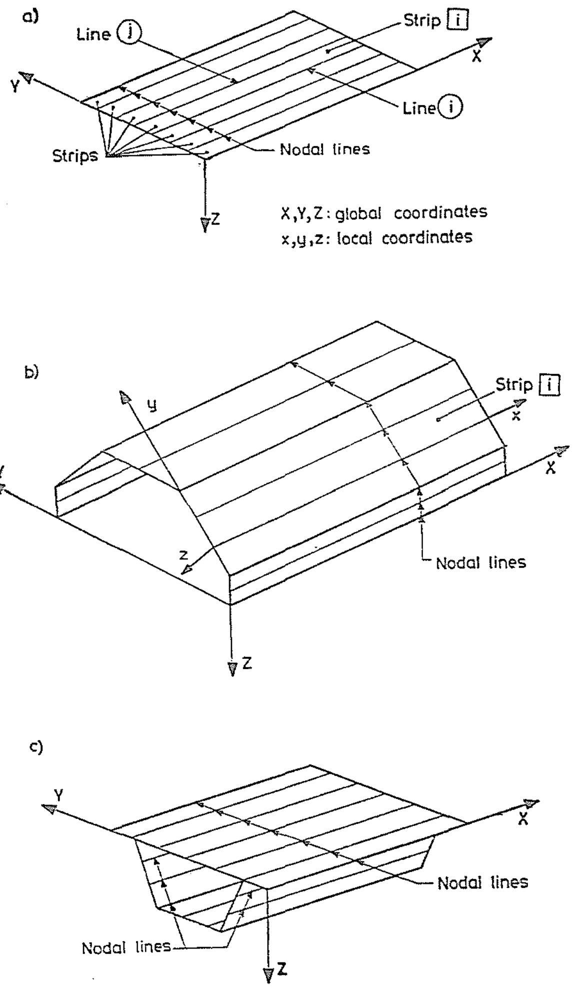

Fig. 1

2. Writing boundary conditions by Galerkin's method

2.1 *Boundary conditions of the plane stress problem*

Constitutive equation of linear elastic orthotropic materials in plane stress in concise, and in detailed matrix form is:

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \quad (1)$$

respectively, where

σ_x and σ_y — normal stresses along x and y, resp.;
τ_{xy} — shear stress component along y acting on a section with normal x;
ε_x and ε_y — normal strains along x and y, resp.;
γ_{xy} — shear strain for normals x and y;
D_{ik} — well-known coefficients of the generalized *Hooke*'s law

$(i = 1, 2, 3;\ k = 1, 2, 3)$.

Strain-displacement relationships of the plane stress:

$$\varepsilon_x = u_x \quad ; \quad \varepsilon_y = v_y \quad ; \quad \gamma_{xy} = u_y + v_x. \tag{2}$$

Displacements along x, y and z will be denoted by u, v and w, and their derivatives with respect to x and y will be indicated by subscripts, e.g.: $\frac{\partial u}{\partial x} = u_x$, $\frac{\partial^2 u}{\partial x \, \partial y} = u_{xy}$, subscripts x and y of other quantities (e.g. σ, ε, q) refer to the direction rather than to the derivative (Fig. 2).

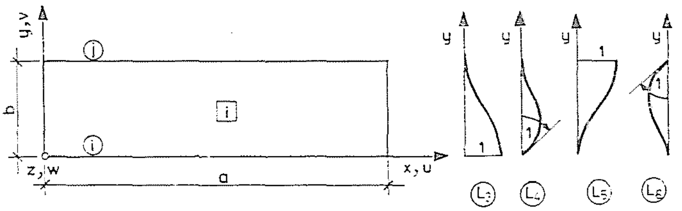

Fig. 2

Lamé's equations for orthotropic materials in plane stress:

$$\left.\begin{aligned} D_{11} u_{xx} + (D_{12} + D_{33}) v_{xy} + D_{33} u_{yy} + q_x = 0 \\ D_{22} v_{yy} + (D_{12} + D_{33}) u_{xy} + D_{33} v_{xx} + q_y = 0 \end{aligned}\right\} \tag{3}$$

where q_x and q_y are specific body forces along x and y, resp.

