DISCRETE ANALYSIS OF STEADY TRANSPORT PROBLEMS IN CASE OF A THRESHOLD GRADIENT

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Introduction

The classic mathematical physics is known to have several, from engineering aspects perfectly distinct problems that can be uniformly handled in the frames of transport theory [7], such as heat conduction, diffusion, seepage, electric current or even elastic stress and strain. Finite element analysis of the simplest problem of transport theory, that of plane steady material flow (ZIENKIEWICZ, 1971) will be discussed below, extending the law of conduction to the case where start of the material flow is bound to a special condition [4]. These problems will be reduced, under different conduction conditions, to one case of linear programming, that of the linear complementarity problem [2], that can be solved on a computer CDC 3300 by means of an available program.

There are several examples in structural mechanics for similar, essentially optimization problems [1], [3], [5].

*Idea underlying the linear complementarity method**

The basic problem of LCM is to solve the algebraic problem

$$\mathbf{Ax} - \mathbf{y} = \mathbf{b} \qquad \mathbf{x} \geq \mathbf{0} \quad \mathbf{y} \leq \mathbf{0} \quad \mathbf{x}^*\mathbf{y} = 0 \tag{1}$$

containing the positive semi-definite matrix **A** for vectors x and y.

In LCM, this problem will be replaced by

$$\mathbf{Ax} - \mathbf{y} + \mathbf{Dz} = \mathbf{b} \qquad \mathbf{x}, \mathbf{z} \geq \mathbf{0} \quad \mathbf{y} \leq \mathbf{0} \quad \mathbf{x}^*\mathbf{y} = 0 \tag{2}$$

$$C = \sum_{i=1}^{m} z_i = \min!$$

* Hereinafter abbreviated as LCM

where $\mathbf{D}$ is a diagonal matrix composed of the unit matrix such that:

$$\text{if} \quad b_i \geq 0, \quad \text{then} \quad d_{ii} = 1$$

$$\text{if} \quad b_i < 0, \quad \text{then} \quad d_{ii} = -1$$

m being order of involved matrices and vectors. Conditions

$$\mathbf{z} \geq \mathbf{0} \qquad C = \sum_{i=1}^{m} z_i = \min!$$

being simultaneously satisfied by vector $\mathbf{z} = \mathbf{z}_0 = \mathbf{0}$ the solution vector couple $\mathbf{x}_0, \mathbf{y}_0$ of the substituting problem is identical to that of the original problem.

Furthermore the substituting problem, rather than the original one, contains an objective function, thus it may be solved by the simplex method.

Omitting side-condition $\mathbf{x}^*\mathbf{y} = 0$ would yield a linear programming problem of the form

$$[\mathbf{A} \mid -\mathbf{E} \mid \mathbf{D}] \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \mathbf{b} \qquad \mathbf{x}, \mathbf{z} \geq \mathbf{0}, \quad \mathbf{y} \leq \mathbf{0} \tag{3}$$

$$C = \sum_{i=1}^{m} z_i = \min!$$

with vector

$$\begin{bmatrix} \mathbf{x}_{(1)} \\ \mathbf{y}_{(1)} \\ \mathbf{z}_{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{D}^{-1}\mathbf{b} \end{bmatrix}$$

as starting basic solution. Further part of the computation would be to gradually reduce the starting objective value $C_{(1)} = \sum_{i=1}^{m} z_{(1),i}$ by basis exchange series. In general, the basis exchange simplex algorithm

1° either minimizes the objective function and seeks the related unique solution vector;

2° or shows the minimum to be delivered by a single-parameter vector manifold;

3° or enounces that under the given conditions, the objective function is unbounded.

Because of side condition $\mathbf{x}^*\mathbf{y} = 0$, the original problem (and so the exchange problem) is that of quadratic programming, though accessible to the simplex method. Namely the side-condition is easy to meet by involving only one at a time of quantities x_j and y_j with common subscript into the basis.

Basic equations of the transport problem

The plane steady transport problem deals with the development of a scalar density field $p = p(x, y)$ of two variables (hence, time-independent) in the domain T, for specified boundary conditions. Denoting the flow density vector and the extensive quantity of source density of the density field by $v(x, y)$ and $q(x, y)$, respectively, leads to the balance equation

$$\operatorname{div} v = q. \tag{4}$$

On the other hand, the flow density is related to the inhomogeneity of the intensive quantity distribution characteristic of the problem. Intensive quantity $\varphi(x, y)$ forms a potential distribution, its inhomogeneity is characterized by a vector $i(x, y)$:

$$i = grad\ \varphi. \tag{5}$$

in classic transp ort problems, these two vectors are in a homogeneous and lnear relation:

$$v = \mathbf{U}i \tag{6}$$

where $\mathbf{U}$ is a conductivity tensor, supposed, for the sake of simplicity, to have a diagonal matrix, that is, its principal directions coincide with those of components v and i.

