NO-TENSION ANALYSIS. PARTICULAR METHODS
OF SOLVING PLAIN STRAIN PROBLEMS

By

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Introduction

The ultimate strength of rocks in compression is many times that in tension, therefore in rock mechanics calculations the material is often supposed to take no tension. Such an analysis in rock mechanics can be performed by various means depending upon the chosen rock-continuum model and upon the method of analysis.

One kind of procedure assumes the place of fissures and joints to be previously determined. Apart from this, the remainder of the bulk can be regarded as absolutely rigid. Obviously, this bulk can be chosen as bricks. Supposing fissures to be characterized by dislocations along the joints, a method of analysis can be developed which seems to be comparatively simple, omits the rather lengthy generation of stiffness matrices, and the analysis is reduced to a linear complementary problem. It is mentioned as a disadvantage that the method is little appropriate for the case of complicated domains or fissure nets.

This drawback can be avoided by the second procedure, which works with general triangular finite elements. The material of the elements is considered elastic, apart from that they can bear no tension. For the generation of element stiffness matrices, a more general extremum principle, for instance the method of least squares, is used since this method can be extended to the investigation of seepage-pressure problems, too. The assumed no-tension is referred to the centre of gravity of each triangular element. Therefore the shape functions, valid for the finite elements, are selected so as to result in displacements at corners and gravity centres and also in derivatives of displacements at the corners. No-tension conditions are guaranteed in the main directions by solving a small linear complementary problem in each step by the complete describing method. This leads again and again to loads out of balance at nodal points, so the problem can be treated by the Newton—Raphson method. The geostatic pressure will be taken into account by an initial stress state.

The problem will be examined in connection with the altered conditions arising after the excavation of a pit (Fig. 1). This is obviously a plain strain

problem. The simplified network for the first procedure is seen in Fig. 2, and for the second one in Fig. 3.

The following problems will briefly be dealt with below:

a) Development of a linear complementary problem and the basis for the solution of the first model.

b) Generation of a refined stiffness matrix.

c) Elimination of tension in the case of the second model.

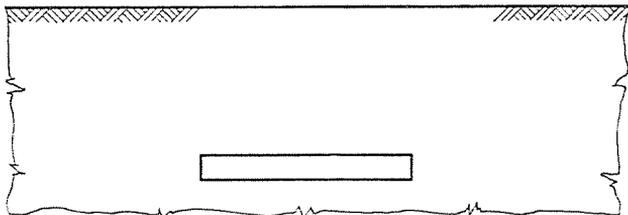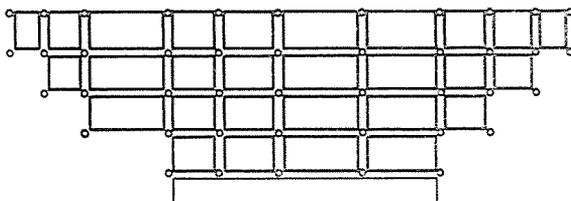d) Numerical example presenting input data and some features of the output.

Fig. 1

Fig. 2
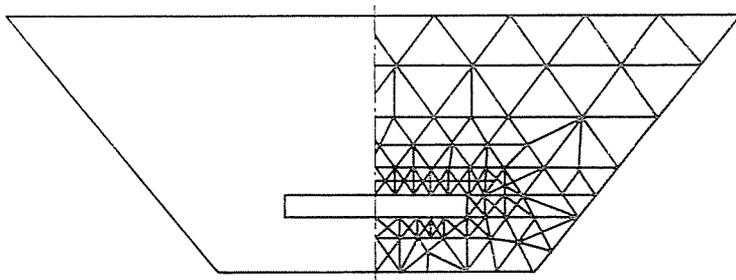

Fig. 3

Linear complementary problem related to the bulk model

Analysis of the bulk model is based upon relationships of rigid-body motion and the equilibrium of the bulk, as well as upon equilibrium at nodes. Vectors introduced in the theory are:

$$\mathbf{u}_i^T = [u_x \quad u_y \quad \varphi_z]_i \qquad \mathbf{u}_N^T = [u_x \quad u_y \quad \varphi_z]_N$$
$$\mathbf{t}_{zi}^T = [t_x \quad t_y \quad \omega_z]_{zi} \qquad \mathbf{v}_{zi}^T = [v_x \quad v_y \quad \psi_z]_{zi}$$

— generalized displacement vector at the centre of gravity of the i-th bulk

— generalized nodal displacement vector at node N

— initial (thermal) strain vector assigned to α_i-th corner of the bulk

— dislocation vector possible at corner α_i.

$\alpha = j, k, l, m$ denotes four corners of an absolutely rigid element with proportions $2a_i \times 2b_i$.

Corresponding to the above-mentioned kinematical vectors, $\mathbf{q}_i$ is defined as the vector of loads referred to the gravity centres, $\mathbf{q}_N$ as the vector of nodal loads and $\mathbf{s}_{\alpha_i}$ as stresses at corner points in the bulk.

Relationship for rigid-body motion:

$$\mathbf{u}_{\alpha_i} = \mathbf{G}_{\alpha_{i,i}} \, \mathbf{u}_i$$

with

$$\mathbf{G}_{j_i,i} = \begin{bmatrix} 1 & & -b_i \\ & 1 & a_i \\ & & 1 \end{bmatrix} \mathbf{G}_{k_i,i} = \begin{bmatrix} 1 & & -b_i \\ & 1 & -a_i \\ & & 1 \end{bmatrix} \mathbf{G}_{l_i,i} = \begin{bmatrix} 1 & & b_i \\ & 1 & -a_i \\ & & 1 \end{bmatrix} \quad (1)$$

$$\mathbf{G}_{m_i,i} = \begin{bmatrix} 1 & & b_i \\ & 1 & a_i \\ & & 1 \end{bmatrix}$$

is valid for every bulk.

