SOME PROBLEMS OF THE NUMERICAL ANALYSIS OF ELLIPTIC PARABOLOID SHALLOW SHELLS*

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1. Introduction

Loads, displacements and membrane forces (stress functions) of shallow shells are related by the Wlassow—Marguerre differential equations:

$$\begin{aligned} D\Delta\Delta w - \Delta_p F &= Q \\ \Delta\Delta F + Eh\Delta_p w &= 0 \end{aligned} \tag{1}$$

where:

$Q = Q(x, y)$, $w = w(x, y)$ and $F = F(x, y)$ are load, vertical displacement and stress function, respectively,

h shell thickness;

E Young's modulus;

ν Poisson' ratio;

$D = \dfrac{Eh^3}{12(1-\nu^2)}$ plate rigidity modulus;

$z = z(x, y)$ function of the unloaded shell median surface;

$\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ Laplace operator;

$\Delta_p = \dfrac{\partial^2 z}{\partial x^2}\dfrac{\partial^2}{\partial y^2} = 2\dfrac{\partial^2 z}{\partial x \partial y}\dfrac{\partial^2}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2}\dfrac{\partial^2}{\partial x^2}$ shell operator.

Wlassow equations permit to deduce both deflection and stress function to load relationships:

$$\left.\begin{aligned} D\Delta\Delta\Delta\Delta w + Eh\Delta_p\Delta_p &= \Delta\Delta Q \\ \frac{K}{Eh}\Delta\Delta\Delta\Delta F + \Delta_p\Delta_p F &= -\Delta_p Q \end{aligned}\right\} \tag{2}$$

In case of an elliptic paraboloid shell (Fig. 1):

$$\frac{\partial^2 z}{\partial x^2} = k_2 \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = k_1, \quad \frac{\partial^2 z}{\partial x \partial y} = 0,$$

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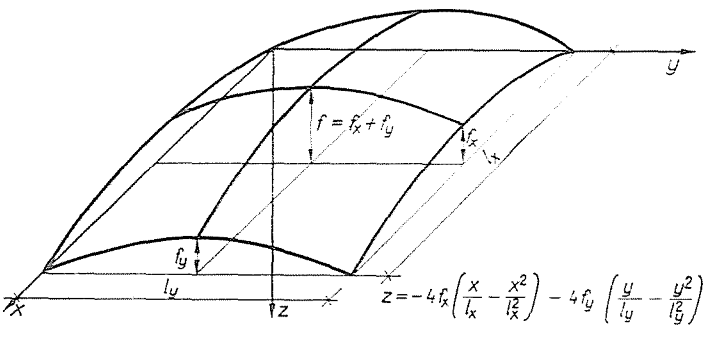


Fig. 1

the shell operator simplifies to:

$$\Delta_p = k_1 \frac{\partial^2}{\partial x^2} + k_2 \frac{\partial^2}{\partial y^2} \,. \tag{3}$$

In this case inhomogeneous partial differential equations (2) of order eight include only difference operators of even order, permitting them to be expediently solved by the tensor product variant of the finite difference method based on the known spectral form $\mathbf{C} = \mathbf{ULU}$ of the second-order difference operator matrix $\mathbf{C}$ [1].

Nonlinear equations of shallow shells (taking into consideration:

a) the effect of deflection on the forces and reactions, and

b) the square terms in the geometric equations)

include an excess term each compared to the Wlassow equations:

$$\left.\begin{aligned} D\Delta\Delta w - \Delta_p F &= Q + \Delta_k F \\ \frac{1}{Eh}\Delta\Delta F + \Delta_p w &= -\frac{1}{2}\Delta_k w \end{aligned}\right\} \tag{4}$$

where:

$$\Delta_k = L(w,\,) = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2}{\partial y^2} - 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2}{\partial x^2}$$

the Kármán operator.

(The two symbols for operator Δ_k have been introduced for unambiguously describing the iteration procedures.)