In our discussion, both scalar equations of the conduction law (6) will be replaced by a relationship similar to those in Figs 1a and 1b.

Figure 1a represents a material flow the start of which is conditioned by given minima of the gradient, to have a transport process in the considered point at all.

Accordingly, the following possibilities of flow exist:

1° For a gradient below the threshold value, there is no flow:

$$v = 0, \quad \text{for} \quad -i^{\mathrm{II}} \leq i \leq i^{\mathrm{I}}.$$

2° For a positive gradient above the threshold value, there is a linear law of conduction:

$$v = -\frac{i - i^{\mathrm{I}}}{\operatorname{tg} \alpha^{\mathrm{I}}} = -\frac{i - i^{\mathrm{I}}}{m^{\mathrm{I}}} \qquad \text{for } i^{\mathrm{I}} \leq i\,.$$

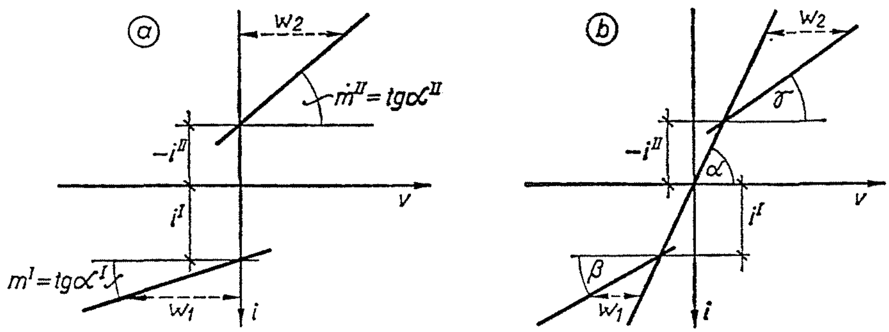

Fig. 1

3° For a negative gradient over the threshold value, the law of conduction is linear again:

$$v = -\frac{i + i^{\mathrm{II}}}{\operatorname{tg}\alpha^{\mathrm{II}}} = -\frac{i + i^{\mathrm{II}}}{m^{\mathrm{II}}} \qquad \text{for } i \leq -i^{\mathrm{II}}\,.$$

The set of these relationships is described by the following matrix equation, connected to side-conditions

$$\left[\begin{array}{c} v \\ \hline -i^{\mathrm{I}} \\ \hline -i^{\mathrm{II}} \end{array}\right] = \left[\begin{array}{c|c|c|c|c} & -1 & 1 & & \\ \hline -1 & m^{\mathrm{I}} & & 1 & \\ \hline 1 & & m^{\mathrm{II}} & & 1 \end{array}\right] \left[\begin{array}{c} i \\ \hline w_1 \\ \hline w_2 \\ \hline h_1 \\ \hline h_2 \end{array}\right] \qquad (7)$$

$$w_1, w_2 \geq 0, \qquad h_1, h_2 \leq 0, \qquad w_1 h_1 + w_2 h_2 = 0.$$

The formula is confirmed by the following: The quoted sign limitations imposed on w_1, w_2, h_1, h_2 involve that the scalar product $w_1 h_1 + w_2 h_2$ may only vanish if all its terms are zero.

Thus:

If $w_1 > 0$, then $h_1 = 0$. If $w_1 = 0$, then either $h_1 = 0$ or $h_1 < 0$.
If $w_2 > 0$, then $h_2 = 0$. If $w_2 = 0$, then either $h_2 = 0$ or $h_2 < 0$.

And :

If $h_1 < 0$, then $w_1 = 0$. If $h_1 = 0$, then either $w_1 = 0$ or $w_1 > 0$.
If $h_2 < 0$, then $w_2 = 0$. If $h_2 = 0$, then either $w_2 = 0$ or $w_2 > 0$.

Taking these into consideration, the matrix relationship may have the following meanings:

1°
$$w_1 = w_2 = 0 \qquad h_1 < 0 \qquad h_2 < 0$$

$$\begin{bmatrix} v \\ -i^{\mathrm{I}} \\ -i^{\mathrm{II}} \end{bmatrix} = \begin{bmatrix} 0 \\ -i + h_1 \\ i + h_2 \end{bmatrix}$$

