DYNAMIC ANALYSIS OF A STRUCTURE WITH COMPONENTS OF DIFFERENT DAMPING CHARACTERISTICS

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The dynamic analysis of structures involves the analysis of vibrating systems with several degrees of freedom. Vibrations underlie damping due to the so-called internal friction arising from the non-linear elasticity of the material. In certain cases (resonance, free vibration, impulse loads), neglect of this damping leads to a basically wrong description of the phenomenon. An algorithm for the analysis of structures with components of different damping characteristics will be presented, together with practical experience.

1. Analysis of systems with one degree of freedom

It can experimentally be demonstrated that in different building materials, the logarithmic decrement characterizing damping may be considered as constant in case of dynamic stresses over a very small fraction of ultimate yield stress. At the same time it depends also on the stress state (torsion — bending) or e.g. for r.c. structures, on the crack development. Systems with one degree of freedom are accessible to the analysis by the equivalent *Kelvin—Voight* model or the SOROKIN complex rigidity [1, 2]. In this latter case, the stress-strain relationship may be written in complex form as:

$$\sigma^* = (u + iv)E\varepsilon^*.$$

(* referring to complex quantities).

In this complex expression,

$$u = \frac{4 - \gamma^2}{4 + \gamma^2}, \quad v = \frac{4\gamma}{4 + \gamma^2}, \quad \gamma = \frac{\delta}{\Pi},$$

δ being the logarithmic decrement.

Analysis of a system with one degree of freedom may start from the differential equation:

$$m\ddot{x}^* + (u + iv)\,kx^* = q^*(t)\ .$$

From the examination of the homogeneous part, after having solved the characteristic equation:

$$x_1^* = B_1^*\,e^{ip^*t} + B_2^*\,e^{-ip^*t},$$

where

$$p^* = \left(1 + i\,\frac{\gamma}{2}\right)p\,, \quad p = \frac{p_o}{\sqrt{1 + \frac{\gamma^2}{4}}}\,, \qquad p_o = \sqrt{\frac{k}{m}}\,.$$

The differential equation of complex displacements has been deduced for damped vibration. General solution of the homogeneous equation meets these conditions only if the first term alone, yielding the decreasing displacements, is reckoned with. Finding also the particular solution of the inhomogeneous equation yields the general solution of the differential equation [1]:

$$x^* = c^*\,e^{ip^*t} + J^*(t)$$

where $c^* = a + ib$ is the constant determinable from the initial conditions, while

$$J^*(t) = \frac{e^{ip^*t}}{2\,ip^*m}\int q^*(t)\,e^{-ip^*t}\,dt$$

and

$$-\frac{e^{-ip^*t}}{2\,ip^*m}\int q^*(t)\,e^{ip^*t}\,dt\,.$$

The displacement will be the real part of the complex solution.

a) *In case of free vibration:*

the homogeneous part will have a solution in the form:

$$\mathrm{Re}\,x^* = e^{-\frac{\gamma}{2}pt}\,(a\ \cos pt - b\sin pt).$$

b) *In case of periodic excitation:*

For the excitation

$$q(t) = \sum_{s=1}^{n} p_s \cos\,(\omega_s t - \nu_s)\,,$$

reckoning with a complex force

$$q^*(t) = \sum_{s=1}^{n} p_s e^{i(\omega_s t - \nu_s)},$$

solution of the inhomogeneous part will be of the form: [1]

$$\operatorname{Re} J^*(t) =$$

$$= \sum_{s=1}^{n} \frac{p_s}{mp^2} \frac{1}{\sqrt{\left(1 - \frac{\omega_s^2}{p^2} - \frac{\gamma^2}{4}\right)^2 + \gamma^2}} \cos\left(\omega_s t - \nu - \operatorname{arctg} \frac{\gamma}{1 - \frac{\omega_s^2}{p^2} - \frac{\gamma^2}{4}}\right).$$