Galerkin's method is a special case of the method of weighted residuals where arbitrary increments of approximate solution functions are chosen as weighting functions [3]. Accordingly, orthogonality conditions of differential equation system (3) for the approximate solution of u and v can be written for a rectangular plate of thickness h, with sides $0 \leq x \leq a$, $0 \leq y \leq b$, subject to in-plane forces (see Fig. 2):

$$\begin{aligned} & h \int_0^b \int_0^a \{D_{11} u_{xx} + D_{12} v_{xy} + D_{33} v_{xy} + D_{33} u_{yy} + q_x\} \, \delta u \, dx \, dy + \\ & + h \int_0^b \int_0^a \{D_{22} v_{yy} + D_{12} u_{xy} + D_{33} u_{xy} + D_{33} v_{xx} + q_x\} \, \delta v \, dx \, dy = 0 \end{aligned} \tag{4}$$

where δu and δv are arbitrary small increments of functions u and v, resp., called virtual displacement systems in engineering mechanics.

Integrating by parts with respect to x the first two terms in the first figure brackets, and the second two terms in the second figure brackets of Eq. (4), and integrating by parts with respect to y the other terms comprising derivatives of u and v leads — after arrangement — to:

$$+\left[h\int_0^b \{(D_{11}u_x + D_{12}v_y)\,\delta u + D_{33}(u_y + v_x)\,\delta v\}\,dy\right]_{x=0}^{x=a} + \\ +\left[h\int_0^a \{(D_{12}u_x + D_{22}v_y)\,\delta v + D_{33}(u_y + v_x)\,\delta u\}\,dx\right]_{y=0}^{y=b} = 0 \tag{5}$$

where terms in square brackets contain boundary conditions $x = \text{const.}$ and $y = \text{const.}$; since Eq. (5) has to be satisfied for any possible δu and δv, the terms containing δu and δv must separately vanish at boundaries. First two terms of (5) containing a double integral are plainly the internal and external virtual work. *Galerkin*'s method is known to lead to equilibrium conditions written as the principle of virtual displacements, and to the boundary conditions to be satisfied in problems of structural mechanics.

Chapter 3 being concerned with analysis of boundaries where $x = \text{const.}$, let us write the relevant boundary conditions:

For $x = 0$ or $x = a$,

$$h\int_0^b (D_{11}u_x + D_{12}v_y)\,\delta u\,dy = 0 \qquad \text{i.e.} \quad h\int_0^b \sigma_x\,\delta u\,dy = 0 \tag{6}$$

and

$$h\int_0^b D_{33}(u_y + v_x)\,\delta v\,dy = 0 \qquad \text{i.e.} \quad h\int_0^b \tau_{xy}\,\delta v\,dy = 0. \tag{7}$$

Let us consider how integrals (6) and (7) can be zero for all possible δu and δv, and what is understood by possible virtual displacement systems δu and δv. Two basic cases may be realized:

a) *There is a prescribed displacement at the considered edge*: for instance, $v = \bar{v}(y)$ for boundary condition (7). Now, $\tau_{xy} \neq 0$ ("reactions"). Integral (7) can only be zero if at the edge, locus of prescribed displacement, $\delta v = 0$. This is obvious for homogeneous boundary condition $v = 0$. But it is clear from (7) that δv cannot be other than zero for the inhomogeneous boundary condition $v = \bar{v}$: of course, function $v(x, y)$ has to be found in a form such as to *a priori* satisfy kinematic condition $v = \bar{v}(y)$ at edges $x = \text{const.}$ This is why in the displacement method kinematic boundary conditions are termed *essential boundary conditions.*

Examination of (6) shows the form of static boundary condition $\sigma_x = 0$ in terms of displacements to be influenced by kinematic boundary condition $v = \bar{v}(y)$. Namely, prescribing edge displacement v involves also its derivative $v_y = \varepsilon_y$ along the edge. Then condition $\sigma_x = 0$ permits to express the prescribed $u_x = \varepsilon_x$ value:

$$u_x(y) = -\frac{D_{12}}{D_{11}} \bar{v}_y(y) \tag{8}$$

form of condition $\sigma_x = 0$ simultaneous to prescribed displacements $\bar{v}(y)$. For $\bar{v} = 0$, (8) simplifies to $u_x = 0$. From (6), displacement function $u(x, y)$ appears to satisfy condition (8), namely in case of the static boundary condition for σ_x, at the edge $u \neq 0$ and $\delta u \neq 0$.

b) *There is a prescribed* (generalized) *force at the considered edge*: e.g. for (6), $\sigma_x = \bar{\sigma}_x(y)$. Then $u \neq 0$, and in general, $\delta u \neq 0$. Under homogeneous condition $\sigma_x = 0$, condition (6) is satisfied. Under inhomogeneous condition $\sigma_x = \bar{\sigma}_x$, term $\int \bar{\sigma}_x \, \delta u \, \delta y$, virtual work of prescribed edge forces, is to be put into the principle of virtual displacements and the resulting function $u(x, y)$ satisfies static boundary condition without having satisfied it in its original form, provided it has been assumed in a proper form. This is why in the displacement method, static boundary conditions are called *non-essential or natural boundary conditions.*