or

$$v = 0 \qquad h_1 = i - i^{\mathrm{I}} \qquad h_2 = -(i + i^{\mathrm{II}}) .$$

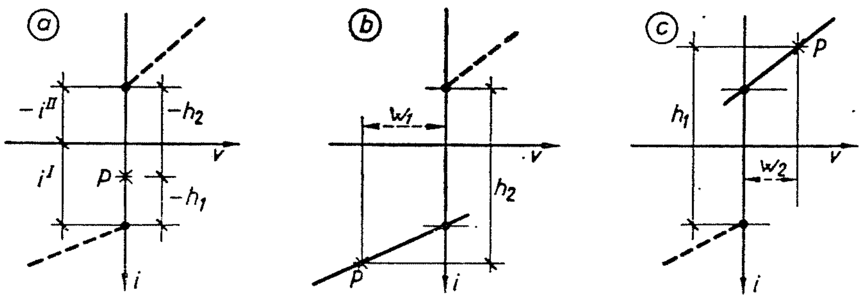

Fig. 2

The gradient is kept inside the threshold value, there is no flow (Fig. 2a).

2°
$$w_2 > 0, \quad h_2 < 0 \qquad w_2 = h_1 = 0$$

$$\begin{bmatrix} v \\ -i^{\mathrm{I}} \\ -i^{\mathrm{II}} \end{bmatrix} = \begin{bmatrix} -w_1 \\ -i + m^{\mathrm{I}} w_1 \\ i + h_2 \end{bmatrix}$$

or

$$v = -\frac{i - i^{\mathrm{I}}}{m^{\mathrm{I}}} \qquad h_2 = -(i + i^{\mathrm{II}}) .$$

The gradient has passed the positive threshold value, flow has started (Fig. 2b).

3°
$$w_2 > 0, \quad h_1 < 0 \qquad w_1 = h_2 = 0$$

$$\begin{bmatrix} v \\ -i^{\mathrm{I}} \\ -i^{\mathrm{II}} \end{bmatrix} = \begin{bmatrix} w_2 \\ -i + h_1 \\ i + m^{\mathrm{II}} w_2 \end{bmatrix}$$

or

$$v = -\frac{i + i^{\mathrm{II}}}{m^{\mathrm{II}}} \qquad h_1 = i - i^{\mathrm{I}}.$$

The gradient has passed the negative threshold value, flow has started (Fig. 2c).

In the case shown in Fig. 2b, where the material transport immediately starts but the resistance is very high up to the threshold gradients, the law of conduction becomes:

$$\begin{bmatrix} v \\ -i^{\mathrm{I}} \\ -i^{\mathrm{II}} \end{bmatrix} = \begin{bmatrix} -u & -1 & 1 & & \\ -1 & m^{\mathrm{I}} & & 1 & \\ 1 & & m^{\mathrm{II}} & & 1 \end{bmatrix} \begin{bmatrix} i \\ w_1 \\ w_2 \\ h_1 \\ h_2 \end{bmatrix} \quad \begin{matrix} w_1, w_2 \geq 0 \\ h_1, h_2 \leq 0 \\ w_1 h_1 + w_2 h_2 = 0 \end{matrix} \tag{8}$$

with

$$u = \operatorname{ctg} \alpha \quad m^{\mathrm{I}} = \frac{\operatorname{tg}\beta - \operatorname{tg}\alpha}{\operatorname{tg}\alpha \operatorname{tg}\beta} \quad m^{\mathrm{II}} = \frac{\operatorname{tg}\gamma - \operatorname{tg}\alpha}{\operatorname{tg}\alpha \operatorname{tg}\gamma}.$$

Outline of the numerical analysis

Field of flow densities and potentials of the transport problem is to be described numerically, determining a finite number of data, therefore an adequate number of points will be marked out inside and on the boundary of the tested domain, dividing it into finite elements. The state variables are assumed to be described by simple functions inside individual elements, linear combinations of given basic functions by unknown numbers. These are the values of state variables valid inside the finite elements (at preferential nodes or throughout the element) or at preferential boundary points of the domain. These parameters have to be determined according to state change conditions at points inside or marginal to the elements. Developing finite state equations of the problem consists in the algebraic formulation of these conditions.

In the simplest and most practical case, the domain is divided into triangles. Now, inside sufficiently small elements, the potential field can be assumed to be element by element linear hence flow density components are constant.

In the local reference frame of the j-th element (Fig. 3).

$$v_{j,\xi} = \mathrm{const} \qquad v_{j,\eta} = \mathrm{const}. \tag{9}$$

As the finite counterpart of Eq. (4) it may be written for element by element that the material quantity leaving at constant flow density equals the material quantity present at these points due either to the effect of adjacent elements, or to concentrated sources, or to sinks. First step of the