The relative displacement of nodes and joints of bulks:

$$\Delta\mathbf{u}_{\alpha_i} = \mathbf{u}_{\alpha_i} - \mathbf{u}_N \quad (2)$$

α_i denoting the joint between the i-th bulk and the N-th node.

Equilibrium equations referring to each bulk:

$$\sum_{\alpha_i} \mathbf{G}_{\alpha_i,i}^T \mathbf{s}_{\alpha_i} + \mathbf{q}_i = \mathbf{0} \quad (3)$$

The summation at nodes

$$\mathbf{q}_N - \sum_i \mathbf{s}_{\alpha_i} = \mathbf{0} \quad (4)$$

will be generated by bulk number i belonging to nodes N and by indices α_i. The constitutive equation at the joints is:

$$\mathbf{s}_{\alpha_i} = \mathbf{S}_{\alpha_i,i}(\Delta\mathbf{u}_{\alpha_i} - \mathbf{t}_{\alpha_i} - \mathbf{v}_{\alpha_i}) \quad (5)$$

taking into consideration also initial strains and possible dislocations. $\mathbf{S}_{\alpha_i,i}$ stands for the stiffness matrix of the α_i-th joint at the i-th bulk.

The above relationships are ordered according to the matrix-displacement method

$$\mathbf{K}_{ii}\mathbf{u}_i + \sum_N \mathbf{K}_{iN}\mathbf{u}_N = \mathbf{q}_i + \mathbf{q}_{i,t} + \mathbf{q}_{i,v} \tag{6}$$

$$\sum_i \mathbf{K}_{Ni}\mathbf{u}_i + \mathbf{K}_{NN}\mathbf{u}_N = \mathbf{q}_N + \mathbf{q}_{N,t} + \mathbf{q}_{N,v}.$$

The first equation holds for each i at the bulk, the second one for each N at nodal points.

Here

$$\mathbf{K}_{ii} = - \sum_{\alpha_i} \mathbf{G}_{\alpha_{i,i}}^T \mathbf{S}_{\alpha_{i,i}} \mathbf{G}_{\alpha_{i,i}}$$

$$\mathbf{K}_{iN} = \mathbf{G}_{\alpha_{i,i}}^T \mathbf{S}_{\alpha_{i,i}} \qquad \mathbf{K}_{NN} = - \sum_i \mathbf{S}_{\alpha_{i,i}}$$

are the stiffness matrices related to the nodes and gravity centres, whereas

$$\mathbf{q}_{i,t} = - \sum_{\alpha_i} \mathbf{G}_{\alpha_{i,i}}^T \mathbf{S}_{\alpha_{i,i}} \mathbf{t}_{\alpha_i} \qquad \mathbf{q}_{N,t} = \sum_i \mathbf{S}_{\alpha_{i,i}} \mathbf{t}_{\alpha_i}$$

$$\mathbf{q}_{i,v} = - \sum \mathbf{G}_{\alpha_{i,i}}^T \mathbf{S}_{\alpha_{i,i}} \mathbf{v}_{\alpha_i} \qquad \mathbf{q}_{N,v} = \sum \mathbf{S}_{\alpha_{i,i}} \mathbf{v}_{\alpha_i}$$

can be regarded as thermal loads.

After the global stiffness matrix is known, the vector

$$\mathbf{s} = \mathbf{s}_e + \mathbf{s}_v \tag{7}$$

can be determined, where $\mathbf{s}_e$ is the elastic stress vector, $\mathbf{s}_v$ the dislocation-stress vector of the global system. From the element stiffness matrices:

$$\mathbf{s}_{v,\alpha_i} = - \mathbf{S}_{\alpha_{i,i}} \mathbf{v}_{\alpha_i}, \tag{8}$$

which holds for each joint.

Applying the hypermatrices

$$\mathbf{s} = \begin{bmatrix} \mathbf{s}_{\alpha_1} \\ \mathbf{s}_{\alpha_2} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}_v \qquad \mathbf{v} = \begin{bmatrix} \mathbf{v}_{\alpha_1} \\ \mathbf{v}_{\alpha_2} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} -\mathbf{S}_{\alpha_1} & & \\ & -\mathbf{S}_{\alpha_2} & \\ & & \cdot \\ & & & \cdot \end{bmatrix}$$

we obtain:

$$\mathbf{s}_v = \mathbf{S}\mathbf{v}, \tag{9}$$

The state equation of the system seems to be hyperstatic since simultaneous stress and dislocation at each joint are unknown. The assumption that some

of the stress components cannot be tensile can be written by introducing a slack variable $\boldsymbol{\Phi}$:

$$\mathbf{I}_0^T \mathbf{s} - \boldsymbol{\Phi} = 0 \tag{10}$$

If $\mathbf{v} \geqq \mathbf{0}, \boldsymbol{\Phi} \leqq \mathbf{0}, \mathbf{v}^T \boldsymbol{\Phi} = O$ holds, then the relationships show that some selected stresses cannot be negative, furthermore, the fissure gaps cannot close, finally, no fissures can arise where the stress is positive.

In generating matrix $\mathbf{I}_0$ in (10), a unit matrix can be started from, and the columns corresponding to the unrestricted stresses will be equal to a zero vector.