Numerical analysis of excess terms (2) demonstrated maxima of excess terms of both equilibrium and compatibility equations to be in the centre and in corner points along the shell ground plan, in conformity with available observation on shell corner disturbances. In the function of relative rise (rise to shell thickness), the compatibility equation and the equilibrium equations have extreme values for a flat slab ($f = 0$), and in the vicinity of unit relative rise ($f = h$), respectively.

2. Solution of non-linear equations

Non-linear equations of the shallow shell (4) are of degree two in the derivatives of the deflection and stress function, hence no explicit writing of deflection to load or stress function to load relationships is possible. Equations are advisably solved by iteration, with initial values $w_0 = 0$, $F_0 = 0$. Iteration formulae are:

$$\left.\begin{aligned} K\Delta\Delta_{i+1} - \Delta_p F_{i+1} &= Q + L(w_i, F_i) \\ \frac{1}{Eh}\Delta\Delta F_{i+1} + \Delta_p w_{i+1} &= -\frac{1}{2}L(w_i, w_i) \end{aligned}\right\} \qquad (5)$$

Here again, equations of order eight are expedient:

$$\left.\begin{aligned} K\Delta\Delta\Delta\Delta w_{i+1} + Eh\Delta_p\Delta_p w_{i+1} &= \Delta\Delta Q + \Delta\Delta L(w_i, F_i) - \frac{Eh}{2}\Delta_p L(w_i, w_i) \\ \frac{K}{Eh}\Delta\Delta\Delta\Delta F_{i+1} + \Delta_p\Delta_p F_{i+1} &= -\Delta_p Q - \Delta_p L(w_i, F_i) - \frac{K}{2}\Delta\Delta L(w_i, w_i) \end{aligned}\right\} \qquad (6)$$

Computation experience showed iteration to rapidly converge for small loads, slowing down with increasing loads and changing to divergence after a limit value of load.

Possibilities to accelerate convergence have been examined. Difference operator applied for determining excess term values was seen from computations to significantly affect the iteration convergence. Three different difference patterns have been applied for the second derivative, namely those involving three, five and nine points (Fig. 2). Numerical outputs showed the nine-point pattern to be the most adequate (Fig. 3). Namely it is most likely that, while the nine-point pattern tends to the exact result from above [3], the three-point operator applied in deflection computations approximates from below, causing the two kinds of error to about compensate each other.

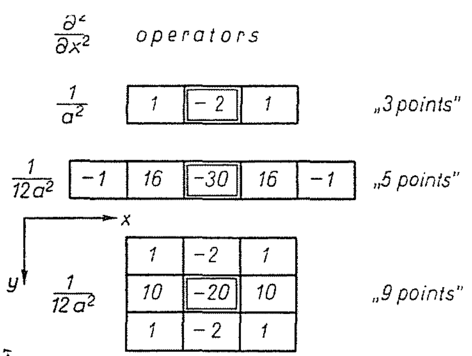


Fig. 2

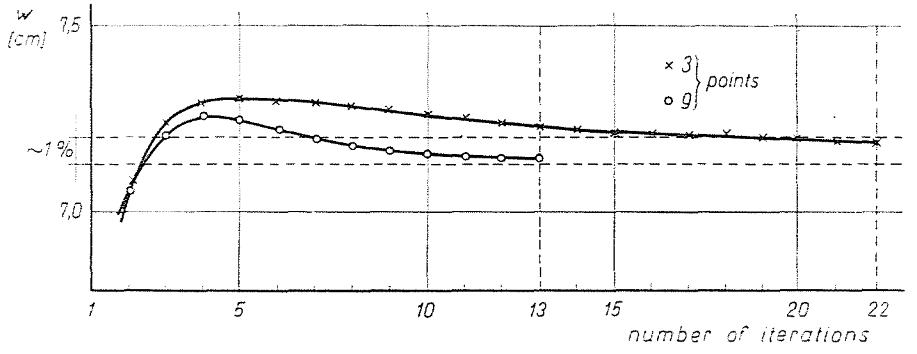


Fig. 3

3. Numerical examples and conclusions

An ALGOL program has been made, applying this computation method, permitting great many computations according to both the linear and the non-linear theory.