2.2 *Boundary conditions of plates in bending*

Constitutive equation of orthotropic plates of thickness h, made of a linear elastic material, in concise and in detailed matrix form is:

$$\mathbf{m} = \mathbf{H}\boldsymbol{\rho} : \begin{bmatrix} m_x \\ m_y \\ m_{xy} \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & 0 \\ H_{12} & H_{22} & 0 \\ 0 & 0 & H_{33} \end{bmatrix} \begin{bmatrix} \varrho_x \\ \varrho_y \\ \varrho_{xy} \end{bmatrix} \tag{9}$$

where

m_x and m_y — specific bending moments acting on sections with normals x and y, resp.;
m_{xy} — specific torque;
ϱ_x and ϱ_y — curvatures along x and y, resp.;
ϱ_{xy} — specific distortion;

$$\mathbf{H} = \frac{h^3}{12} \mathbf{D}.$$

Strain-displacement relationships of the plate:

$$\varrho_x = -w_{xx} \; ; \; \varrho_y = -w_{yy} \; ; \; \varrho_{xy} = -2w_{xy}. \tag{10}$$

Differential equation of orthotropic plates:

$$H_{11} w_{xxxx} + 2(H_{12} + 2H_{33}) w_{xxyy} + H_{22} w_{yyyy} - p = 0 \tag{11}$$

where $p = p(x, y)$ is distributed load in direction z.

Writing orthogonality condition according to *Galerkin*'s method and rearranging at the same time differential equation (11):

$$\int_0^b \int_0^a \{H_{11} w_{xxxx} + H_{12} w_{xxyy} + 2H_{33} w_{xxyy} + 2H_{33} w_{xxyy} + H_{12} w_{xxyy} + + H_{22} w_{yyyy} - p\} \; \delta w \, dx \, dy = 0. \tag{12}$$

Let us integrate by parts twice with respect to x, the first and second terms in figure brackets, the third term with respect to x, then to y, the fourth term with respect to y, then to x, the fifth and the sixth terms twice with respect to y. Thereafter integrating by parts terms obtained from the third and the fourth terms and containing simple integral of w_{xy}, multiplying the equation by —1, then rearranging yields:

$$\begin{aligned} &- \int_0^b \int_0^a \{(H_{11} w_{xx} + H_{12} w_{yy}) \, \delta w_{xx} + (H_{12} w_{xx} + H_{22} w_{yy}) \, \delta w_{yy} + 4H_{33} w_{xy} \, \delta w_{xy}\} \, dx \, dy + \\ &+ \int_0^b \int_0^a p \, \delta w \, dx \, dy + 4D_{33} \Big[w_{xy} \delta w \Big]_{x=0, \; y=0}^{x=a, \; y=b} + \\ &+ \Big[\int_0^b \{(-H_{11} w_{xxx} - H_{12} w_{xyy} - 4H_{33} w_{xyy}) \, \delta w + (H_{11} w_{xx} + H_{12} w_{yy}) \, \delta w_x\} \, dy \Big]_{x=0}^{x=a} + \\ &+ \Big[\int_0^a \{(-H_{22} w_{xxx} - H_{12} w_{xxy} - 4H_{33} w_{xxy}) \, \delta w + (H_{22} w_{yy} + H_{12} w_{xx}) \, \delta w_y\} \, dx \Big]_{y=0}^{y=b} = 0. \end{aligned} \tag{13}$$

Again, terms containing double integrals are internal and external virtual work, while terms in square brackets are boundary conditions. The first among them is the virtual work of concentrated forces resulting from the torque acting at plate corners.

Chapter 4 will concern conditions for edges $x =$ const., such as:

For $x = 0$ and $x = a$,

$$\int_0^b \{-H_{11} w_{xxx} - (H_{12} + 4H_{33}) w_{xyy}\} \, \delta w \, dy = 0 \tag{14}$$

that is

$$\int_0^b b_x \, \delta w \, dy = 0$$

and

$$\int_0^b (H_{11}w_{xx} + H_{12}w_{yy}) \quad \delta w_x \, dy = 0 \tag{15}$$

that is

$$\int_0^b m_x \, \delta w_x \, dy = 0$$

b_x being the so-called *Kirchhoff*'s shear force.

Boundary conditions have two basic cases:

a) *There is a prescribed displacement at the considered edge*: for instance, $w = \overline{w}(y)$ for (14). Now $b_x \neq 0$, hence δw cannot be other than zero at the edge, and function $w(x, y)$ has *a priori* to satisfy kinematic boundary condition $w = \overline{w}(y)$, an essential boundary condition in the displacement method.