Vectorial equations referring to slack variables can be written as

$$\mathbf{I}_0^T \mathbf{S} \mathbf{v} - \boldsymbol{\Phi} = -\mathbf{I}_0^T \mathbf{s}_e \tag{11}$$

with limiting conditions:

$$\mathbf{v} \geqq 0 \qquad \boldsymbol{\Phi} \leqq 0 \qquad \mathbf{v}^T \cdot \boldsymbol{\Phi} = O.$$

The problem can be solved by introducing a new variable vector w, which is non-negative, and its co-ordinates denote fictitious stresses. In addition, an objective function will be attached to the problem, met only when the fictitious forces disappear:

$$\mathbf{I}_0^T \mathbf{S} \mathbf{v} - \boldsymbol{\Phi} + \mathbf{D} \mathbf{w} = -\mathbf{I}_0^T \mathbf{s}_e$$

$$\Sigma w_i = \min ! \tag{12}$$

$$\mathbf{v}, \mathbf{w} \geqq 0; \qquad \boldsymbol{\Phi} \leqq 0 \qquad \mathbf{v}^T \cdot \boldsymbol{\Phi} = O.$$

Here matrix $\mathbf{D}$ originates from the unit matrix with elements of altered signs, if necessary, so that by choosing $\mathbf{v} = O, \boldsymbol{\Phi} = O$ we get a trivial basis for the simplex procedure.

The problem can be solved with the simplex method, the condition $\mathbf{v}^T \cdot \boldsymbol{\Phi} = O$ must be taken into account only by base exchanges. Technically, the procedure is interpreted so that the gaps will be closed first by adequate forces, then these forces will be removed from the system in a suitable sequence.

Refined stiffness matrix of a triangular element

In generating the refined stiffness matrix of the second model, the triangular elements and the local or global reference frames of Fig. 4 will be made use of. The matrix is developed on the basis of the differential equation of the plain strain problem:

$$\mathbf{D} \mathbf{u} + \mathbf{q} = \mathbf{f}(\mathbf{u}) = 0. \tag{13}$$

Here:

$$
\mathbf{D} = \left[\begin{array}{c|c} S_{11} \dfrac{\partial^2}{\partial x^2} + S_{33} \dfrac{\partial^2}{\partial y^2} & (S_{12} + S_{33}) \dfrac{\partial^2}{\partial x \partial y} \\ \hline (S_{12} + S_{33}) \dfrac{\partial^2}{\partial x \partial y} & S_{33} \dfrac{\partial^2}{\partial x^2} + S_{22} \dfrac{\partial^2}{\partial y^2} \end{array} \right]
$$

$$
\mathbf{u} = \begin{bmatrix} u_x(x, y) \\ u_y(x, y) \end{bmatrix} \qquad \mathbf{q} = \begin{bmatrix} q_x(x, y) \\ q_y(x, y) \end{bmatrix}
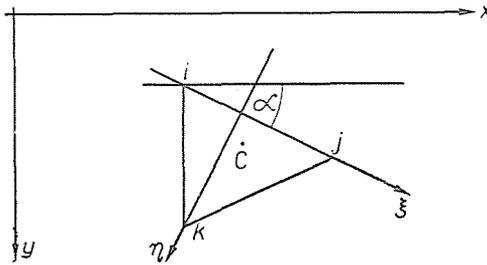$$

Fig. 4

$\mathbf{u}$ denotes displacements and $\mathbf{q}$ denotes loads. Furthermore S_{ij} stands for the elements of the material stiffness matrix in

$$
\begin{bmatrix} \sigma_x \\ \sigma_y \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \end{bmatrix}
$$

and involves the effect of plain strain as well.

The displacement vector is approximated by

$$
\mathbf{u} = \mathbf{U}(\xi, \eta)\mathbf{v} \tag{14}
$$

where

$$
\mathbf{v} = \begin{bmatrix} \mathbf{v}_y \\ \mathbf{v}_y \end{bmatrix}
$$

assigns a constant vector containing 20 parameters.

$$
\mathbf{v}_x^T = \left[u_{x,i} \ \ \frac{\partial u_{x,i}}{\partial x} \ \ \frac{\partial u_{x,i}}{\partial y} \ \ u_{x,j} \ \ \frac{\partial u_{x,j}}{\partial x} \ \ \frac{\partial u_{x,j}}{\partial y} \ \ u_{x,k} \ \ \frac{\partial u_{x,k}}{\partial x} \ \ \frac{\partial u_{x,k}}{\partial y} \ \ u_{x,c} \right]
$$

and $\mathbf{v}_y^T$ is a similar one.

$\mathbf{U}$ is a variable matrix, the structure of which is determined starting from the assumption

$$
u_x(\xi, \eta) = \mathbf{z}^T(\xi, \eta)\mathbf{a} \qquad u_y(\xi, \eta) = \mathbf{z}^T(\xi, \eta)\mathbf{b} \tag{15}
$$

where

$$\mathbf{z}^T = \begin{bmatrix} 1 & \xi & \eta & \xi^2 & \xi\eta & \eta^2 & \xi^3 & \xi^2\eta & \xi\eta^2 & \eta^3 \end{bmatrix} \tag{16}$$

is the basis of the interpolation, furthermore $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

Now, the condition that the elements of $\mathbf{v}$ contain the substitution values of u_x, u_y and their derivatives, respectively, has to be met. Then the local frame must be replaced by the global one by orthogonal transformation of the co-ordinates. Thus

$$\mathbf{U} = \begin{bmatrix} \mathbf{z}^T & \\ & \mathbf{z}^T \end{bmatrix} \begin{bmatrix} \mathbf{R}^{-1} & \\ & \mathbf{R}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{T}_{\xi,x} & \mathbf{T}_{\xi,y} \\ -\mathbf{T}_{\xi,y} & \mathbf{T}_{\xi,x} \end{bmatrix} \tag{17}$$

where

$$\mathbf{R} = \begin{bmatrix}
1 & \xi_i & & \xi_i^2 & & & \xi_i^3 & & & \\
1 & & 2\xi_i & & & & 3\xi_i^2 & & & \\
& & 1 & & \xi_i & & & \xi_i^2 & & \\
1 & \xi_j & & \xi_j^2 & & & \xi_j^3 & & & \\
1 & & 2\xi_j & & & & 3\xi_j^2 & & & \\
& & 1 & & \xi_j & & & \xi_j^2 & & \\
1 & \eta_k & & & \eta_k^2 & & & & & \eta_k^3 \\
1 & & & \eta_k & & & & & \eta_k^2 & \\
& & 1 & & 2\eta_k & & & & & 3\eta_k^2 \\
1 & \xi_c & \eta_c & \xi_c^2 & \xi_c\eta_c & \eta_c^2 & \xi_c^3 & \xi_c^2\eta_c & \xi_c\eta_c^2 & \eta_c^3
\end{bmatrix}$$

