INTERPOLATION DEFLECTION OF THE VERTICAL BASED ON TORSION BALANCE RESULTS

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Introduction

Knowledge of the deflections of the vertical helps to solve two great problems: partly it yields important data for the precise determination of the geoid, partly gives information to geoscientists about the distribution of underground, covered masses. In both instances a very dense net of deflections of the vertical is necessary. Astrogeodetic determination of deflections of the vertical is extremely lengthy and expensive, therefore in practice a sparser net of astronomical stations has to be put up with and this astrogeodetic net is interpolated by different methods.

Interpolation is either based on gravity anomalies (gravimetric interpolation method) or on the curvature deviations of the gravity level surface, or by other methods.

In the following, the interpolation deflections of the vertical measured by torsion balance curvature data will be discussed.

Principle of the interpolation method

In the following, the fundamental conceptions concerning deflections of the vertical and quantities of torsion balance observations are supposed as known and — omitting mathematic deductions — only the fundamental relationships are written.

Our computations are related to a Cartesian co-ordinate system the origin of which is an arbitrary point inside the area to be investigated and the axes $+x$ and $+y$ point to the north and to the west, respectively, in the horizontal plane of the origin. If the investigated area is not too large, of an extension of 0.5° by 0.5° at maximum, a uniform co-ordinate system may be used for the entire area [1].

The computation of deflection of the vertical is performed along the adjacent triangles. Let section between points P_1 and P_2 be one side of an arbitrary triangle (Fig. 1), and the two deflection components ξ_1 and η_1 in point P_1; ξ_2 and η_2 in point P_2 — their difference being: $\Delta\xi_{21} = \xi_2 - \xi_1$ and $\Delta\eta_{21} = \eta_2 - \eta_1$.

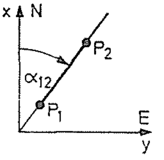

Fig. 1

Now a very simple relationship can be written for relating the change of the deflection components between two points and the curvature gradients of the torsion balance observations:

$$\Delta\xi_{21} \sin \alpha_{12} - \Delta\eta_{21} \cos \alpha_{12} =$$
$$= \frac{n_{12}}{4g} \{[(W_\Delta - U_\Delta)_1 + (W_\Delta - U_\Delta)_2] \sin 2\alpha_{12} +$$
$$+ [2(W_{xy} - U_{xy})_1 + 2(W_{xy} - U_{xy})_2] \cos 2\alpha_{12} \tag{1}$$

where n_{12} is the distance between points P_1 and P_2, g is the value of the average gravity acceleration between these two points,

$$W_\Delta = \frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2} \quad \text{and} \quad W_{xy} = \frac{\partial^2 W}{\partial x \partial y}$$

are characteristics for the curvature gradients of the level surface determined by torsion balance; U_Δ and U_{xy} are the curvature gradients in the normal gravity field of the Earth.

Two further relations can be written, similarly to (1) if also a third point P_3 is given, forming a triangle with the previous points P_1 and P_2.

Since around the triangle formed by points P_1 P_2 P_3 the total deflection component difference has to be zero, beside the three relationships type (1), two more can be written:

$$\Delta\xi_{21} + \Delta\xi_{32} + \Delta\xi_{13} = 0$$
$$\Delta\eta_{21} + \Delta\eta_{32} + \Delta\eta_{13} = 0. \tag{2}$$

So in any singular triangle there are six unknowns: $\Delta\xi_{21}$, $\Delta\xi_{32}$, $\Delta\xi_{13}$, $\Delta\eta_{21}$, $\Delta\eta_{32}$, $\Delta\eta_{13}$ for which, according to the above, five independent equations can be written (three of type (1) and two of type (2)).

Let us now investigate the interpolation net in Fig. 2, consisting of n points.

The n points form a chain of altogether $n-2$ triangles with $2n-3$ sides, with two unknown deflection component differences along each, hence for the entire net there are $4n-6$ unknowns. At the same time for the $n-2$ triangles $2n-3$ equations type (1) and $2n-4$ ones type (2) can be written, so that for the $4n-6$ unknowns there are altogether $4n-7$ equations. For an unambiguous solution of the problem, a further information, independent of the former, is needed.