2. Analysis of systems with several degrees of freedom

Provided the entire structure has one damping characteristic, the problem can be solved by the equivalent *Kelvin—Voight* model, using the solution of the real eigenvalue problem of the undamped case [2]. A solution is found in [3] for the case of different damping characteristics where the differential equation will be written in the form:

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{K}_R + i \operatorname{sgn} \omega \, \mathbf{K}_i)\, \mathbf{x} = \mathbf{q}$$

and the solution will be determined by writing a double-size complex matrix eigenvalue problem. A method will be presented below, yielding complex displacements similar to those in the former item, hence it requires the solution of a complex eigenvalue problem of only the original order number. For instance, let us consider a system of bars of different materials. In this case the differential equation for individual bars can be written in the form:

$$\mathbf{M}_j \ddot{\mathbf{x}}_j^* + (u_j + iv_j)\mathbf{K}_j = \mathbf{q}_j$$

but for the bar system the magnitude $u + iv$ cannot be factored out any more. Differential equation of the bar system will be of the form:

$$\mathbf{M}\ddot{\mathbf{x}}^* + \mathbf{K}^*\mathbf{x}^* = \mathbf{q}^*$$

where

$$\mathbf{K}^* = \mathbf{K}_R + i\mathbf{K}_i.$$

Solution of the differential equation will be considered for the cases outlined in the introduction.

2.1 *Free vibration*

Let us have the differential equation

$$\mathbf{M}\ddot{\mathbf{x}}^* + \mathbf{K}^*\mathbf{x}^* = 0.$$

Seeking the solution in the form

$$\mathbf{x}^* = \mathbf{v}^* e^{ip^*t}$$

as the first step, the complex eigenvalue problem

$$\mathbf{M}^{-1}\mathbf{K}^*\mathbf{v}^* = \mathbf{p}^{*2}\mathbf{v}^*$$

has to be solved.

If eigenvalues and eigenvectors are known:

$$\mathbf{x}^* = \sum_{j=1}^{n} c_j^* \mathbf{v}_j^* e^{p_j^* t}.$$

Introducing notations

$$c_j^* = a_j + bi_j,$$

$$\mathbf{v}_j^* = \mathbf{v}_{jr} + i\mathbf{v}_{ji},$$

$$p_j^* = \left(1 + i\frac{\gamma_j'}{2}\right) p_j,$$

real part of the solution can be written in the form:

$$\mathbf{x} = [\mathbf{V}_r \mathbf{D}_1(t) + \mathbf{V}_i \mathbf{D}_2(t)]\mathbf{a} + [-\mathbf{V}_i \mathbf{D}_1(t) + \mathbf{V}_r \mathbf{D}_2(t)]\mathbb{b}$$

where **D** are diagonal matrices:

$$\mathbf{D}_1(t) = \langle e^{-\frac{\gamma_j'}{2} p_j t} \cos p_j t \rangle$$

$$\mathbf{D}_2(t) = \langle -e^{-\frac{\gamma_j'}{2} p_j t} \sin p_j t \rangle.$$

Velocities can be written as:

$$\dot{\mathbf{x}} = \mathbf{v} = [\mathbf{V}_r \mathbf{D}_3(t) + \mathbf{V}_i \mathbf{D}_4(t)]\mathbf{a} + [-\mathbf{V}_i \mathbf{D}_3(t) + \mathbf{V}_r \mathbf{D}_4(t)]\mathbb{b}$$

where

$$\mathbf{D}_3(t) = \left\langle p_j e^{-\frac{\gamma_j'}{2} p_j t} \left(-\frac{\gamma_j'}{2} \cos p_j t - \sin p_j t\right) \right\rangle$$

$$\mathbf{D}_4(t) = \left\langle p_j e^{-\frac{\gamma_j'}{2} p_j t} \left(\frac{\gamma_j'}{2} \sin p_j t - \cos p_j t\right) \right\rangle.$$