and the submatrices $\mathbf{T}$ are built up as follows:

$$c = \cos\alpha \qquad s = \sin\alpha$$

$$\mathbf{T}_{\xi,x}^0 = \begin{bmatrix} c & & \\ & c^2 & sc \\ & -sc & c^2 \end{bmatrix} \qquad \mathbf{T}_{\xi,y}^0 = \begin{bmatrix} s & & \\ & sc & s^2 \\ & -s^2 & sc \end{bmatrix}$$

$$\mathbf{T}_{\xi,x} = \begin{bmatrix} \mathbf{T}_{\xi,x}^0 & & & \\ & \mathbf{T}_{\xi,x}^0 & & \\ & & \mathbf{T}_{\xi,x}^0 & \\ & & & c \end{bmatrix} \qquad \mathbf{T}_{\xi,y} = \begin{bmatrix} \mathbf{T}_{\xi,y}^0 & & & \\ & \mathbf{T}_{\xi,y}^0 & & \\ & & \mathbf{T}_{\xi,y}^0 & \\ & & & s \end{bmatrix}$$

Returning to the stiffness matrix, let us state the principle of least squares:

$$\int_A \mathbf{f}^T(\mathbf{u}) \cdot \mathbf{f}(\mathbf{u}) dA = \min ! \tag{18}$$

(A = area of the triangle).

Hence, making use of (13):

$$\int_A \mathbf{U}^T \mathbf{D}^T \mathbf{D} \mathbf{U} \, dA \cdot \mathbf{v} + \int_A \mathbf{U}^T \mathbf{D}^T \mathbf{q} \, dA = 0.$$

Thus the matrix to be determined is

$$S = \int_A U^T D^T D U \, dA. \tag{19}$$

Now U contains the variable elements of z, and D denotes operators related to the global co-ordinate system. Therefore D must be treated in detail

$$
\begin{bmatrix} \dfrac{\partial}{\partial x} \\[2mm] \dfrac{\partial}{\partial y} \end{bmatrix}
=
\begin{bmatrix} c & -s \\ s & c \end{bmatrix}
\begin{bmatrix} \dfrac{\partial}{\partial \xi} \\[2mm] \dfrac{\partial}{\partial \eta} \end{bmatrix}.
$$

Introducing notations:

$$
d^T = \begin{bmatrix} \dfrac{\partial}{\partial \xi} & \dfrac{\partial}{\partial \eta} \end{bmatrix}
$$

$$
\mathbb{T}_{x,x} = \begin{bmatrix} c^2 & -cs \\ -cs & s^2 \end{bmatrix}
\qquad
\mathbb{T}_{x,y} = \begin{bmatrix} sc & c^2 \\ -s^2 & -sc \end{bmatrix}
\qquad
\mathbb{T}_{y,y} = \begin{bmatrix} s^2 & sc \\ sc & c^2 \end{bmatrix}
$$

we have

$$
\mathbb{D} = \begin{bmatrix} d^T & \\ & d^T \end{bmatrix}
\begin{bmatrix} S_{11}\mathbb{T}_{x,x} + S_{33}\mathbb{T}_{y,y} & (S_{11} + S_{33})\mathbb{T}_{x,y} \\ (S_{11} + S_{33})\mathbb{T}_{x,y}^T & S_{33}\mathbb{T}_{x,x} + S_{22}\mathbb{T}_{y,y} \end{bmatrix}
\begin{bmatrix} d & \\ & d \end{bmatrix}. \tag{20}
$$

What we have to do is to evaluate S using (19), (17), (20) and perform several integrals.

The final form of S is

$$
S = \begin{bmatrix} \mathbb{T}_{\xi,x}^T & \mathbb{T}_{\xi,y}^T \\ -\mathbb{T}_{\xi,y}^T & \mathbb{T}_{\xi,x}^T \end{bmatrix}
\begin{bmatrix} R^{T,-1} & \\ & R^{T,-1} \end{bmatrix}
\begin{bmatrix} H_{1,1} & H_{1,2} \\ H_{2,1} & H_{2,2} \end{bmatrix}
\begin{bmatrix} R^{-1} & \\ & R^{-1} \end{bmatrix}
\begin{bmatrix} \mathbb{T}_{\xi,x} & \mathbb{T}_{\xi,y} \\ -\mathbb{T}_{\xi,y} & \mathbb{T}_{\xi,x} \end{bmatrix}
$$

where the elements of submatrices H depend on the blocks

$$
\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}
= \begin{bmatrix} S_{11}c^2 + S_{33}s^2 & (S_{33} - S_{11})sc \\ (S_{33} - S_{11})sc & S_{11}s^2 + S_{33}c^2 \end{bmatrix}
$$

$$
\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}
= \begin{bmatrix} a_3 & c_3 \\ b_3 & d_3 \end{bmatrix}
= \begin{bmatrix} (S_{11} + S_{33})sc & (S_{11} + S_{33})c^2 \\ -(S_{11} + S_{33})s^2 & -(S_{11} + S_{33})sc \end{bmatrix}
$$

$$
\begin{bmatrix} a_4 & b_4 \\ c_4 & d_4 \end{bmatrix}
= \begin{bmatrix} S_{22}s^2 + S_{33}c^2 & (S_{22} - S_{33})sc \\ (S_{22} - S_{33})sc & S_{22}c^2 + S_{33}s^2 \end{bmatrix}
$$