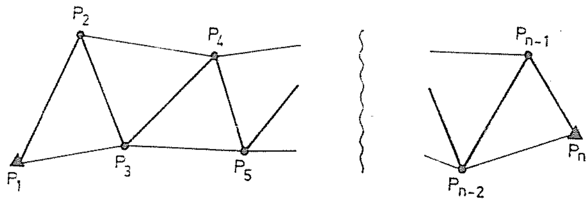

Fig. 2

If the value of the deflection component difference between two extreme points is known, then — as this is a further independent information — the system of $4n-6$ equations can be solved by any of the procedures below:

(a) The complete coefficient matrix belonging to the $4n-6$ unknowns is inverted ([1], [2], [3], [4]).

(b) Only the submatrix belonging to the wanted $2n-2$ unknowns of the above coefficient matrix has to be inverted.

(c) The unknowns are determined step by step [5].

In the following, only method (b) will be discussed.

Practical solution of the interpolation

According to the fundamental idea of our procedure, the system containing the $4n-6$ unknowns is parted into two groups by eliminating the redundant unknowns. One group contains only the necessary unknowns (e.g. the net in Fig. 2 only the component differences pertaining to sides P_1P_2, P_2P_3, $P_3P_4 \ldots$) — while the other group the redundant unknowns (i.e. the component differences for the remaining sides P_1P_3, P_2P_4, $P_3P_5 \ldots$ in Fig. 2). The system containing the second group of unknowns can be ignored further on, and only the coefficient matrix of the equation system concerning the necessary unknowns has to be inverted. This is, however, only of size $(2n-2)$ by $(2n-2)$ essentially smaller than the coefficient matrix $(4n-6)$ by $(4n-6)$ under (a).

Equations needed to be written according to the above method are outlined in the following:

Looking again at Fig. 2, for the first triangle ($P_1P_2P_3$), eliminating the values $\Delta\xi_{31}$ and $\Delta\eta_{31}$ from Eqs (1) and (2):

$$g\Delta\xi_{21}\sin\alpha_{12} - g\Delta\eta_{21}\cos\alpha_{12} = T_{12} \tag{3}$$

$$g\Delta\xi_{32}\sin\alpha_{23} - g\Delta\eta_{32}\cos\alpha_{23} = T_{23} \tag{4}$$

$$-g\Delta\xi_{21}\sin\alpha_{31} + g\Delta\eta_{21}\cos\alpha_{31} - g\Delta\xi_{32}\sin\alpha_{31} + g\Delta\eta_{32}\cos\alpha_{31} = T_{31} \tag{5}$$

there are two more equations for each other triangle:

$$g\Delta\xi_{i+2,i+1}\sin\alpha_{i+1,i+2} - g\Delta\eta_{i+2,i+1}\cos\alpha_{i+1,i+2} = T_{i+1,i+2} \tag{6}$$

$$-g\Delta\xi_{i+1,i}\sin\alpha_{i+2,i} + g\Delta\eta_{i+1,i}\cos\alpha_{i+2,i} - g\Delta\xi_{i+2,i+1}\sin\alpha_{i+2,i} + g\Delta\eta_{i+2,i+1}\cos\alpha_{i+2,i} = T_{i+2,i} \tag{7}$$

where: $i = 2, 3, 4, \ldots, n-2$.

$T_{i,j}$ in Eqs (3) to (7) stands for the right-side term of Eq. (1).

These are altogether $2n-3$ equations with $2n-2$ unknowns, to which other equation(s) can be written, if the deflection components ξ and η in the end points of the interpolation net are known.

As along the interpolation line the separate sums of each of the two deflection component differences have to equal the deflection component differences at the end points, two further so-called condition equations can be written:

$$\xi_n - \xi_1 = \sum_{i=1}^{n-1} \Delta\xi_{i+1,i}\,, \tag{8}$$

$$\eta_n - \eta_1 = \sum_{i=1}^{n-1} \Delta\eta_{i+1,i}\,. \tag{9}$$

If only one deflection component is known at the end points, hence, if only one of the condition equations (8) and (9) can be given — then exactly as many equations can be written as many unknowns there are, therefore the system containing $2n-2$ unknowns can be solved unambiguously.

However, if both condition equations can be written, there are totally $2n-1$ equations, i.e. one more than the number of unknowns. In this case the operation is redundant, the most probable value of the unknowns is determined by adjustment.

Adjustment can be carried out in two different ways: either indirectly, by the known method (through setting up and solving the normal equations) or directly (by the orthonormalization method) [6].

The orthonormalization method was applied in our computations beside its other very favourable properties, chiefly because of a high numerical stability.