Surfaces under moment and membrane force of the elliptic paraboloid shell of the ground plan shown in Fig. 4 ($f_x = f_y = 0.5$ m, $h = 0.06$ m, $v = 0$, $E = 250\,000$ kp/cm^2, uniform load $p = 0.1$ Mp/m^2) are seen axonometrically but heavily distorted in Fig. 5. Figure scales differ as indicated. For a better survey, only quarter surfaces have been traced.

Deflections and stresses of axes of symmetry of two-way symmetric shells of different relative rises, 10 by 10 m in ground plan and 10 cm in thickness, are seen in Fig. 6. Results according to non-linear and linear theories are shown in heavy and thin lines, respectively. Stresses due to membrane forces have been plotted in continuous lines, and those due to bending in dashed lines. The load is uniform but of different value for each case, about 95% of the load causing the shell form to diverge.

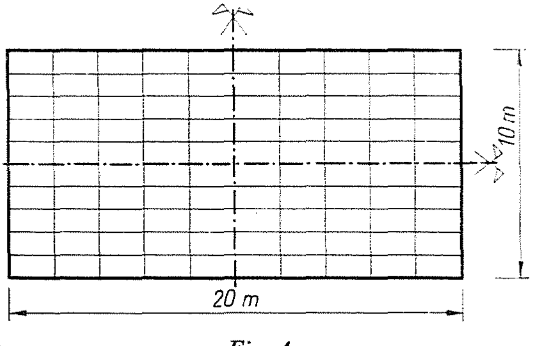
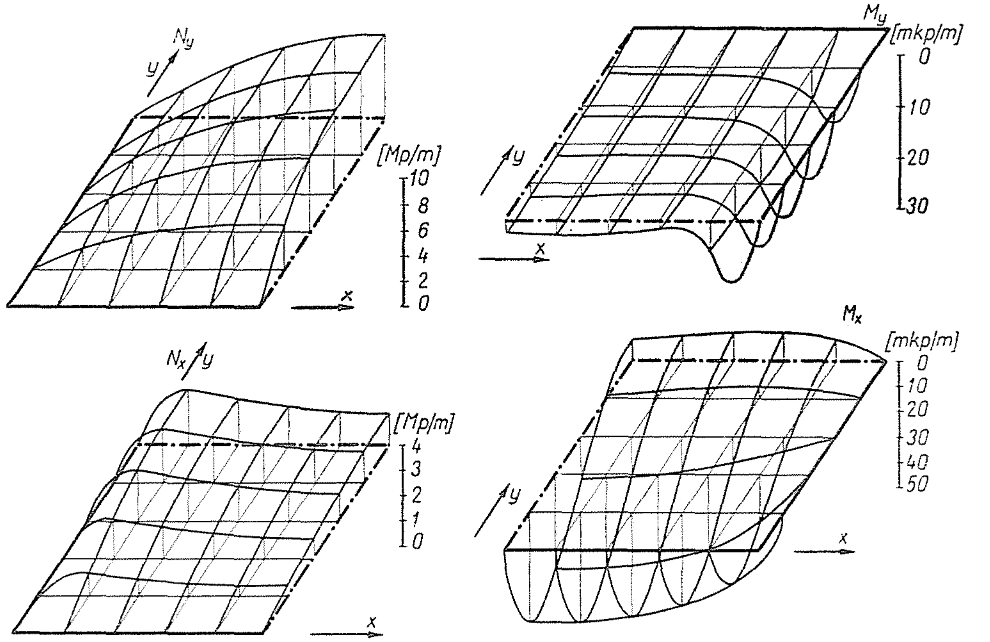
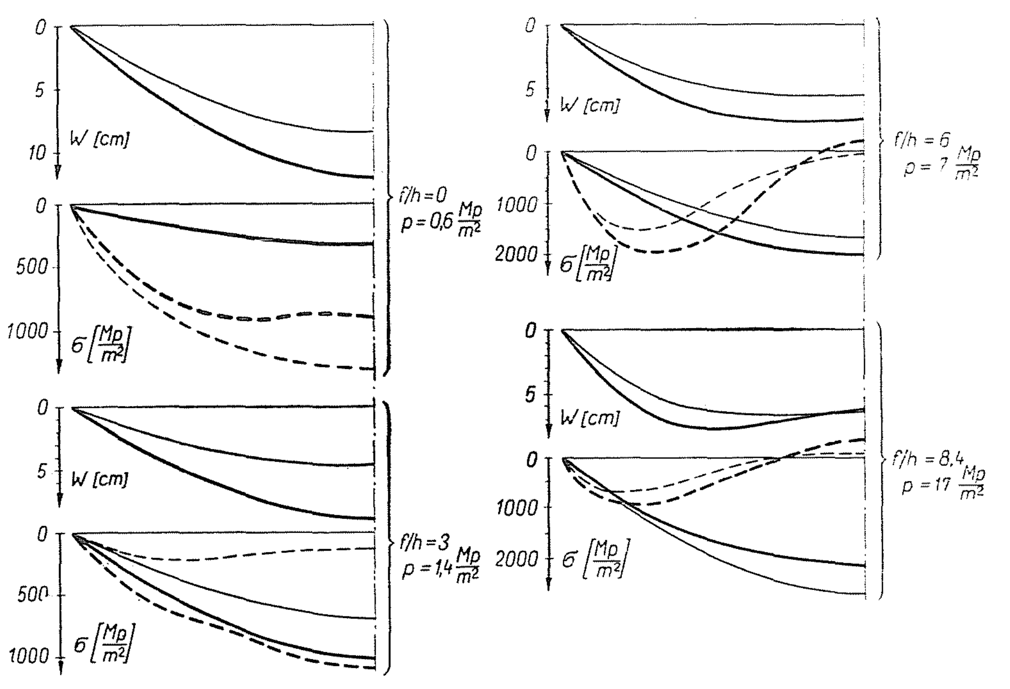