$H_{1,2}$ is a quasisymmetric matrix, the elements of which are functions of the subscripts of the parameters so that

$$
H_{ij}^{1,2} = H_{ij}^{1,2}(1,3;2,4) \qquad H_{ji}^{1,2} = H_{ij}^{1,2}(2,4;1,3)
$$

$H_{1,1}$ and $H_{2,2}$ are symmetric matrices:

$$
H_{ij}^{1,1} = H_{ji}^{1,1} = H_{ij}^{1,2}(1,3;1,3) \qquad H_{ij}^{2,2} = H_{ji}^{2,2} = H_{ij}^{1,2}(2,4;2,4)
$$

The first three rows and the first three columns of each matrix contain merely 0 elements. Moreover it is sufficient to describe the lower left triangle part of $\mathbf{H}_{1,2}$. According to Table 1:

Table 1

$H_{4,4}$	$4\{(a_1 + c_1)\ (a_2 + c_1) + (a_3 + c_3)\ (a_4 + c_4)\}A$
$H_{5,4}$	$2\{(b_1 + c_1)\ (a_2 + c_2) + (b_3 + c_3)\ (a_4 + c_4)\}A$
$H_{5,5}$	$\{(b_1 + c_1)\ (b_2 + c_2) + (b_3 + c_3)\ (b_4 + c_4)\}A$
$H_{6,4}$	$4\{d_1(a_2 + c_2) + d_3(a_4 + c_4)\}$
$H_{6,5}$	$2\{d_1(b_2 + c_2) + d_3(b_4 + c_4)\}A$
$H_{6,6}$	$4(d_1d_2 + d_3d_4)A$
$H_{7,4}$	$12\{a_1(a_2 + c_2) + a_3(a_4 + c_4)\}S_\eta$
$H_{7,5}$	$6\{a_1(b_2 + c_2) + a_3(b_4 + c_4)\}S_\eta$
$H_{7,6}$	$12(a_1d_2 + a_3d_4)S_\eta$
$H_{7,7}$	$36(a_1a_2 + a_3a_4)J_\eta$
$H_{8,4}$	$4\{a_1(a_2 + c_2) + a_3(a_4 + c_4)\}S_\xi + 4\{(b_1 + c_1)\ (a_2 + c_2) + (b_3 + c_3)\ (a_4 + c_4)\}S_\eta$
$H_{8,5}$	$2\{a_1(b_2 + c_2) + a_3(b_4 + c_4)\}S_\xi + 2\{(b_1 + c_1)\ (b_2 + c_2) + (b_3 + c_3)\ (b_4 + c_4)\}S_\eta$
$H_{8,6}$	$4(a_1d_2 + a_3d_4)S_\xi + 4\{(b_1 + c_1)d_2 + (b_3 + c_3)d_4\}S_\eta$
$H_{8,7}$	$12(a_1a_2 + a_3a_4)C_{\xi\eta} + 12\{(b_1 + c_1)a_2 + (b_3 + c_3)a_4\}J_\eta$
$H_{8,8}$	$4(a_1a_2 + a_3a_4)J_\xi + 4\{a_1(b_2 + c_2) + a_2(b_1 + c_1) + a_3(b_4 + c_4) + a_4(b_3 + c_3)\}C_{\xi\eta} + 4\{(b_1 + c_1)\ (b_2 + c_2) + (b_3 + c_3)\ (b_4 + c_4)\}J_\eta$
$H_{9,4}$	$4\{(b_1 + c_1)\ (a_2 + c_2) + (b_3 + c_3)\ (a_4 + c_4)\}S_\xi + 4\{d_2(a_2 + c_2) + d_3(a_4 + c_4)\}S_\eta$
$H_{9,5}$	$2\{(b_1 + c_1)\ (b_2 + c_2) + (b_3 + c_3)\ (b_4 + c_4)\}S_\xi + 2\{d_1(b_2 + c_2) + d_3(b_4 + c_4)\}S_\eta$
$H_{9,6}$	$4\{(b_1 + c_1)d_2 + (b_3 + c_3)d_4\}S_\xi + 4(d_1d_2 + d_3d_4)S_\eta$
$H_{9,7}$	$12\{(b_1 + c_1)a_2 + (b_3 + c_3)a_4\}C_{\xi\eta} + 12(d_1a_2 + d_3a_4)J_\eta$
$H_{9,8}$	$4\{(b_1 + c_1)a_2 + (b_3 + c_3)a_4\}J_\xi + 4\{(b_1 + c_1)\ (b_2 + c_2) + (b_3 + c_3)\ (b_4 + c_4) + d_1a_2 + d_3a_1\}C_{\xi\eta} + 4\{d_1(b_2 + c_2) + d_3(b_4 + c_4)\}J_\eta$
$H_{9,9}$	$4\{(b_1 + c_1)\ (b_2 + c_2) + (b_3 + c_3)\ (b_4 + c_4)\}J_\xi + 4\{(b_1 + c_1)d_2 + ((b_2 + c_2)d_1 + (b_3 + c_3)d_4 + (b_4 + c_4)d_3\}C_{\xi\eta} + 4(d_1d_2 + d_3d_4)J_\eta$
$H_{10,4}$	$12\{d_3(a_2 + c_2) + d_3(a_4 + c_4)\}S_\xi$
$H_{10,5}$	$6\{d_1(b_2 + c_2) + d_3(b_4 + c_4)\}S_\xi$
$H_{10,6}$	$12(d_1d_2 + d_3d_4)S_\xi$
$H_{10,7}$	$36(d_1a_2 + d_3a_4)C_{\xi\eta}$
$H_{10,8}$	$12(d_1a_2 + d_3a_4)J_\xi + 12\{(b_2 + c_2)d_1 + (b_4 + c_4)d_3\}C_{\xi\eta}$
$H_{10,9}$	$12\{d_1(b_2 + c_2) + d_3(b_4 + c_4)\}J_\xi + 12(d_1d_2 + d_3d_4)C_{\xi\eta}$
$H_{10,10}$	$36(d_1d_2 + d_3d_4)J_\xi$

A, S_ξ, S_η, J_ξ, J_η, $C_{\xi\eta}$ are the usual geometric quantities of the triangle.