Elements of vectors **a** and **b** can be determined from the initial conditions by solving a linear system of equations. Specifying displacements and velocities $(\mathbf{x}_0, \mathbf{y}_0)$ at a time t_0:

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{Z}^{-1}(t_0) \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{v}_0 \end{bmatrix},$$

where

$$\mathbf{Z}(t_0) = \left[\begin{array}{c|c} \mathbf{V}_r \mathbf{D}_1(t_0) + \mathbf{V}_i \mathbf{D}_2(t_0) & -\mathbf{V}_i \mathbf{D}_1(t_0) + \mathbf{V}_r \mathbf{D}_2(t_0) \\ \hline \mathbf{V}_r \mathbf{D}_3(t_0) + \mathbf{V}_i \mathbf{D}_4(t_0) & -\mathbf{V}_i \mathbf{D}_3(t_0) + \mathbf{V}_r \mathbf{D}_4(t_0) \end{array} \right].$$

2.2 *Effect of instantaneous impulse*

Be the i-th mass point acted upon by impulse $S_i = \int_{t_1}^{t_1+\Delta t} P(\tau) d\tau$ where $\Delta \rightarrow 0$. The impulse causes a velocity increment at point i:

$$v_i = \frac{S_i}{m_i}.$$

Characteristics of displacement started at t_0 are at time t_1:

$$\mathbf{x}_0(t_1) \quad \text{and} \quad \mathbf{v}_0(t_1).$$

Further movement can be considered as a free vibration with the initial condition:

$$\begin{bmatrix} \mathbf{x}_{10} \\ \mathbf{v}_{10} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0(t_1) \\ \mathbf{v}_0(t_1) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ v_{i1} \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{x}_{10} \\ \mathbf{v}_{10} \end{bmatrix} = \mathbf{Z}(t_1 - t_0) \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{b}_0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ v_{i1} \\ \vdots \\ 0 \end{bmatrix},$$

and having the constants:

$$\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{b}_1 \end{bmatrix} = \mathbf{Z}^{-1}(0) \begin{bmatrix} \mathbf{x}_{10} \\ \mathbf{v}_{10} \end{bmatrix},$$

the continued movement being:

$$\mathbf{x}_1 = \left[\begin{array}{c|c} \mathbf{V}_r \mathbf{D}_1(t - t_1) + & -\mathbf{V}_i \mathbf{D}_1(t - t_1) + \\ + \mathbf{V}_i \mathbf{D}_2(t - t_1) & + \mathbf{V}_r \mathbf{D}_2(t - t_1) \end{array} \right] \begin{bmatrix} \mathbf{a}_1 \\ \hline \mathbf{b}_1 \end{bmatrix}.$$

After the j-th impulse:

$$\mathbf{x}_j = \left[\begin{array}{c|c} \mathbf{V}_r \mathbf{D}_1(t - t_j) + & -\mathbf{V}_i \mathbf{D}_1(t - t_j) + \\ + \mathbf{V}_i \mathbf{D}_2(t - t_j) & + \mathbf{V}_r \mathbf{D}_2(t - t_j) \end{array} \right] \begin{bmatrix} \mathbf{a}_j \\ \hline \mathbf{b}_j \end{bmatrix},$$

where:

$$\begin{bmatrix} \mathbf{a}_j \\ \mathbf{b}_j \end{bmatrix} = \mathbf{Z}^{-1}(0)\ \mathbf{Z}(t_j - t_{j-1}) \begin{bmatrix} \mathbf{a}_{j-1} \\ \mathbf{b}_{j-1} \end{bmatrix} + \mathbf{Z}^{-1}(0) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ v_{ij} \\ \vdots \\ 0 \end{bmatrix}.$$