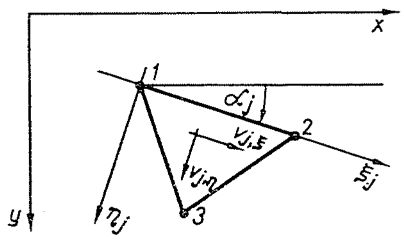

Fig. 3

computation will be to determine material quantities leaving through the sides, thereafter they are reduced to nodes in equal proportions. Thereby Eq. (4) will be replaced for any element by matrix equation

$$\begin{bmatrix} \eta_{j,2}-\eta_{j,3} & \xi_{j,3}-\xi_{j,2} \\ \eta_{j,3}-\eta_{j,1} & \xi_{j,1}-\xi_{j,3} \\ \eta_{j,1}-\eta_{j,2} & \xi_{j,2}-\xi_{j,1} \end{bmatrix} \begin{bmatrix} v_{j,\xi} \\ v_{j,\eta} \end{bmatrix} + 2 \begin{bmatrix} g_{j,1} \\ g_{j,2} \\ g_{j,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

where $g_{j,k}$ is material quantity entering at nodes but originating from an other than j-th element, or the distributed source density value reduced from the element to the node. Counterpart of Eq. (5) will be obtained by stressing that the potential field is linear from element to element. Applying Taylor's theorem, according to the principle of contragradiency:

$$\begin{bmatrix} i_{j,\xi} \\ i_{j,\eta} \end{bmatrix} = \frac{1}{2A_j} \left[\begin{array}{c|c|c} \eta_{j,2}-\eta_{j,3} & \eta_{j,3}-\eta_{j,1} & \eta_{j,1}-\eta_{j,2} \\ \hline \xi_{j,3}-\xi_{j,2} & \xi_{j,1}-\xi_{j,3} & \xi_{j,2}-\xi_{j,1} \end{array} \right] \begin{bmatrix} \varphi_{j,1} \\ \varphi_{j,2} \\ \varphi_{j,3} \end{bmatrix} \tag{11}$$

A_j being the element area.

Writing Eqs (7) and (8) combined with (10) and (11) for every element considering common quantities at common nodes to be identical, changing to global co-ordinates of the system, and compiling the equations into one set, yields the equation of the transport problem in a form with finite degrees of freedom.

For instance, in case of a single element:

$$w_{\xi,1}, w_{\xi,2}, w_{\eta,1}, w_{\eta,2} \geq 0\,; \quad h_{\xi,1}, h_{\xi,2}, h_{\eta,1}, h_{\eta,2} \leq 0$$

$$w_{\xi,1}h_{\xi,1} + w_{\xi,2}h_{\xi,2} + w_{\eta,1}h_{\eta,1} + w_{\eta,2}h_{\eta,2} = 0\,.$$

$$(c = \cos \alpha_j\,; \quad s = \sin \alpha_j)$$

The linear system contains $3 + 2 + 2 + 4 = 11$ equations with $3 + 2 + + 2 + 4 + 4 = 15$ possible unknowns, composed, for the sake of comprehensiveness, in a row vector above the coefficient matrix columns. Side-conditions yield four further relationships, permitting the problem to be solved unambiguously, but generally this is not the case without meeting the boundary conditions.

To meet the boundary conditions requires to interchange some elements φ and v by certain elements g.

It can be stated in general, as it appears from the tabulation, that:

— first group of the equations contains material flow continuity equations for each node;

— the second group relates components of nodal potential values to those of the gradient vector for each element, and within them, for each component;

— the third group relates flow density components to gradient components;

— the fourth group expresses relations between gradients, possible flow density excesses and gradient deficiencies, equally for triangles, and within them, for each component.

In hypermatrix form:

$$\begin{bmatrix} & \mathbf{D}^* & & & \\ \mathbf{D} & & \mathbf{I} & & \\ & \mathbf{I} & \mathbf{U} & \mathbf{L}^* & \\ & & \mathbf{L} & \mathbf{M} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi} \\ \mathbf{v} \\ \mathbf{i} \\ \mathbf{w} \\ \mathbf{h} \end{bmatrix} + \begin{bmatrix} \mathbf{g} \\ \\ \\ \mathbf{j} \end{bmatrix} = \mathbf{0}\,. \tag{12}$$

$$\mathbf{w} \geq \mathbf{0} \qquad \mathbf{h} \leq \mathbf{0} \qquad \mathbf{w}^*\mathbf{h} = 0$$

In case of several elements, conduction law of several sections or in case of a spatial problem, the content of blocks in the formula changes and becomes more extensive.

As concerns reckoning with boundary conditions, let us refer to [6].