No-tension test of a single element

By applying elastic triangular elements, the no-tension requirement concerning the principal stresses of each element has to be fulfilled separately. This can be performed by a linear complementary problem that may be solved by complete description since it is simple enough.

The principal strains due to the displacement are known in each step of the computation. Principal stresses and the possible dislocations caused by

cracks have to be computed. So the equation of the problem is

$$
\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_z \end{bmatrix} = -\frac{1}{E} \begin{bmatrix} 1 & -1/m & -1/m \\ -1/m & 1 & -1/m \\ -1/m & -1/m & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_z \end{bmatrix} + \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_z \end{bmatrix} \tag{21}
$$

where compression is assumed to be positive.

E denotes the Young's modulus of the material and m denotes the reciprocal of Poisson's ratio, δ is the symbol of dislocation strains.

Since the order of the principal strains is not known in advance, those located in the plane of the analysis are specified by numbers 1 and 2, the third one receives the subscript z.

For a plain strain, $\varepsilon_z = \delta_z = 0$ must be considered. Besides, neither the stresses nor the dislocation strains can be negative, furthermore they can only occur alternatively. Thus

$$
\boldsymbol{\sigma} \geq 0 \qquad \boldsymbol{\delta} \geq 0 \qquad \boldsymbol{\sigma}^T \boldsymbol{\delta} = 0. \tag{22}
$$

There are only four possibilities of selecting unrestricted variables so as to fulfill (22), in particular:

a) $\sigma_1 = 0 \qquad \delta_2 = 0$ \qquad\qquad b) $\sigma_2 = 0 \qquad \delta_1 = 0$ \qquad (23)

c) $\sigma_1 = 0 \qquad \sigma_2 = 0$ \qquad\qquad d) $\delta_1 = 0 \qquad \delta_2 = 0.$

Let us see some cases.

a)

$$
\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 0 \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1/m & 1/m & -E \\ -1 & 1/m & \\ 1/m & -1 & \end{bmatrix} \begin{bmatrix} \sigma_2 \\ \sigma_z \\ \delta_1 \end{bmatrix}
$$

hence we have the constraint

$$
\begin{bmatrix} \sigma_1 \\ \sigma_z \\ \delta_1 \end{bmatrix} = \begin{bmatrix} & Em^2/(1-m^2) \\ & Em/(1-m^2) \\ 1 & 1/(1-m) \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \geq \mathbf{0} .
$$

Now $m \geq 2$, thus the relationships are true for:

$$
\varepsilon_2 \leq 0 \qquad \varepsilon_2 \geq (1-m)\varepsilon_1 \tag{24}
$$

(Fig. 5a).

b) and c) are not detailed here. Obviously

$$
\varepsilon_1 \leq 0 \qquad \varepsilon_1 \geq (1-m)\varepsilon_2 \tag{25}
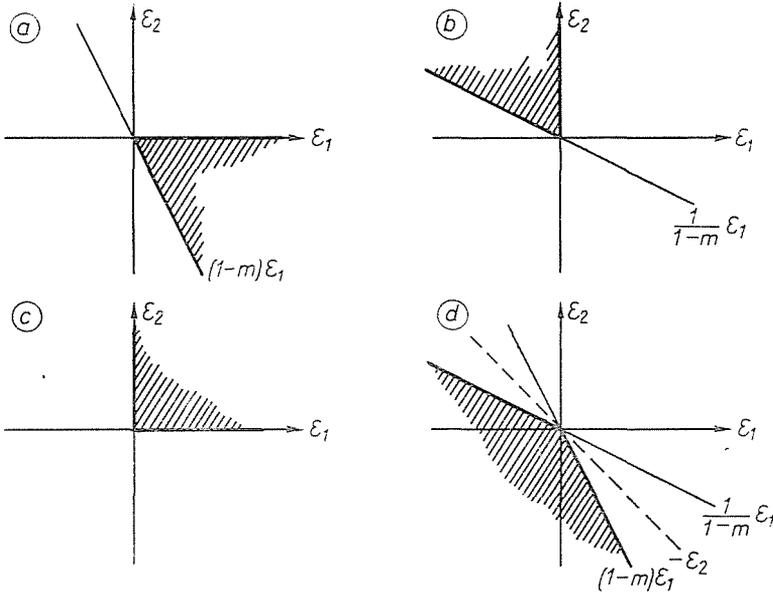$$

Fig. 5

(Fig. 5b), and

$$\varepsilon_1 \geqq 0 \qquad \varepsilon_2 \geqq 0 \tag{26}$$

(Fig. 5c).

Finally

d)

$$\begin{bmatrix} m-1 & 1 & 1 \\ 1 & m-1 & 1 \\ 1 & 1 & m-1 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_z \end{bmatrix} \geqq \mathbf{0}$$

Thus

$$\varepsilon_2 \leqq (1-m)\varepsilon_1 \qquad \varepsilon_1 \leqq (1-m)\varepsilon_2 \qquad \varepsilon_1 \leqq -\varepsilon_2 \tag{27}$$

(Fig. 5d). As $m \geqq 2$, the third condition is implied by the first two relationships.

The plane $(\varepsilon_1, \varepsilon_2)$ is covered by the sets resulting in the four cases considered, so the solution is unique. The non-negative stresses and dislocations can be computed if the principal strains are given.