2.3 *Harmonic excitation*

Be the function of excitation

$$\mathbf{p}(t) = \mathbf{q}(t) \cos \omega t$$

and the relevant set of differential equations:

$$\mathbf{M}\ddot{\mathbf{x}}^* + \mathbf{K}^*\mathbf{x}^* = \mathbf{q}^*$$

where

$$\mathbf{q}^* = \mathbf{q}\, e^{i\omega t}.$$

In course of solving the complex eigenvalue problem, the eigenvectors are normed to give

$$\overline{\mathbf{V}}^{*T}\mathbf{M}\mathbf{V}^* = \mathbf{E}$$

and substituted:

$$\mathbf{x}^* = \mathbf{V}^*\mathbf{z}^*.$$

After having multiplied by matrix $\mathbf{V}^{*T}$:

$$\ddot{\mathbf{z}}^* + \overline{\mathbf{V}}^{*T}\mathbf{K}^*\mathbf{V}^*\mathbf{z}^* = \overline{\mathbf{V}}^{*T}\mathbf{q}^*.$$

Since

$$\overline{\mathbf{V}}^{*T}\mathbf{K}^*\mathbf{V}^* = \langle p_j^{*2} \rangle$$

the system of n degrees of freedom is decomposed into n systems of one degree of freedom:

$$\ddot{z}_j^* + p_j^{*2} z_j = \sum_{s=1}^{n} \bar{v}_{js}^{*T} q_s^* = f_j^* e^{i\omega t}$$

to be solved as:

$$z_j = (a_j + ib_j) e^{-\frac{\gamma_j}{2} p_j t} e^{ip_j t} + f_j^* \frac{A_j - iB_j}{A_j^2 + B_j^2} e^{i\omega t}.$$

In the term z_j, a_j and b_j are by the moment unknown constants,

$$A_j = \left(1 - \frac{\gamma_j^2}{4}\right) p_j^2 - \omega^2 \quad \text{and} \quad B_j = \gamma_j p_j^2.$$

Multiplication by matrix $\mathbf{V}^*$ yields $\mathbf{x}^*$ with a real and an imaginary part; after proper transformations:

$$\mathbf{x} = [\mathbf{V}_r \mathbf{D}_1(t) + \mathbf{V}_i \mathbf{D}_2(t) \mid -\mathbf{V}_i \mathbf{D}_1(t) + \mathbf{V}_r \mathbf{D}_2(t) \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} +$$

$$+ \sum_{j=1}^{n} z_j \{ \mathbf{v}_{rj} [f_{rj} \cos(\omega t - \varphi) - f_{ij} \cos(\omega t + \varphi)] -$$

$$- \mathbf{v}_{ij} [f_{ij} \cos(\omega t - \varphi) + f_{rj} \cos(\omega t + \varphi)] \}.$$

where

$$z_j = \frac{1}{p_j^2 \sqrt{\left(1 - \frac{\omega^2}{p_j^2} - \frac{\gamma_j^2}{4}\right) + \gamma_j^2}}, \qquad \varphi = \operatorname{arctg} \frac{\gamma_j}{1 - \frac{\omega}{p_j^2} - \frac{\gamma_j^2}{4}},$$

$$f_{rj} = \sum_{s=1}^{n} \bar{v}_{rjs}^T q_s, \qquad f_{ij} = \sum_{s=1}^{n} \bar{v}_{ijs}^T q_s.$$

Constants $\mathbf{a}$ and $\mathbf{b}$ in the solution may be obtained from the initial conditions.

3. Numerical results

Solution of the complex eigenvalue problem is numerically the most difficult part of this method. In knowledge of complex eigenvectors and eigenvalues, frequencies, damping characteristics $\frac{\gamma'_j}{2}$ and solutions meeting the initial conditions can be determined as seen above.