Fig. 4



N_y
y
[Mp/m]
10
8
6
4
2
0
x
M_y
[mkp/m]
0
10
20
30
y
x
N_x
y
[Mp/m]
4
3
2
1
0
x
M_x
[mkp/m]
0
10
20
30
40
50
y
x

Fig. 5



W [cm]
0
5
10
σ [Mp/m²]
0
500
1000
f/h = 0
p = 0,6 Mp/m²
W [cm]
0
5
σ [Mp/m²]
0
500
1000
f/h = 3
p = 1,4 Mp/m²
W [cm]
0
5
σ [Mp/m²]
0
1000
2000
f/h = 6
p = 7 Mp/m²
W [cm]
0
5
σ [Mp/m²]
0
1000
2000
f/h = 8,4
p = 17 Mp/m²

Fig. 6

Flexural and membrane stresses of the mid-point of a shell 10 by 10 cm in ground plan and 10 cm thick are seen in Fig. 7 to lin—log scale, linear and non-linear theory results being plotted in thin and heavy lines, respectively. Bending and membrane stresses are to the left and to the right from the vertical axis, respectively. The linear case refers to unit load, while in the non-linear case pre-critical load effect has been reduced to unit load.

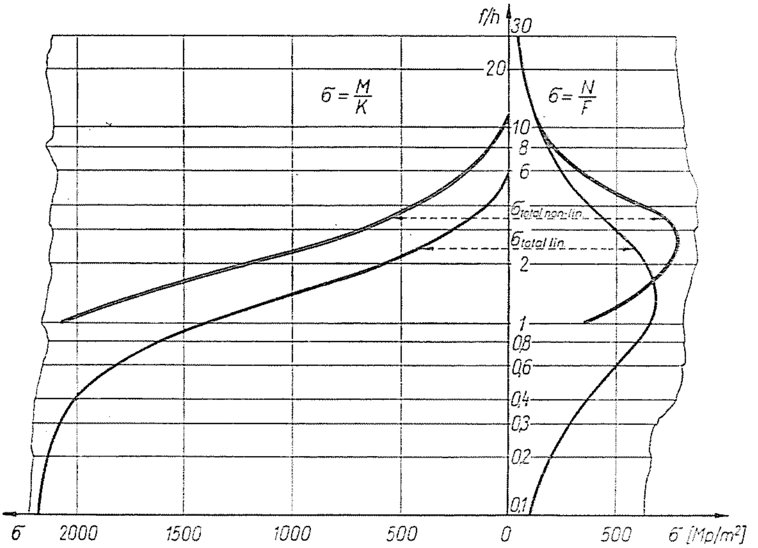


Fig. 7

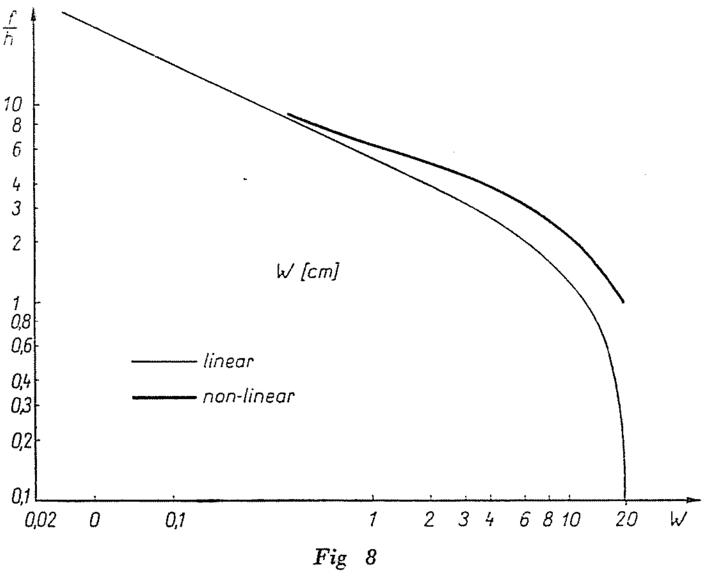


Fig 8

Also Fig. 8 shows mid-point deflections of a shell of the above geometry, but to log–log scale, plotted as indicated above, reduced to unit load. Numerical examples permitted to conclude for shells of double curvature:

— In a partial range ($f < l_{min}/5$) of the range of validity of shallow shell equations ($f/h > 6$) membrane stresses are prevalent in the shell. Hence, in this range the shell is advisably analysed as a membrane shell, considering bending effects as boundary disturbances.