Numerical result

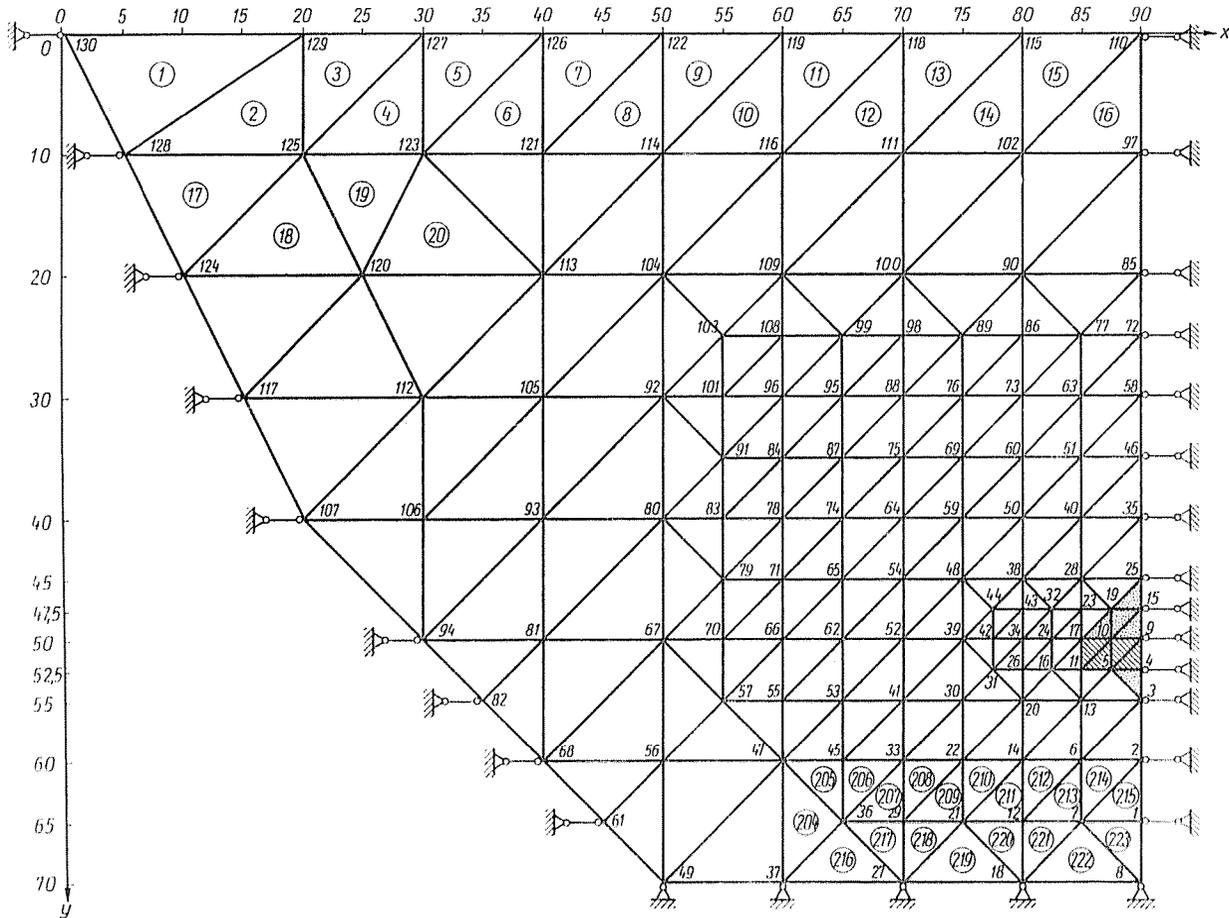A plain strain problem is seen in Fig. 6. Because of symmetry, only half of the plane is taken into account.

Fig. 6. Plain strain rock mechanics problem — division into elements; Shaded area — part already extracted; Dotted area — cracked zone

We shall consider the problem of stress distribution in the vicinity of a 10 m wide, 2.5 m high pit lying 50 m under the surface.

The pit is provided with timbering, the timbers are spaced at $t_1 = 2.0$ m in longitudinal and $t_2 = 2.5$ m in transverse direction. The cross-sectional area of a timber is $F = 500$ cm²; the Young's modulus of timber material:

$$E_t = 10^5 \frac{Mp}{m^2}.$$

Rock properties are:

Young's modulus $\qquad\qquad E_s = 10^4$ Mp/m²,
Poisson's ratio $\qquad\qquad\quad \nu = 0.16$,
density $\qquad\qquad\qquad\quad \gamma_s = 1.8$ Mp/m³
lateral pressure coefficient $\varkappa = \mathrm{tg}^2(45° - \varphi/2) = 0.4$ ($\varphi =$ angle of friction).

The timbering can be replaced by a substitute homogeneous material, of a different thickness such as:

$$v = \frac{E_t}{E_s} \frac{F_t}{t_1 t_2}$$

(soil thickness is regarded to be 1.0 m).

Thus a program for a refined plain strain finite element with no tensile resistance can be used. Fig. 6 shows our finite element discretization with 130 nodal points and 223 elements together with the boundary conditions. To check the convergence of the iteration, the Euclidean norm of the vector of unbalanced nodal loads has been calculated in each step. Table 2 shows the change of norm during the iterative process.
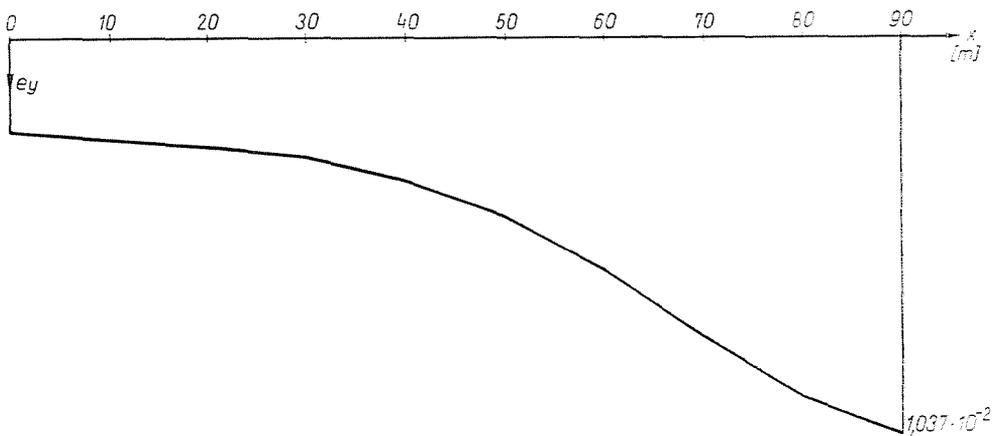