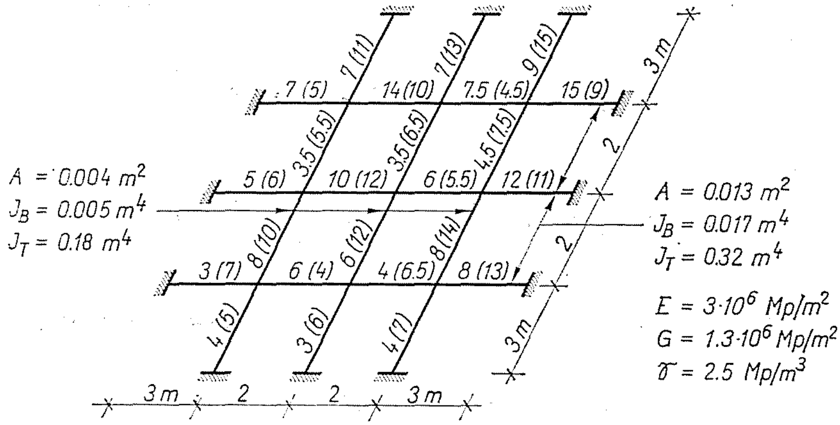

Fig. 1

Let us consider the grid system in Fig. 1 (where also the bar cross section areas, moments of inertia in bending and in torsion, as well as hundred times the γ values belonging to bending (torsion) for each bar have been indicated).

Let bar units have damping characteristics (γ_i values) different in bending and in torsion, ranging from $3_{10} - 2$ to $15_{10} - 2$, or have a uniform value of 0.1 assigned. In this latter case, solution of the complex eigenvalue problem will yield real eigenvectors, and the γ'_j values will be uniformly 0.1. Performing the computation by means of the checked algorithm, the case of mixed damping will feature the following:

a) The real part of eigenvectors is distorted compared to that of the identical damping (or undamped) case. Figs 2 and 3 show real parts of eigenvectors for the lowest and highest frequencies for $\gamma = 0.1$ and for mixed damping (underlined values). Distorsion is seen to be higher for higher frequencies (1% and 6%).

b) The imaginary part of eigenvectors is important at higher frequencies, 2% and 24% of the highest absolute valued real terms for the presented two modes.

c) γ'_j values belonging to particular vibration modes range between the two limits (0.03 to 0.015) indicated above, and have values from 0.055 to 0.085.

The numerical example shows the complex rigidity analysis to be viable for structures with components of mixed damping. In case of impulses and excited vibration, results obtained for the γ'_j values much affecting the amplitudes will correspond to the distribution of damping.

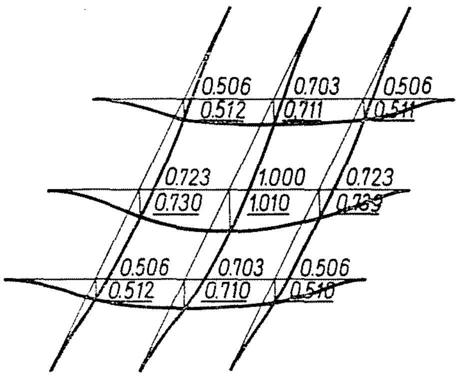

Fig. 2

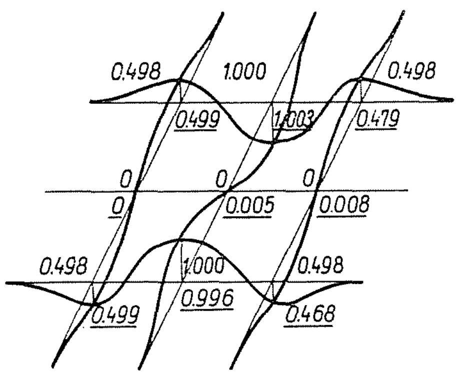

Fig. 3

Summary

An algorithm has been developed for structures with components of different damping characteristics exposed to free vibration, periodic excitation and impulse loads. Complex rigidity analysis — using the solution of a complex eigenvalue problem — gave results truly expressing the damping characteristics.

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