For the interpolation procedure described, a computer program has been developed. The programs were written in the ALGOL 60 language for a computer type ODRA—1204.

The operations of the program can be outlined as follows:

1. Reads the number of net points and also the deflection values of ξ and η in the initial and end points of the chain.
2. Reads the serial number, the co-ordinates x and y, as well as the data W_Δ and W_{xy} for each point of the net.
3. Using relationships (3) to (9), produces the so-called enlarged coefficient matrix of the pertaining equation system and the vector of absolute terms.
4. Orthogonalizes the enlarged coefficient matrix according to (6) resulting for each net point in the deflection component values ξ and η, and their mean square errors.
5. Prints the input data and the above determined values.

Data of the experimental computations

For practically testing the method, the plane and hilly 1200 km^2 territory in Fig. 3 surveyed in detail by torsion balance observations was chosen, including three adjacent points (1, 2, 3) of the first order triangulation net of Hungary — with an average spacing of 40 km. For these points the values of both the relative and the absolute deflection components were available, based on astrogravimetric data. In the shown area the deflections of the vertical between the three astrogeodetic points were interpolated based on torsion balance observations. For comparing or checking the results, the astrogravimetrically determined values for the three points (13, 14, 27) inside the area were used.

It appears from Fig. 3 that the torsion balance observation points were not uniformly distributed, observations were more concentrated in areas of increased gradients, in "more perturbed" areas. This has an importance in the development of interpolation nets, because the nets on territories of increased gradients are advisably established so that the change of the second potential derivatives of two adjacent points can still be considered as linear. (This approximation was utilized to deduce relationship (1)).

Our experimental computations were carried out along the three principal interpolation lines, marked by a heavy line, the further six smaller concentration chains in Fig. 3 were established to calculate geoid undulations.