(15) shows the form of static boundary condition $m_x = 0$ expressed in terms of displacements to be influenced by kinematic condition $w = \overline{w}(y)$, it being decisive for the edge curvature, namely $w_{yy} = \overline{w}_{yy}(y)$. Now, condition $m_x = 0$ permits to determine the counterpart of the curvature normal to the edge:

$$w_{xx}(y) = -\frac{H_{12}}{H_{11}} \overline{w}_{yy}(y) \tag{16}$$

condition $m_x = 0$ in form simultaneous to the prescribed displacement $\overline{w}(y)$. For $\overline{w} = 0$, (16) simplifies to $w_{xx} = 0$. Obviously from (15), displacement function $w(x, y)$ has *a priori* to satisfy condition (16), namely, under static boundary condition for m_x, $w_x \neq 0$ and $\delta w_x \neq 0$.

b) *There is a prescribed force at the considered edge*: in (14) e.g. $m_x = \overline{m}_x(y)$. Now, at the edge $w_x \neq 0$ and $\delta w_x \neq 0$. Under inhomogeneous condition $m_x = \overline{m}_x$, term $\int m_x \, \delta w_x \, dy$ as virtual work of prescribed edge forces has to be put into the principle of virtual displacements, and the resulting $w(x, y)$ satisfies static boundary condition $m_x = \overline{m}_x$, a non-essential or natural boundary condition in the displacement method.

3. Satisfying various boundary conditions of plane stress strips

3.1 *Homogeneous boundary conditions*

Cheung [1] was the first to apply the finite strip method for the analysis of prismatic folded plates where edges normal to the generatrices are connected to so-called rigid diaphragms, walls infinitely stiff in their plane, and perfectly flexible normally to it. Corresponding boundary conditions for plane stress strips are homogeneous (see Fig. 2):

$$\left.\begin{array}{l} \text{For} \quad x = 0 \quad \text{or} \quad x = a, \\ v \;\; = 0 \\ \sigma_x = 0 \;\; \text{i.e.} \;\; u_x = 0 \end{array}\right\} . \tag{17}$$

and

CHEUNG applied displacement function (18) satisfying homogeneous kinematic and static boundary conditions (17):

$$\begin{bmatrix} u \\ v \end{bmatrix} = \sum_{m=1}^{M} \begin{bmatrix} \cos k_m x & 0 \\ 0 & \sin k_m x \end{bmatrix} \begin{bmatrix} L_1 & 0 & L_2 & 0 \\ 0 & L_1 & 0 & L_2 \end{bmatrix} \begin{bmatrix} u_{im} \\ v_{im} \\ u_{jm} \\ v_{jm} \end{bmatrix} \tag{18}$$

or, in concise form:

$$\mathbf{u} = \sum_{(m)} \mathbf{G}_m \mathbf{N} \mathbf{e}_m$$

where

M — number of *Fourier* terms used for analysis;

$k_m = \frac{m\pi}{a}$;

L_1 and L_2 — linear interpolation polynomials [2], [4];

u_{im} and u_{jm} — m-th cosine *Fourier* coefficients of displacement functions u of nodal lines i and j, resp.;

v_{im} and v_{jm} — m-th sine *Fourier* coefficients of displacement functions v of nodal lines i and j, resp.

Expanding loads q_x and q_y into cosine and sine *Fourier* series, respectively, and substituting both these and displacement function (18) into (5) for virtual work yields the equilibrium equation system of the plane stress strip, decomposing into *Fourier* terms. The m-th equilibrium equation system is of the form:

$$\mathbf{K}'_m \mathbf{e}_m + \mathbf{t}'_m = 0 \tag{19}$$

where

$\mathbf{K}'_m$ — m-th stiffness matrix of the plane stress strip;

$\mathbf{t}_m$ — m-th load vector of the plane stress strip,

formulae see in [2].