— Geometric non-linearity is manifest only for high stresses of the order of ultimate concrete stresses. Even for such high stresses, the effect of geometric non-linearity has only to be considered for $2 < f/h < 8$. Neglect of geometric non-linearity for $f/h > 8$ is on the side of safety.

4. Determination of critical load

In case of small loads, the iteration procedure in the previous item is rapidly converging. With increasing loads convergence slows down to change to divergence at a given limiting value.

The critical load causing divergence will be determined by a geometrical method based on energy considerations. Changing the one-parameter load in small increments, deflection field w_i, w_{i+1} ... will be determined for successive loads $\alpha_i q$, $\alpha_{i+1} q$ ... 2. In knowledge of deflection field and load, work done by the load in each load step can be determined (Fig. 9):

$$\Delta L_{i+1} = L_{i+1} - L_i = \frac{1}{2} \int_F (\alpha_{i+1} + \alpha_i)\, q(w_{i+1} - w_i) dF . \tag{7}$$

Shell equilibrium is considered as stable when the sign of load parameter change is the same as that of the load potential energy change, that is:

$$\Delta L_{i+1}(\alpha_{i+1} - \alpha_i) > 0. \tag{8}$$

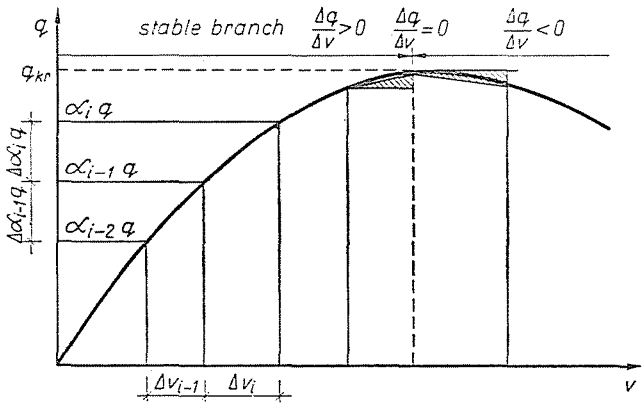


Fig. 9

Shell equilibrium is unstable when the sign of load parameter change differs from that of the load potential energy change:

$$\Delta L_{i+1}(\alpha_{i+1} - \alpha_i) < 0 \tag{9}$$

(with other words, deflection increase is concomitant to the decrease of equilibrium load parameter).

With notations in Fig. 9, the critical load (extreme value of the load–load potential energy function) will be at:

$$\frac{\Delta \alpha_i}{\Delta v_i} = 0\,. \tag{10}$$

5. Total stiffness matrix of shallow shells

The outlined computation method holds only up to the critical load. Further analyses require the knowledge of the so-called total stiffness matrix of the shell.

Shell operator

$$\Delta_p = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2}{\partial y^2} - 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2}\frac{\partial^2}{\partial x^2}$$

and Kármán operator

$$\Delta_k = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2}{\partial y^2} - 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2}{\partial x^2}$$

being of perfectly identical structure, non-linear differential equations of shallow shells (4) can be written as:

$$\left.\begin{aligned} & D\Delta\Delta w - L(z + w, F) = Q \\ & \frac{1}{Eh}\Delta\Delta F + L\left(z + \frac{1}{2}w, w\right) = 0 \end{aligned}\right\}. \tag{11}$$

When applying the difference method, the differential operators have been approximated by matrix difference operators. Let us denote

matrix $\mathbf{A}$ the matrix difference operator corresponding to biharmonic operator;

matrix $\mathbf{B}$ the matrix difference operator corresponding to shell operator;

matrix $\mathbf{C}$ the matrix difference operator corresponding to the Kármán operator $\mathbf{C} = \mathbf{C}(\mathrm{w})$.

At a difference from matrices $\mathbf{A}$ and $\mathbf{B}$, matrix $\mathbf{C}$ depends on $\mathbf{w}$, hence continuously varies during the loading-deformation process. With the introduced notations, the non-linear shallow shell equations can be concisely written as:

$$\left[\begin{array}{c|c} K\mathbb{A} & -(\mathbb{B}+\mathbb{C}) \\ \hline \frac{1}{Eh}\left(\mathbb{B}+\frac{1}{2}\mathbb{C}\right) & \mathbb{A} \end{array}\right]\begin{bmatrix} w \\ F \end{bmatrix}=\begin{bmatrix} Q \\ 0 \end{bmatrix}. \tag{11}$$

Expressing stress function from the second row and substituting it in the first row:

$$\left[K\mathbb{A}+(\mathbb{B}+\mathbb{C})Eh\mathbb{A}^{-1}\left(\mathbb{B}+\frac{1}{2}\mathbb{C}\right)\right]w=\mathbb{K}w=Q \tag{12}$$

where the term in square brackets is the total stiffness matrix of shallow shells. Expanded and arranged:

$$\mathbb{K}=\underbrace{K\mathbb{A}}_{\alpha}+\underbrace{Eh\,\mathbb{B}\mathbb{A}^{-1}\mathbb{B}}_{\beta}+\underbrace{Eh\left(\mathbb{C}\mathbb{A}^{-1}\mathbb{B}+\frac{1}{2}\mathbb{B}\mathbb{A}^{-1}\mathbb{C}+\frac{1}{2}\mathbb{C}\mathbb{A}^{-1}\mathbb{C}\right)}_{\gamma} \tag{13}$$

where:

α plate effect,

β shallow shell effect,

γ effect of geometric non-linearity.

Term γ varies throughout the loading-deformation process, $\mathbb{C}$ depending on w.