Fig. 7. Vertical displacements along the surface. Displacements are to a scale 5000 times that of horizontal distance. Scale: 1 : 500

Table 2

Iteration number	1	3	5	10
Square of Euclidean norm	175.1	100.6	63.77	17.25

Vertical displacements at the surface and at the pit top as well as the stress diagrams along the axis of symmetry are illustrated in Figs 7, 8, 9. The fissured zone is shown in Fig. 6 by dotted area.
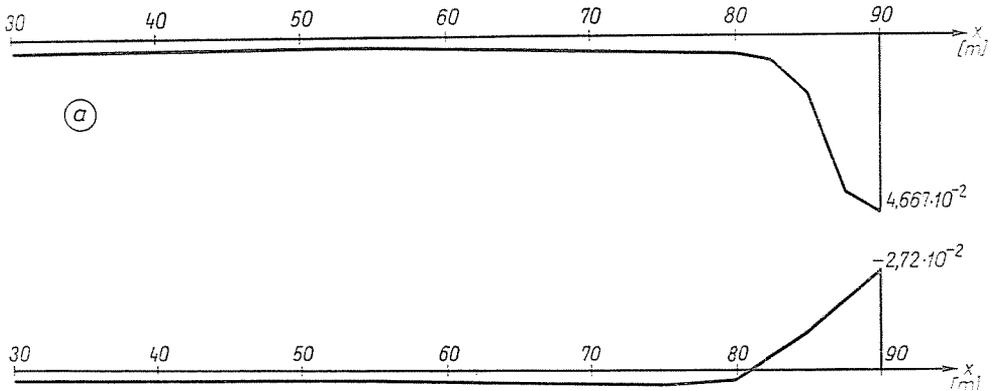

Fig. 8. a) Vertical displacements at the pit top; b) Vertical displacements along section at $y = 55$ m; Displacements are to a scale 2000 times that of horizontal distance. Scale: 1 : 200
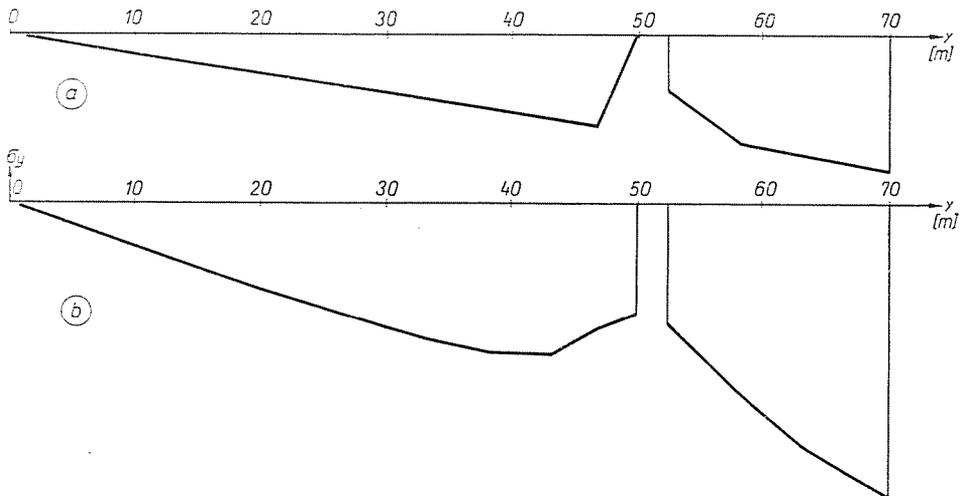

Fig. 9. Distribution of stresses a) σ_x; b) σ_y along the section at $x = 90$ m; 1 cm = 10 Mp/m². Scale: 1 : 200

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Summary

No-tension materials can be analyzed by several means. The paper deals with two particular procedures of analysis: first, a linear complementary problem is established, considering the region as a set of rigid bulks connected by elastic joints; second, the Newton—Raphson iteration process is applied, supposing the finite elements to be continuously elastic, and using a rather refined stiffness matrix. Some computation results are presented.

References

1. ZIENKIEWICZ, O. C.,—VALLIAPPAN, S.,—KING I. P.: Elasto-plastic Solutions of Engineering Problems with Initial Stress by Finite Element Method. Int. Journal for Numerical Methods in Engineering Vol. 1. (1969) pp. 75—100.
2. ZIENKIEWICZ, O. C.,–CHEUNG, Y. K.: The Finite Element Method in Structural and Continuum Mechanics. McGraw-Hill, London 1971.
3. MEEK, J. L.: Excavation in Rock. An Appreciation of the Finite Element Method of Analysis. "Theory and Practice in Finite Element Structural Analysis" Proceedings of the 1973 Tokyo Seminar on Finite Element Analysis. University of Tokyo Press 1973.
4. GOODMAN, R. E.,–TAYLOR, R. L.,–BREKKE, T. L.: A Model for the Mechanics of Jointed Rock. Journal of Soil Mech. ASCE Vol. 94. SM3. 1968.
5. ROLLER, B.,–SZENTIVÁNYI, B.: Die Berechnung von Tragwerken mit bedingten Stützen und Verbindungen durch quadratische Programmierung. Per. Pol. Civ. Eng. Vol. 19. No. 3—4. Budapest, 1975.
6. ASSZONYI-RICHTER: Rock Mechanics.✻ Nehézipari Dokumentációs Központ, Budapest (1975).

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