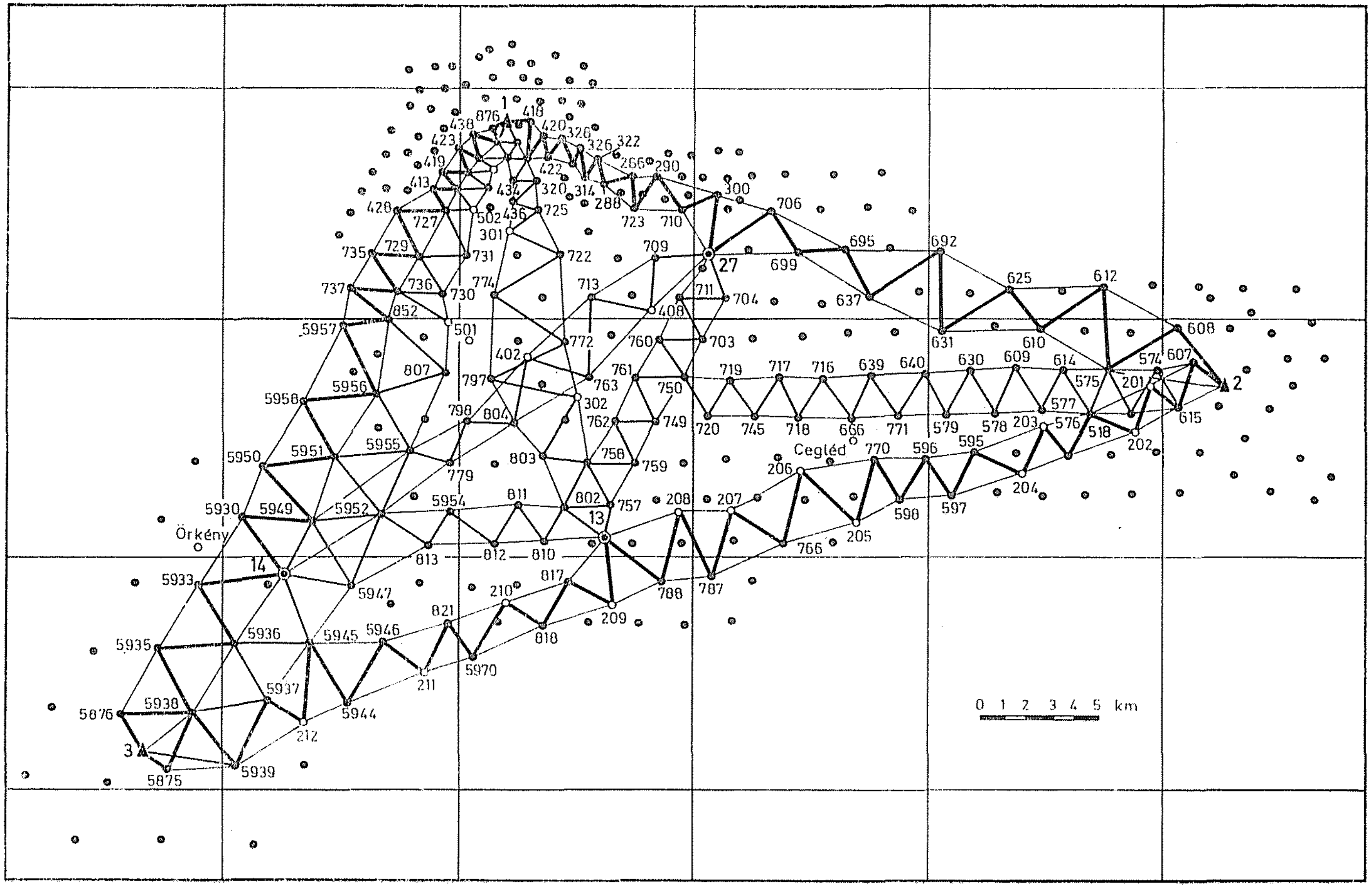

Fig. 3

Summary of the experimental computations

Results of the described computations are shown in Fig. 4. Arrows represent vectors, which may be considered either horizontal force components, or direct deflection values $\vartheta = \sqrt{\xi^2 + \eta^2}$. (Former differ from latter only by multiplication by vector g).

In our interpretation also the deflection values may be taken as vectors considering the direction from the ellipsoid zenith toward the astronomical one as positive, and the absolute value of the deflection in the given point as the vector length.

In this way, applying the adequate scale, both the deflection values and the horizontal force components can be read off the same figure.

The deflection values are marked by heavy lines (vectors) in the initial and check points, the interpolated values are plotted with thin lines.

It can be stated that except for point 27 the given deflection values in the check points agree with the interpolated values within the graphic accuracy.

In connection with Fig. 4 it should be noted that in order to establish a completer picture of the whole territory, in addition to the main interpolation lines, computations have also been made for six smaller chains plotted between the check points in Fig. 3, the results of which are also shown here.

Deviations in the check points are summarized in the following table:

mark	check		interpolated		difference	
	ξ''	η''	ξ''	η''	ξ''	η''
27	+4.02	+4.05	+5.05	+2.99	−1.03	+1.06
13	+5.31	+3.12	+5.52	+3.64	−0.21	−0.52
14	+5.27	+3.30	+6.01	+3.49	−0.74	−0.19

Mean square error of interpolated deflection components from the deviations:

$$\mu_\xi = \pm 0.74'' \qquad \text{and} \qquad \mu_\eta = \pm 0.67'' .$$

To check reliability of our computations, a further investigation was carried out. Namely between the astrogeodetic points marked 3 and 1 two interpolation chains were established as seen in Fig. 3. Chain "A" was plotted between the points $3 \rightarrow 5876 \rightarrow 5938 \rightarrow 5935 \rightarrow \ldots \rightarrow 1-$, "B" between points $3 \rightarrow 5939 \rightarrow 5938 \rightarrow 5937 \rightarrow \ldots \rightarrow 1$. Chains 3—1/A and 3—1/B are seen in Fig. 3 to have fourteen common points along the corresponding sides.

Theoretically, identical deflection values should be obtained with the same interpolation method in the corresponding points of the two separate