6. Post-critical analysis

Introducing notations

$$\mathbb{X}=K\mathbb{A}+Eh\,\mathbb{B}\mathbb{A}^{-1}\mathbb{B} \tag{14}$$

$$\mathbb{Y}=Eh(\mathbb{C}\mathbb{A}^{-1}\mathbb{B}+\frac{1}{2}\mathbb{B}\mathbb{A}^{-1}\mathbb{C}+\frac{1}{2}\mathbb{C}\mathbb{A}^{-1}\mathbb{C}), \tag{15}$$

Eq. (13) simplifies to:

$$(\mathbb{X}+\mathbb{Y})w=Q.$$

In the case of "direct iteration" (conditioned by the fulfilment of non-singularity of matrix $\mathbb{X}$ from strength reasons) two subsequent steps of iteration are:

$$w_{i+1}=\mathbf{X}^{-1}q-\mathbf{X}^{-1}\mathbf{Y}_i w_i \tag{16}$$

$$w_i=\mathbf{X}^{-1}q-\mathbf{X}^{-1}\mathbf{Y}_{i-1}w_{i-1}.$$

From their difference:

$$w_{i+1} - w_i = \mathbf{X}^{-1}(\mathbf{Y}_i w_i - \mathbf{Y}_{i-1} w_{i-1}). \tag{17}$$

Provided iteration started from the stable range of the load-displacement relationship, in case of sufficiently small steps

$$\mathbf{Y}_i \approx \mathbf{Y}_{i-1} \quad \text{and} \quad \| \mathbf{Y}_i \| \geq \| \mathbf{Y}_{i-1} \|$$

may be assumed, permitting (17) to be re-written as;

$$(w_{i+1} - w) \approx \mathbf{X}^{-1}\mathbf{Y}_i(w_i - w_{i-1}) \tag{18}$$

and iteration convergence has the condition $\| \mathbf{X}^{-1}\mathbf{Y}_i \| < 1$, stricter than the real condition of convergence because of (18). In post-critical range this condition is not met, requiring a different method, that of step-by-step loading, to be applied. Indicating below the load step by a superscript, a load step involves the following variations:

$$\begin{aligned} w^{(i+1)} &= w^{(i)} + \delta w \\ Q^{(i+1)} &= Q^{(i)} + \delta Q \\ F^{(i+1)} &= F^{(i)} + \delta F \\ \mathbf{C}^{(i+1)} &= \mathbf{C}^{(i)} + \delta \mathbf{C} \end{aligned} \tag{19}$$

δ being the symbol of increment). Equilibrium equation for two subsequent oad steps:

$$(\mathbf{X} + \mathbf{Y}^{(i)} + \delta\mathbf{Y})\,(w^{(i)} + \delta w) = Q^{(i)} + \delta Q$$
$$(\mathbf{X} + \mathbf{Y}^{(i)})\,w^{(i)} = Q^{i},$$

deduced from each other:

$$\mathbf{X}\delta w + \mathbf{Y}^{(i)}\delta w + \delta\mathbf{Y} w^{(i)} + \delta\mathbf{Y}\delta w = \delta Q. \tag{20}$$

Considering the product of increments to be small of second order, and arranging:

$$\delta w = (\mathbf{X} + \mathbf{Y}^{(i)})^{-1}(\delta Q - \delta\mathbf{Y}_w^{(i)}) \tag{21}$$

a term lending itself to follow the overall load-deformation process. Stability or instability of the concerned range is seen from the sign of the energy increment variation:

$\delta E = \delta w \cdot \delta Q > 0$ stable range,
$\delta E = \delta w \cdot \delta Q = 0$ critical load,
$\delta E = \delta w \cdot \delta Q < 0$ unstable range.

Computer development of the procedure presented in this item is being performed.

Summary

The tensor product variant of the method of finite differences has been developed into an iterative way of solution of the non-linear shallow shell equations. The critical load parameter is determined by a geometric-energetical procedure. Application fields both of shallow shell and membrane shell equations are concluded on from numerical examples. Discussion of the deduced total stiffness matrix of the shallow shell is followed by some considerations on its post-critical behaviour.

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* In Hungarian