3.2 *Inhomogeneous static boundary conditions*

In the analysis of continuous structures [4] it is essential to calculate the influence of distributed forces Q_0 and Q_a on the strip ends (see Fig. 3). For sufficiently narrow strips, these forces may be of strip-wise constant intensity. For Q_0 and Q_a acting on the edge supported by a rigid diaphragm described in the previous chapter, boundary conditions become:

$$\left.\begin{aligned} &\text{For } x = 0 \quad \text{or} \quad x = a, \\ &\qquad v = 0 \\ \text{and} \quad &\sigma_x = \frac{Q_0}{h} \quad \text{or} \quad \sigma_x = \frac{Q_a}{h} \end{aligned}\right\} . \tag{20}$$

Substitution of the virtual work of end forces (6) into the principle of virtual displacements yields the load vector in the m-th equilibrium equation system type (19) due to end forces, in the form:

$$\mathfrak{t}'_{mQ} = \frac{b}{2}\{Q_0 - Q_a(-1)^m\}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} . \tag{21}$$

Remind that diaphragm supports are unable to reaction along x thus load projection sum along x has to be zero. If also displacements along x have to be determined, then the zeroth *Fourier* term of shape function has to be taken into consideration, and the relevant equation system type (19) to be solved. This problem has comprehensively been dealt with in [4].

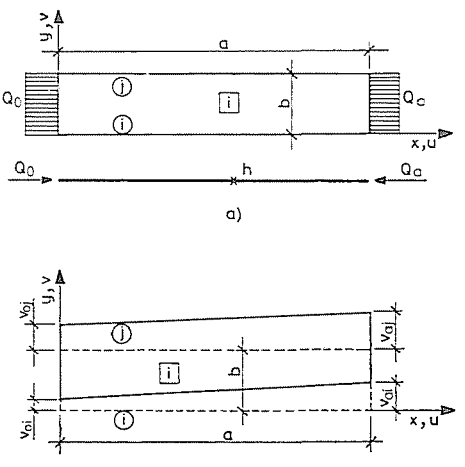

Fig. 3

3.3 *Inhomogeneous kinematic boundary conditions*

For the edge supported by a rigid diaphragm under 3.1, displacements along y may be prescribed [5]. Now, boundary conditions become:

$$\left.\begin{matrix} \text{For } x = 0 \quad \text{or} \quad x = a, \\ v = \bar{v}_0(y) \quad \text{or} \quad v = \bar{v}_a(y) \\ \sigma_x = 0 \end{matrix}\right\} . \tag{22}$$

and

This latter can be written for edges $x = 0$ or $x = a$ according to (8) as:

$$u_{0x}(y) = -\frac{D_{12}}{D_{11}} \bar{v}_{0y}(y) \quad \text{or} \quad u_{ax}(y) = -\frac{D_{12}}{D_{11}} \bar{v}_{ay}(y). \tag{23}$$

Support displacement functions $\bar{v}_0(y)$ and $\bar{v}_a(y)$ are given as seen in Fig. 3b, in terms of discrete values at the nodal lines, assuming the displacement to linearly vary between nodal lines:

$$\bar{v}_0(y) = L_1 v_{0i} + L_2 v_{0j} = v_{0i} + \frac{v_{0j} - v_{0i}}{b} y$$

$$\bar{v}_a(y) = L_1 v_{ai} + L_2 v_{aj} = v_{ai} + \frac{v_{aj} - v_{ai}}{b} y$$

permitting conditions (23) to be written as:

$$u_x = -\frac{D_{12}}{D_{11}} \frac{v_{0j} - v_{0i}}{b} \qquad u_{ax} = -\frac{D_{12}}{D_{11}} \frac{v_{aj} - v_{ai}}{b} . \tag{24}$$

In the case of displacements prescribed according to comments on boundary conditions (6) and (7) under 2.1, the shape function has to satisfy both conditions (22) and (24). Hence to function (18) will be added a function $\mathbf{u}^{(1)}$ so as their sum satisfies (22), and a function $\mathbf{u}^{(2)}$ so as their sum satisfies conditions (22) and (24). $\mathbf{u}^{(1)}$ is advisably linear function of x, and $\mathbf{u}^{(2)}$ is independent of y:

$$\mathbf{u}^{(1)} = \mathbf{N}\left\{\mathbf{e}_0 + \frac{x}{a}(\mathbf{e}_a - \mathbf{e}_0)\right\} \tag{25}$$

where

$$\mathbf{e}_0 = \begin{bmatrix} 0 \\ v_{0i} \\ 0 \\ v_{0j} \end{bmatrix} \qquad \mathbf{e}_a = \begin{bmatrix} 0 \\ v_{ai} \\ 0 \\ v_{aj} \end{bmatrix}$$

$$\mathbf{u}^{(2)} = \begin{bmatrix} \dfrac{\varepsilon_{ax} - \varepsilon_{0x}}{2a} + \varepsilon_{0x} x - \dfrac{a}{8}(3\varepsilon_{0x} + \varepsilon_{ax}) \\ 0 \end{bmatrix} . \tag{26}$$