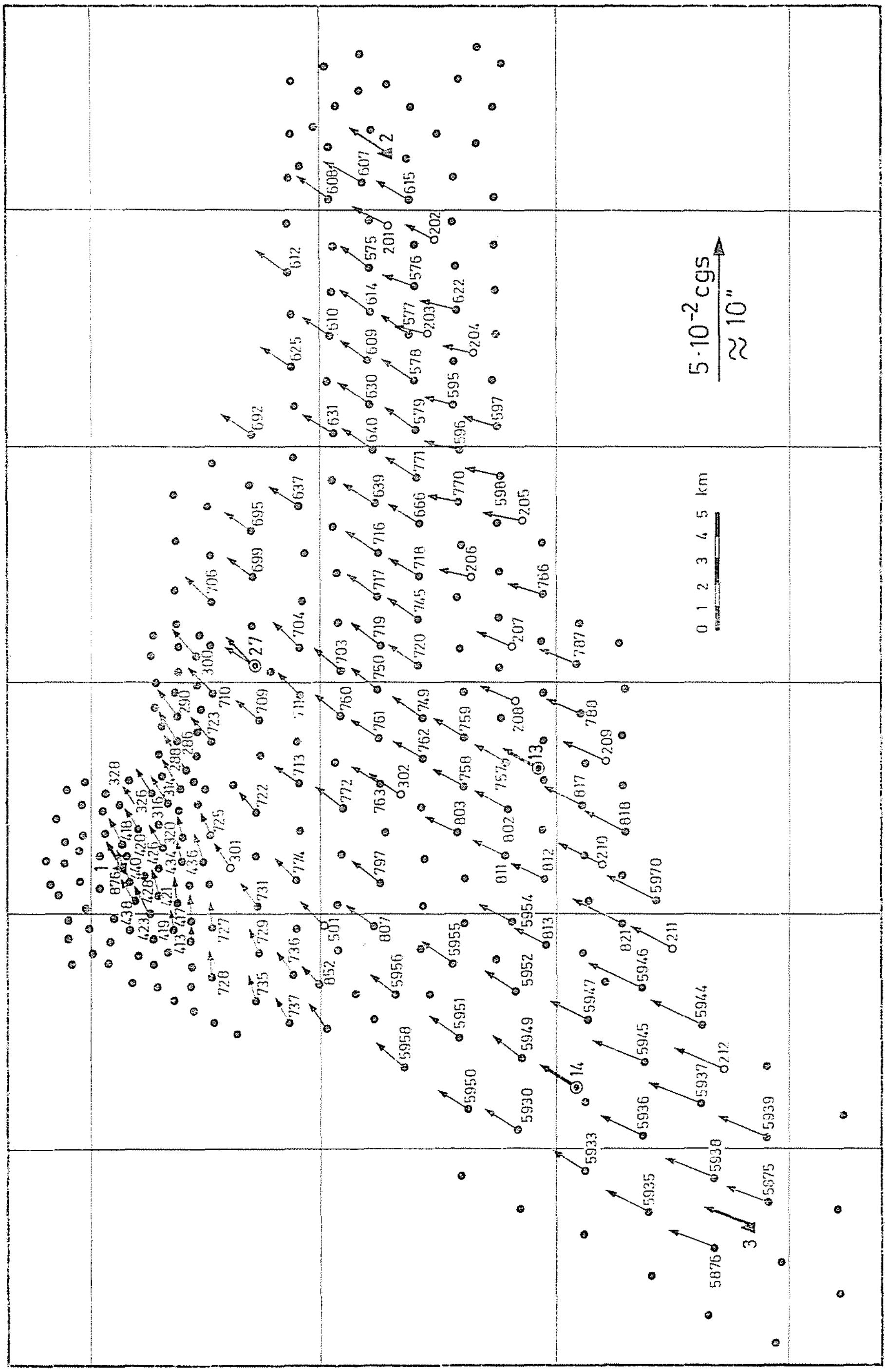

Fig. 4

chains. In reality, however, these components differ by a certain value from each other. The magnitude of the difference characterizes the reliability of the method.

The deflection components computed for the two adjacent chains are shown in Fig. 5.

The corresponding components agree very well, differences exceed the value ±0.5″ only in four points.