φ_1	φ_2	φ_3	v_ξ	v_η	i_ξ	i_η	$w_{\xi,1}$	$w_{\xi,2}$	$w_{\eta,1}$	$w_{\eta,2}$	$h_{\xi,1}$	$h_{\xi,2}$	$h_{\eta,1}$	$h_{\eta,2}$			
			$s(x_3-x_2)+$ $+c(y_2-y_3)$	$c(x_3-x_2)+$ $+s(y_3-y_2)$												$2g_1$	
			$s(x_1-x_3)+$ $+c(y_3-y_1)$	$c(x_1-x_3)+$ $+s(y_1-y_3)$												$2g_2$	
			$s(x_2-x_1)+$ $+c(y_1-y_2)$	$c(x_2-y_1)+$ $+s(y_2-y_1)$												$2g_3$	
$s(x_3-x_2)+$ $+c(y_2-y_3)$	$s(x_1-x_3)+$ $+c(y_3-y_1)$	$s(x_2-x_1)+$ $+c(y_2-y_2)$			$-2A$												
$c(x_3-x_2)+$ $+s(y_3-y_2)$	$c(x_1-x_3)+$ $+s(y_1-y_3)$	$c(x_2-x_1)+$ $+s(y_2-y_1)$				$-2A$									$+$		$=0$
			$-2A$		$-2Au_\xi$		$-2A$	$2A$									
				$-2A$		$-2Au_\eta$			$-2A$	$2A$							
					$-2A$		$2Am_\xi^{\mathrm{I}}$				$2A$					$2Ai_\xi^{\mathrm{I}}$	
					$2A$			$2Am_\xi^{\mathrm{II}}$				$2A$				$2Ai_\xi^{\mathrm{II}}$	
						$-2A$			$2Am_\eta^{\mathrm{I}}$				$2A$			$2Ai_\eta^{\mathrm{I}}$	
						$2A$				$2Am_\eta^{\mathrm{II}}$				$2A$		$2Ai_\eta^{\mathrm{II}}$	

Reduction algorithms

In examining the algorithms, the boundary conditions are considered not to be omissible, thus, φ and $\mathbf{g}$ contain only really unknown, and only known elements, respectively.

Starting from the particular case of a problem without threshold gradient, now the problem is a linear one in fact.

$$\begin{bmatrix} & \mathbf{D}^* & \\ \mathbf{D} & & \mathbf{I} \\ & \mathbf{I} & \mathbf{U} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi} \\ \mathbf{v} \\ \mathbf{i} \end{bmatrix} + \begin{bmatrix} \mathbf{g} \\ \\ \end{bmatrix} = \mathbf{0}. \tag{13}$$

Eliminating vector i, it is reduced to the usual form:

$$\begin{bmatrix} & \mathbf{D}^* \\ \mathbf{D} & \mathbf{F} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi} \\ \mathbf{v} \end{bmatrix} + \begin{bmatrix} \mathbf{g} \\ \end{bmatrix} = \mathbf{0} \tag{14}$$

where

$$\mathbf{F} = -\mathbf{I}\mathbf{U}^{-1}\mathbf{I}. \tag{15}$$

Solution of the hypermatrix equation (14) will be obtained by means of the state change matrix

$$\mathbf{D}^*\mathbf{F}^{-1}\mathbf{D} = \mathbf{H} \tag{16}$$

of the entire domain from the transport equation

$$\mathbf{H}\boldsymbol{\varphi} = \mathbf{g} \tag{17}$$

as

$$\boldsymbol{\varphi} \equiv \boldsymbol{\varphi}_0 = \mathbf{H}^{-1}\mathbf{g} \qquad \mathbf{v} \equiv \mathbf{v}_0 = -\mathbf{F}^{-1}\mathbf{D}\mathbf{H}^{-1}\mathbf{g} \tag{18}$$

If the domain contains no finite element so as to be perfectly impermeable in one or the other direction, thus, det $\mathbf{U}=0$, also Eq. (12) can be reduced as before:

$$\mathbf{i} = -\mathbf{U}^{-1}(\mathbf{I}\mathbf{v} + \mathbf{L}^*\mathbf{w})$$

$$\mathbf{D}\boldsymbol{\varphi} - \mathbf{F}\mathbf{v} - \mathbf{I}\mathbf{U}^{-1}\mathbf{L}^*\mathbf{w} = \mathbf{0}$$

$$-\mathbf{L}\mathbf{U}^{-1}\mathbf{I}\mathbf{v} + (\mathbf{M} - \mathbf{L}\mathbf{U}^{-1}\mathbf{L}^*)\mathbf{w} + \mathbf{Z}\mathbf{h} + \mathbf{j} = \mathbf{0}.$$