The total shape function is sum of three functions:

$$\mathbf{u}^{(T)}(x, y) = \mathbf{u}(x, y) + \mathbf{u}^{(1)}(x, y) + \mathbf{u}^{(2)}(x). \tag{27}$$

Applying this function $\mathbf{u}^{(T)}$ for writing virtual work principle (5), to the load vector of the m-th equation system type (19) a term due to support displacements is superposed:

$$\mathbf{t}'_{mv} = -\frac{ah}{m\pi}\{\varepsilon_{0y} - (-1)^m \varepsilon_{ay}\}\begin{bmatrix} 0 \\ D_{22} - \dfrac{D_{12}^2}{D_{11}} \\ 0 \\ \dfrac{D_{12}^2}{D_{11}} - D_{22} \end{bmatrix}. \tag{28}$$

This load vector can be demonstrated to be zero in case of rigid-body displacements and puıe shear strains of the strip.

4. Satisfying different boundary conditions of plate strips in bending

4.1 *Homogeneous boundary conditions*

The diaphragm support under 3.1 corresponds to simply supported edges of plates under homogeneous boundary conditions (see Fig. 2):

$$\left.\begin{array}{c} \text{For} \quad x = 0, \quad \text{or} \quad x = a, \\ w = 0 \\ \text{and} \qquad m_x = 0 \ \text{i.e.}\ w_{xx} = 0 \end{array}\right\}. \tag{29}$$

Cheung applied shape function (30) satisfying these conditions:

$$w = [L_3\, L_4\, L_5\, L_6] \sum_{(m)} \begin{bmatrix} w_{im} \\ \theta_{im} \\ w_{jm} \\ \theta_{jm} \end{bmatrix} \sin k_m x \tag{30}$$

or, in concise form:

$$w = \mathbf{c}^* \sum_{(m)} \mathbf{w}_m \sin k_m x$$

where

L_3, L_4, L_5, L_6 — cubic *Hermitian* interpolation polynomials (see Fig. 2);
w_{im} and w_{jm} — m-th sine Fourier coefficient of displacement function w of i-th and j-th nodal lines, resp.;

θ_{im} and θ_{jm} — m-th sine *Fourier* coefficients of rotation function of nodal lines i and j, resp., parallel to plane yz.

Expanding also load p into a sine *Fourier* series and substituting both it and displacement function (30) into (13) for virtual work yields the m-th equilibrium equation system of the plate strip in bending:

$$\mathbf{K}''_m \mathbf{w}_m + \mathbf{t}''_m = 0 \tag{31}$$

where

$\mathbf{K}''_m$ — m-th stiffness matrix of the plate strip in bending;
$\mathbf{t}''_m$ — m-th load vector of the strip,

formulae see in [2].

4.2 *Inhomogeneous static boundary conditions*

Analysis of continuous structures [4] has to reckon with strip-end distributed couple systems R_0 and R_a (Fig. 4). For sufficiently narrow strips, R_0 and R_a may be uniformly distributed. Simply supported edge with R_0 and R_a involves the boundary conditions:

$$\left.\begin{array}{c} \text{For} \quad x = 0 \quad \text{or} \quad x = a, \\ w = 0 \\ \text{and} \quad m_x = R_0 \quad \text{or} \quad m_x = R_a \end{array}\right\} . \tag{32}$$

Substituting virtual work (15) of these couple systems into the principle of virtual displacements yields the load vector in (31) due to strip end couples:

$$\mathbf{t}''_{mR} = \{R_0 - (-1)^m R_a\} \int_0^b \mathbf{c}\, dy \, . \tag{33}$$

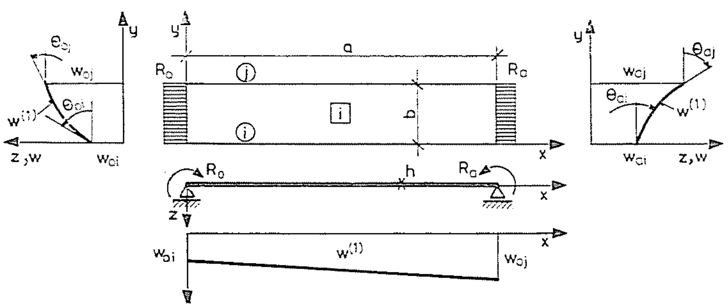

Fig. 4

4.3 *Inhomogeneous kinematic boundary conditions*

For simply supported edges, displacements normal to the middle surface may be prescribed [5]. Now, boundary conditions become:

$$\left.\begin{array}{c} \text{For} \quad x = 0 \quad \text{or} \quad x = a, \\ w = \bar{w}_0(y) \ \text{or} \ w = \bar{w}_a(y) \\ m_x = 0 \end{array}\right\}. \tag{34}$$

and

This latter can be written for edges $x = 0$ and $x = a$, resp., according to (16) as:

$$w_{oxx} = -\frac{H_{12}}{H_{11}} \bar{w}_{oyy}(y) \quad \text{or} \quad w_{axx} = -\frac{H_{12}}{H_{11}} \bar{w}_{ayy}(y). \tag{35}$$