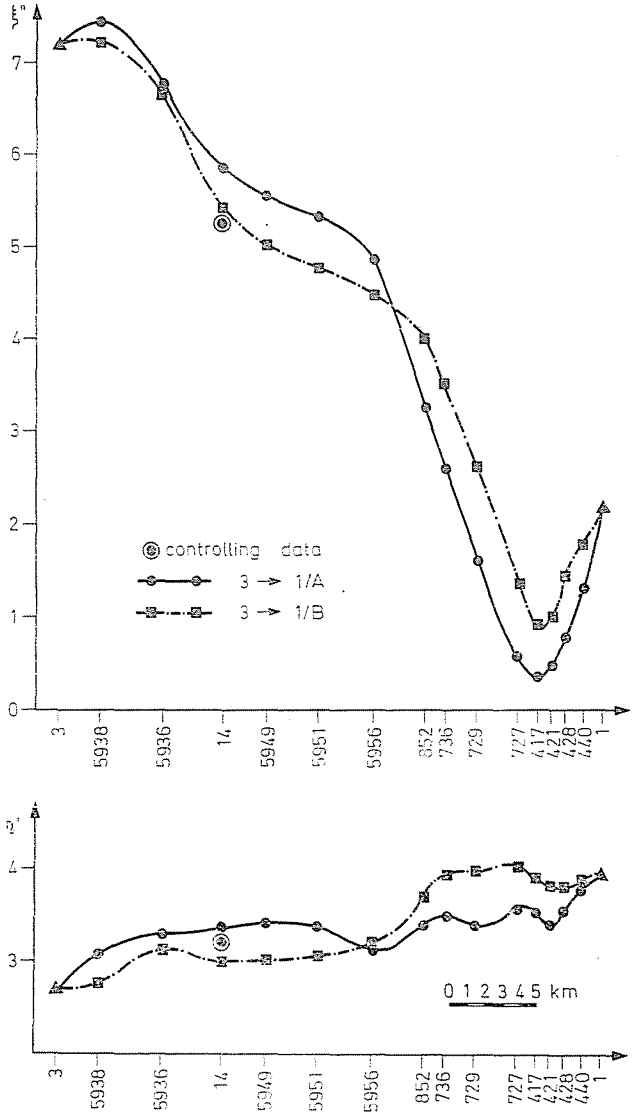

Fig. 5

At the same time, except for two net points, the results show the confidence of the interpolated deflection component values to be the smallest about midchain. The confidence of these points can be increased by reducing the distance between the interpolation end points, as a consequence, in areas where a higher density of deflection values obtained by astrogeodetic measurements is available, confidence of the interpolated values is higher.

Summarizing the above, it can be stated that the mean square error of interpolated deflection components changes also within the same interpolation chain from point to point, this fluctuation is, however, slight enough and the mean square error remains in general below $\pm 0.7''$ if the circumstances agree with those prevailing in the test area.

Applications

As already mentioned, the deflections of the vertical help to solve two problems: partly they yield important data for the precise determination of the geoid, partly they give information about the distribution of subsurface masses. In the following only the geoid computations are discussed.

One of the common problems of geophysics and theoretical geodesy is the precise determination of the geoid, the mathematical form of Earth. Today, based on up-to-date measurement results, contour line maps of the main forms of the geoid for nearly the whole Earth surface are available — these maps, however, do not contain the "fine structure" of the geoid.

As known, there exists a definite mathematic relationship between the geoid undulations and the deflection values. Between any points P_i and P_{i+1}, the geoid height change is

$$\Delta N = \int_{P_i}^{P_{i+1}} (\xi \cos \alpha + \eta \sin \alpha) \, ds$$

where α stands for the azimuth of the line of length s connecting the two points.

If for any point of the investigated area the initial value of geoid height N_0 is known, further if adequately interpolated deflection values are available for this particular area, the detailed map of the geoid can be obtained for the given area.

Based on the above, computations were carried out for the experimental area shown in Fig. 3, using the deflection components interpolated previously.

As on this territory no geoid height related to the same reference surface as the given deflection components was available, the geoid height N_0 for the experimental area in Fig. 3 was chosen arbitrarily to be zero in the astrogeodetic point 1, to serve as basis for determining geoid heights of further points.