Introducing matrices

$$\mathbf{N} = -\mathbf{L}\mathbf{U}^{-1}\mathbf{I}; \qquad \mathbf{P} = \mathbf{M} - \mathbf{L}\mathbf{U}^{-1}\mathbf{L}^{*} \tag{19}$$

yields the hypermatrix state equation of the problem:

$$\begin{bmatrix} & \mathbf{D}^{*} & & \\ \mathbf{D} & \mathbf{F} & \mathbf{N}^{*} & \\ & \mathbf{N} & \mathbf{P} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi} \\ \mathbf{v} \\ \mathbf{w} \\ \mathbf{h} \end{bmatrix} + \begin{bmatrix} \mathbf{g} \\ \\ \mathbf{j} \end{bmatrix} = \mathbf{0} \tag{20}$$

$$\mathbf{w} \geq 0 \qquad \mathbf{h} \leq 0 \qquad \mathbf{w}^{*}\mathbf{h} = 0.$$

This relationship cannot be directly solved as (18) but must be transformed to the form (1) and the result can only be calculated after applying LCM, by modifying (18). Thus:

$$\mathbf{A}\mathbf{w} - \mathbf{h} = \mathbf{b} \quad \mathbf{w} \geq 0 \quad \mathbf{h} \leq 0 \quad \mathbf{w}^{*}\mathbf{h} = 0 \tag{21}$$

where:

$$\mathbf{b} = \mathbf{z}^{-1}\ (\mathbf{i} - \mathbf{N}\mathbf{F}^{-1}\mathbf{D}\mathbf{H}^{-1}\mathbf{g})$$

$$\mathbf{U}_{\text{red}} = \mathbf{F}^{-1} - \mathbf{F}^{-1}\mathbf{D}\mathbf{H}^{-1}\mathbf{D}^{*}\mathbf{F}^{-1}$$

(this latter being a reduced conductivity matrix). Using vectors $\mathbf{w}$ and $\mathbf{h}$:

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}_0 - \mathbf{H}^{-1}\mathbf{D}^{*}\mathbf{F}^{-1}\mathbf{N}^{*}\mathbf{w} \tag{22}$$

$$\mathbf{v} = \mathbf{v}_0 - \mathbf{U}_{\text{red}}\mathbf{N}^{*}\mathbf{w}.$$

If the domain includes perfectly impermeable finite elements, the fundamental particular problem (13) can be written, — after suitable rearrangement of rows and columns, decomposed into blocks, — as:

$$\left[\begin{array}{c|c|c|c|c} & \mathbf{D}_1^* & \mathbf{D}_2^* & & \\ \hline \mathbf{D}_1 & & & \mathbf{I}_1 & \\ \hline \mathbf{D}_2 & & & & \mathbf{I}_2 \\ \hline & \mathbf{I}_1 & & \mathbf{U}_1 & \\ \hline & & \mathbf{I}_2 & & \mathbf{0} \end{array}\right] \left[\begin{array}{c} \boldsymbol{\varphi} \\ \hline \mathbf{v}_1 \\ \hline \mathbf{v}_2 \\ \hline \mathbf{i}_1 \\ \hline \mathbf{i}_2 \end{array}\right] + \left[\begin{array}{c} \mathbf{g} \\ \hline \\ \hline \\ \hline \\ \hline \\ \end{array}\right] = \mathbf{0} \qquad (23)$$

subscripts 1 and 2 referring to permeable, and impermeable elements, respectively.

Since obviously $\mathbf{v}_2 = \mathbf{0}$, the system can be reduced:

$$\left[\begin{array}{c|c|c|c} & \mathbf{D}_1^* & & \\ \hline \mathbf{D}_1 & & \mathbf{I}_1 & \\ \hline \mathbf{D}_2 & & & \mathbf{I}_2 \\ \hline & \mathbf{I}_1 & \mathbf{U}_1 & \end{array}\right] \left[\begin{array}{c} \boldsymbol{\varphi} \\ \hline \mathbf{v}_1 \\ \hline \mathbf{i}_1 \\ \hline \mathbf{i}_2 \end{array}\right] + \left[\begin{array}{c} \mathbf{g} \\ \hline \\ \hline \\ \hline \\ \end{array}\right] = \mathbf{0}$$

or

$$\left[\begin{array}{c|c|c} & \mathbf{D}_1^* & \\ \hline \mathbf{D}_1 & & \mathbf{I}_1 \\ \hline & \mathbf{I}_1 & \mathbf{U}_1 \end{array}\right] \left[\begin{array}{c} \boldsymbol{\varphi} \\ \hline \mathbf{v}_1 \\ \hline \mathbf{i}_1 \end{array}\right] + \left[\begin{array}{c} \mathbf{g} \\ \hline \\ \hline \\ \end{array}\right] = \mathbf{0} \qquad \mathbf{D}_2\boldsymbol{\varphi} + \mathbf{I}_2\mathbf{i}_2 = \mathbf{0}.$$

The first relationship yields:

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}_{1,0} = \mathbf{H}^{-1}\mathbf{g} \qquad \mathbf{v}_1 = \mathbf{v}_{1,0} = -\mathbf{F}_1^{-1}\mathbf{D}_1\mathbf{H}_1^{-1}\mathbf{g} \qquad (24)$$

and the second one:

$$\mathbf{i}_2 = -\mathbf{I}_2^{-1}\mathbf{D}_2\boldsymbol{\varphi}.$$

Here, for instance:

$$\mathbf{H}_1 = \mathbf{D}_1^*\mathbf{F}_1^{-1}\mathbf{D}_1.$$

If the conductivity law is valid in generalized form, but some elements are perfectly impermeable up to the threshold gradient, then the state equation decomposed into blocks reads:

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c} & \mathbf{D}_1^* & \mathbf{D}_2^* & & & & & & \\ \hline \mathbf{D}_1 & & & \mathbf{I}_1 & & & & & \\ \hline \mathbf{D}_2 & & & & \mathbf{I}_2 & & & & \\ \hline & \mathbf{I}_1 & & \mathbf{U}_1 & & \mathbf{L}_{11}^* & \mathbf{L}_{21}^* & & \\ \hline & & \mathbf{I}_2 & & \mathbf{0} & \mathbf{L}_{12}^* & \mathbf{L}_{22}^* & & \\ \hline & & & \mathbf{L}_{11} & \mathbf{L}_{12} & \mathbf{M}_1 & & \mathbf{Z}_1 & \\ \hline & & & \mathbf{L}_{21} & \mathbf{L}_{22} & & \mathbf{M}_2 & & \mathbf{Z}_2 \end{array}\right] \left[\begin{array}{c} \boldsymbol{\varphi} \\ \hline \mathbf{v}_1 \\ \hline \mathbf{v}_2 \\ \hline \mathbf{i}_1 \\ \hline \mathbf{i}_2 \\ \hline \mathbf{w}_1 \\ \hline \mathbf{w}_2 \\ \hline \mathbf{h}_1 \\ \hline \mathbf{h}_2 \end{array}\right] + \left[\begin{array}{c} \mathbf{g} \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \mathbf{j}_1 \\ \hline \mathbf{j}_2 \end{array}\right] = \mathbf{0} \tag{25}$$

$$\mathbf{w}_1, \mathbf{w}_2 \geq \mathbf{0} \qquad \mathbf{h}_1, \mathbf{h}_2 \leqq 0 \qquad \mathbf{w}_1^* \mathbf{h}_1 + \mathbf{w}_2^* \mathbf{h}_2 = 0$$

or reduced:

$$\left[\begin{array}{c|c|c|c} & \mathbf{D}_1^* & \mathbf{N}_1^* & \\ \hline \mathbf{D}_1 & \mathbf{F}_1 & \mathbf{N}_2^* & \\ \hline \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{P} & \mathbf{Z} \end{array}\right] \left[\begin{array}{c} \boldsymbol{\varphi} \\ \hline \mathbf{v}_1 \\ \hline \mathbf{w} \\ \hline \mathbf{h} \end{array}\right] + \left[\begin{array}{c} \mathbf{g} \\ \hline \\ \hline \mathbf{j} \end{array}\right] = \mathbf{0} \tag{26}$$

$$\mathbf{w} \geq \mathbf{0} \qquad \mathbf{h} \leq \mathbf{0} \qquad \mathbf{w}^* \mathbf{h} = 0$$

with

$$\mathbf{F}_1 = -\mathbf{I}_1 \mathbf{U}_1^{-1} \mathbf{I}_1$$

$$\mathbf{N}_1 = -\left[\begin{array}{c} \mathbf{L}_{12} \\ \hline \mathbf{L}_{22} \end{array}\right] \mathbf{I}_2^{-1} \mathbf{D}_2 \qquad \mathbf{N}_2 = -\left[\begin{array}{c} \mathbf{L}_{11} \\ \hline \mathbf{L}_{21} \end{array}\right] \mathbf{U}_1^{-1} \mathbf{I}_1 \tag{27}$$

$$\mathbf{P} = \left[\begin{array}{c|c} \mathbf{M}_1 - \mathbf{L}_{11}\mathbf{U}_1^{-1}\mathbf{L}_{11}^* & -\mathbf{L}_{11}\mathbf{U}_1^{-1}\mathbf{L}_{21}^* \\ \hline -\mathbf{L}_{11}\mathbf{U}_1^{-1}\mathbf{L}_{11}^* & \mathbf{M}_2 - \mathbf{L}_{21}\mathbf{U}_1^{-1}\mathbf{L}_{21}^* \end{array}\right].$$