Support displacement functions $\bar{w}_0$ and $\bar{w}_a$ are given in terms of discrete values at nodal lines ($w_{0i}, w_{0j}, w_{ai}, w_{aj}$) and of derivatives with respect to y ($\theta_{0i}, \theta_{0j}, \theta_{ai}, \theta_{aj}$) as seen in Fig. 4. For sufficiently narrow strips, edge curvatures may be assumed to be strip-wise constant:

$$\bar{w}_{oyy} = \frac{\theta_{0j} - \theta_{0i}}{b}; \quad \bar{w}_{ayy} = \frac{\theta_{aj} - \theta_{ai}}{b}.$$

Accordingly, conditions (35) become:

$$\begin{aligned} \varrho_{ox} &= -w_{axx} = \frac{H_{12}}{H_{11}} \frac{\theta_{oj} - \theta_{oi}}{b} \\ \varrho_{ax} &= -w_{axx} = \frac{H_{12}}{H_{11}} \frac{\theta_{aj} - \theta_{ai}}{b}. \end{aligned} \tag{36}$$

For displacements prescribed in conformity with comments on boundary conditions (14) and (15) under 2.2, the shape function has to satisfy both conditions (34) and (36). Hence to function w in (30) is added a function $w^{(1)}$ so as to have their sum satisfy (34), and a function $w^{(2)}$ to have their sum satisfy both (34) and (36). $w^{(1)}$ is advisably linear function of x and $w^{(2)}$ independent of y:

$$w^{(1)} = \mathbf{c}^* \left\{ \mathbf{w}_0 + \frac{x}{a} (\mathbf{w}_a - \mathbf{w}_0) \right\} = \mathbf{c}^* \sum_{(m)} \mathbf{w}_m^{(1)} \sin k_m x \tag{37}$$

where

$$\mathbf{w}_0 = \begin{bmatrix} w_{oi} \\ \theta_{oi} \\ w_{oj} \\ \theta_{oj} \end{bmatrix} \quad \mathbf{w}_a = \begin{bmatrix} w_{ai} \\ \theta_{ai} \\ w_{aj} \\ \theta_{aj} \end{bmatrix}$$

$\mathbf{w}_m^{(1)}$ being column vector composed of sine *Fourier* coefficients of functions in figure brackets.

$$w^{(2)} = \frac{\varrho_{ax} - \varrho_{ox}}{6a} x^3 + \frac{\varrho_{ox}}{2} x^2 - a \left(\frac{\varrho_{ox}}{3} + \frac{\varrho_{ax}}{6} \right) x . \tag{38}$$

Thus, the total shape function will be sum of three functions:

$$w^{(T)}(x, y) = w(x, y) + w^{(1)}(x, y) + w^{(2)}(x). \tag{39}$$

Writing the principle of virtual work (13) in terms of this function $w^{(T)}$ yields for the load vector in Eqs (31) due to the prescribed displacement:

$$\begin{aligned} \mathbf{t}''_{mw} = & \left\{ -H_{12} k_m^2 \int_0^b \mathbf{c} \frac{d^2 \mathbf{c}^*}{dy^2} dy + H_{22} \int_0^b \frac{d^2 \mathbf{c}}{dy^2} \frac{d^2 \mathbf{c}^*}{dy^2} dy \right\} \mathbf{w}_m^{(1)} + \\ & + \left\{ -H_{11} k_m^2 \int_0^b \mathbf{c}\, dy + H_{12} \int_0^b \frac{d^2 \mathbf{c}}{dy^2} dy \right\} \varrho_{xm} \end{aligned} \tag{40}$$

where ϱ_{xm} is the m-th sine *Fourier* coefficient of function $w_{xx}^{(2)}$.

This load vector can be shown to be zero for rigid body displacements and pure distortion of the plate strip.

5. Numerical results

The presented methods permit efficient computer treatment of continuous folded plates and box girders exposed to arbitrary loads and support displacements. Numerical examples for loads are found in [4]. The Author did not find any published numerical problem for stresses in continuous structures due to support displacements, therefore here a problem will be presented, the results of which can partly be checked by manual approximate analysis.