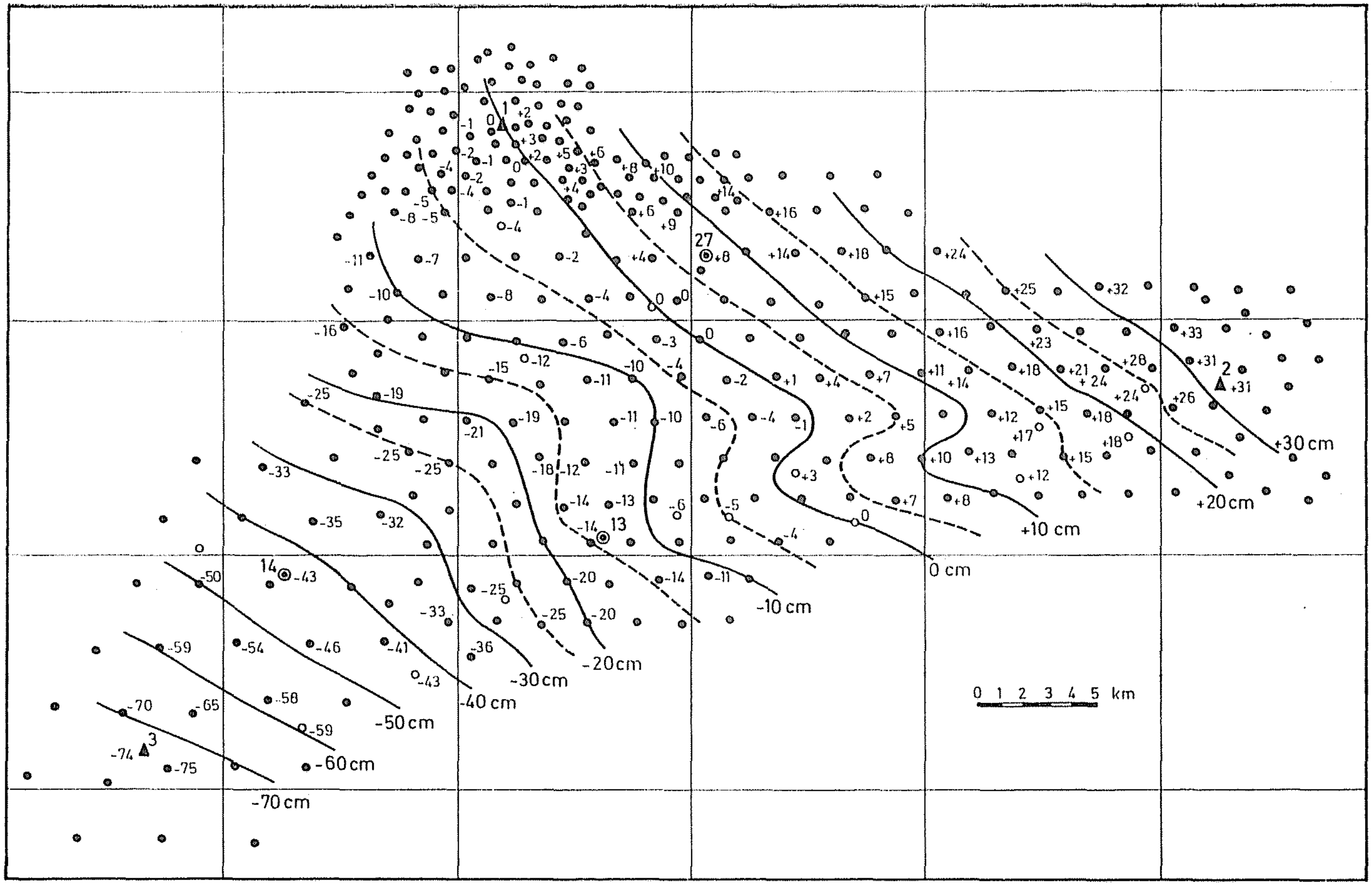
+30 cm
+20 cm
+10 cm
0 cm
-10 cm
-20 cm
-30 cm
-40 cm
-50 cm
-60 cm
-70 cm
0 1 2 3 4 5 km

Fig. 6

The mentioned geoid map is presented in Fig. 6. The map is in good agreement with the distribution of deflection vectors $\vartheta = \sqrt{\xi^2 + \eta^2}$ seen in Fig. 4.

It is characteristic for the accuracy of the computations that going along the chains $1 \rightarrow 2$, $2 \rightarrow 3$, $3 \rightarrow 1$, and returning to point 1, instead of the initial value $N_0 = 0$ cm, $N_0 = -7$ cm was obtained, — i.e. going around the chain length of about 115 km, a misclosure of only 7 cm was obtained. This misclosure is characteristic not only of the accuracy of the geoid height, but is at the same time an excellent possibility to check the confidence of the interpolated ξ and η values.

Therefore deflection values interpolated from torsion balance observations can be stated to give very economically highly reliable geoid maps which are most suitable to study local details.

Summary

The interpolation principle based on gravity gradient values of deflections of the vertical is outlined, then a very simple solution procedure is discussed, followed by description of the experimental computations carried out to test the method in practice. Results of the experimental computations show mean square errors below $\pm 0.7''$ of the deflection components interpolated in this way. Finally, attention is drawn to the fact that with this method, applying interpolated deflection values, very detailed and exact geoid maps can be prepared.

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* In Hungarian.