The relationship transformed into the basic LCM problem is again of the form (21) but here:

$$\begin{aligned} \mathbf{A} &= \mathbf{Z}^{-1}\{\mathbf{N}_2\mathbf{F}_1^{-1}\mathbf{N}_2 - (\mathbf{N}_1 - \mathbf{N}_2\mathbf{F}_1^{-1}\mathbf{D}_1)\,\mathbf{H}_1^{-1}(\mathbf{N}_1^* - \mathbf{D}_1^*\mathbf{F}_1^{-1}\mathbf{N}_2^*) - \mathbf{P}\} \\ \mathbf{b} &= \mathbf{Z}^{-1}\{\mathbf{j} + (\mathbf{N}_1 - \mathbf{N}_2\mathbf{F}_1^{-1}\mathbf{D}_1)\,\mathbf{H}^{-1}\mathbf{g}\}. \end{aligned} \tag{28}$$

Having the vectors $\mathbf{w}$ and $\mathbf{h}$:

$$\begin{aligned} \varphi &= \varphi_{1,0} + \mathbf{H}_1^{-1}(\mathbf{N}_1^* - \mathbf{D}_1^*\mathbf{F}_1^{-1}\mathbf{N}_2^*)\,\mathbf{w} \\ \mathbf{v}_1 &= \mathbf{v}_{1,0} - \{\mathbf{F}^{-1}\mathbf{D}_1\mathbf{H}_1^{-1}(\mathbf{N}_1^* - \mathbf{D}_1^*\mathbf{F}_1^{-1}\mathbf{N}_2^*) + \mathbf{F}_1^{-1}\mathbf{N}_2^*\}\,\mathbf{w}. \end{aligned} \tag{29}$$

And from the original set:

$$\mathbf{v}_2 = -\mathbf{I}_2^{-1}(\mathbf{L}_{12}^*\mathbf{w}_1 + \mathbf{L}_{22}^*\mathbf{w}_2). \tag{30}$$

This algorithm may be replaced by choosing permeabilities of insulating elements as disproportionally less than those of the others, rather than to be zeroed, anything else being kept invariable.

If the domain contains perfectly permeable finite elements, the corresponding state change values will be omitted, and so will be the relationships of conductivity, but marginal potentials will be specified and material discharges will be considered as unknown.

Finally, if up to the threshold gradient, every element is perfectly impermeable, the state equation becomes:

$$\left[\begin{array}{c|c|c|c|c} & \mathbf{D}^* & & & \\ \hline \mathbf{D} & & \mathbf{I} & & \\ \hline & \mathbf{I} & & \mathbf{L}^* & \\ \hline & & \mathbf{L} & \mathbf{M} & \mathbf{Z} \end{array}\right] \left[\begin{array}{c} \varphi \\ \hline \mathbf{v} \\ \hline \mathbf{i} \\ \hline \mathbf{w} \\ \hline \mathbf{h} \end{array}\right] + \left[\begin{array}{c} \mathbf{g} \\ \hline \\ \hline \\ \hline \mathbf{j} \end{array}\right] = \mathbf{0} \tag{31}$$

$$\mathbf{w} \geq \mathbf{0}, \quad \mathbf{h} \leq \mathbf{0}, \quad \mathbf{w}^*\mathbf{h} = 0.$$

For $\mathbf{w} = \mathbf{h} = \mathbf{0}$ this problem has no solution, so we have to write

$$\left[\begin{array}{c|c|c} & \mathbf{D}^* & \\ \hline \mathbf{D} & & \mathbf{I} \\ \hline & \mathbf{I} & \end{array}\right] = \mathbf{X} \qquad \left[\begin{array}{c} \\ \hline \\ \hline \mathbf{L}^* \end{array}\right] = \mathbf{Y}^* \qquad \left[\begin{array}{c} \boldsymbol{\varphi} \\ \hline \mathbf{v} \\ \hline \mathbf{i} \end{array}\right] = \mathbf{x} \qquad \left[\begin{array}{c} \mathbf{g} \\ \hline \\ \hline \\ \end{array}\right] = \mathbf{p}.$$

Thus, the relationships transform the LCM problem as follows:

$$\left[\begin{array}{c|c|c} \mathbf{X} & \mathbf{Y}^* & \\ \hline \mathbf{Y} & \mathbf{M} & \mathbf{Z} \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{w} \\ \hline \mathbf{h} \end{array}\right] + \left[\begin{array}{c} \mathbf{p} \\ \hline \mathbf{j} \end{array}\right] = \mathbf{0} \tag{32}$$

$$\mathbf{w} \geq \mathbf{0} \qquad \mathbf{h} \leq 0 \qquad \mathbf{w}^*\mathbf{h} = 0 .$$

Summary

The basic problem of transport theory, steady material flow has been discussed for the case where start of the flow is controlled by a separate threshold gradient condition. Using finite elements and several conductivity conditions, the problem will be reduced to the linear complementarity problem similar to the state change equation of structures.

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