Two-span continuous plate in Fig. 5 has free edges $Y =$ const., and simply supported edges $X =$ const. The plate of a thickness $h = 0.48$ m is made of an isotropic material with a *Young*'s modulus of 30 000 MPa, and a *Poisson*'s ratio of 1/6. The intermediate support of the plate at $X = 10$ m is displaced vertically by 0.1 m at its end point of coordinate $Y = 8$ m, and by zero at its end $Y = 0$, linearly varying in between. Diagram m_Y of section $X = 5$ m, and distribution of moments m_X along the intermediate support have been plotted in Fig. 5. The problem being symmetrical about the straight axis of the support, it was sufficient to plot half of the deflection and moment diagrams w and m_X of free edges $Y = 0$ and $Y = 8$ m. The plate was divided into 16 strips of equal width, and 20 *Fourier* terms were taken into consideration.

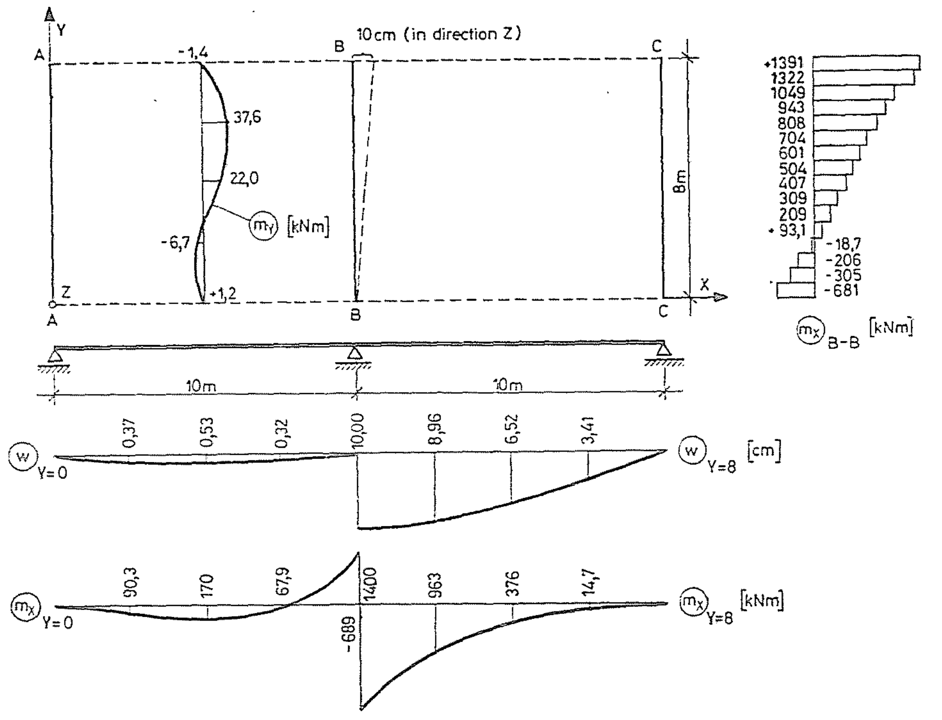

Fig. 5

Summary

Galerkin's method has been applied to write the equilibrium equation of rectangular plates with in-plane forces and in bending, and boundary conditions in general form. At the same time, the way of satisfying inhomogeneous kinematic and static boundary conditions is examined.

Thereafter the most frequent inhomogeneous boundary conditions of rectangular plate strips, in particular, strip end forces, strip end couples and prescribed displacements are examined. Displacement functions keep their orthogonality, permitting the *Fourier* term by term solution of equilibrium equation systems. The presented method permits computer analysis of continuous folded plates and box girders exposed to arbitrary loads and support displacements.

At last, a numerical example for prescribed support displacements of a continuous plate is given.

References

1. Cheung, Y. K.: Folded plate structures by finite strip method. Proc. ASCE, Vol **95.** (1969), ST 12, pp. 2963–2979
2. Cheung, Y. K.: Finite Strip Method in Structural Analysis. Pergamon Press. Oxford 1976
3. Connor, J. J., Brebbia, C. A.: Finite Element Techniques for Fluid Flow. Newnes-Butterworths, London 1977
4. Szilágyi, Gy.: Quelques applications de la méthode des bandes finies. Mémoires, AIPC, Zurich, Vol. **34-II.** (1974), pp. 149–168
5. Szilágyi, Gy.: Application of the finite strip method in the case of support displacements. (In Hungarian.) Műszaki Tudomány, Vol. **54.** (1978), No. 1–2., pp. 263–269

Dr. György Szilágyi, H-1521 Budapest