ELASTO-PLASTIC ANALYSIS OF FRAMES BY WOLFE'S SHORT ALGORITHM

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1. Introduction

The first-order quasi-static analysis of linear elastic structures presents no difficulty to the engineer even in case of large systems. Neither does plastic limit analysis since linear programming codes are available in large computer centres, only that the latter gives no information about the deformations of the structure prior to collapse. Nonetheless, in several engineering problems it is important to know the progress of plastic strain and displacement states as the loads increase beyond the elastic limit.

The elasto-plastic analysis presented in this paper includes the determination of the structural response to given external actions by considering a sequence of incremental problems. This makes possible the study of elasto-plastic response of structures with general loading histories taking the irreversibility of plastic behaviour into consideration.

The following assumptions are made:

a) a discrete structural model for frames is valid where the load, stress, strain, displacement, plastic potential and plastic multiplier fields are characterized by finite dimensional vectors (so-called generalized quantities),

b) the loading is quasi-static with piecewise constant load rates,

c) displacements are small and no instability phenomenon occurs,

d) the material is linear elastic—perfect plastic,

e) the yield surface is piecewise linearized (convex polyhedron),

f) associated flow rules according to KOITER's generalized plastic potential theory are valid [1].

2. Basic relations

In [2] the following matrix equation is given for linear elastic frameworks:

$$\begin{bmatrix} \cdot & \mathbf{G}^* \\ \mathbf{G} & \mathbf{F} \end{bmatrix}\begin{bmatrix} \mathbf{u} \\ \mathbf{s} \end{bmatrix} + \begin{bmatrix} \mathbf{q} \\ \mathbf{t} \end{bmatrix} = \mathbf{0} \tag{1}$$

where

$\mathbf{G}^*$ is the full rowrank equilibrium matrix,

$\mathbf{G}$ is the compatibility matrix (transpose of $\mathbf{G}^*$),

$\mathbf{F}$ is the block diagonal positive definite flexibility matrix and $\mathbf{u}$, $\mathbf{s}$, $\mathbf{q}$, $\mathbf{t}$ are vectors of nodal displacements, generalized stresses, nodal loads and generalized initial strains (kinematical loads), respectively.

The whole matrix is indefinite and non-singular. The blocks of its inverse (the influence matrices) have the following properties:

$$\begin{bmatrix} \cdot & \mathbf{G}^* \\ \mathbf{G} & \mathbf{F} \end{bmatrix}^{-1} \begin{matrix} \}m \\ \}n \end{matrix} = \begin{bmatrix} \mathbf{X} & \mathbf{Y}^* \\ \mathbf{Y} & \mathbf{Z} \end{bmatrix} \tag{2}$$
$$\quad m \quad n$$

$\mathbf{X}$ is a symmetric negative definite matrix,

$\mathbf{Y}^*$ is a full rowrank matrix,

$\mathbf{Z}$ is a symmetric positive semidefinite matrix of rank $n - m$.

To study the response of a structure under a given loading history one has to consider the loads, displacements, stresses and strains as functions of time. Due to the first part of assumption b), the time variable can be replaced by a suitably chosen monotonously increasing scalar variable, for example by the load parameter.

The relations between the aforementioned quantities at a specified state are governed by the equilibrium equation:

$$\mathbf{G}^*\mathbf{s} + \mathbf{q} = \mathbf{0}; \tag{3}$$

the compatibility equation:

$$\mathbf{G}\mathbf{u} + \mathbf{F}\mathbf{s} + \mathbf{p} + \mathbf{t} = \mathbf{0} \tag{4}$$

where $\mathbf{p}$ designates the generalized plastic strains; and the piecewise linearized yield criterion [3]:

$$\mathbf{N}^*\mathbf{s} - \mathbf{k} - \boldsymbol{\varphi} = \mathbf{0}, \qquad \boldsymbol{\varphi} \leqq \mathbf{0} \tag{5}$$

where

$\mathbf{N}^*$ is the block-diagonal yield matrix grouping the normal vectors of the hyperplanes determining the convex yield polyhedron at appropriately chosen points of the structure,

$\mathbf{k}$ is the vector containing the distances of these planes from the origin,

$\boldsymbol{\varphi}$ is the vector of plastic potentials.

As a consequence of the piecewise linearization of the yield surface, the vector of generalized plastic strains may be expressed by:

$$\mathbf{p} = \mathbf{N}\boldsymbol{\lambda}, \qquad \boldsymbol{\lambda} \geqq \mathbf{0} \tag{6}$$

where $\boldsymbol{\lambda}$ designates the vector of plastic multipliers.

Relations (3) to (6) are not sufficient to determine the state variables $\mathbf{s}, \mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\lambda}$. Further relations containing the rates of the variables in question should be taken into account, such as
the rate equation for equilibrium:

$$\mathbf{G}^*\dot{\mathbf{s}} + \dot{\mathbf{q}} = \mathbf{0} \tag{7}$$

the rate equation for compatibility:

$$\mathbf{G}\dot{\mathbf{u}} + \mathbf{F}\dot{\mathbf{s}} + \dot{\mathbf{p}} + \dot{\mathbf{t}} = \mathbf{0} \tag{8}$$

and the flow rules:

$$\dot{\mathbf{p}} = \mathbf{N}^J\dot{\boldsymbol{\lambda}}_J, \quad \mathbf{N}_J^*\dot{\mathbf{s}} - \dot{\boldsymbol{\varphi}}_J = \mathbf{0}, \quad \dot{\boldsymbol{\lambda}}_J \geqq \mathbf{0}, \quad \dot{\boldsymbol{\varphi}}_J \leqq \mathbf{0}, \quad \dot{\boldsymbol{\lambda}}^{*J}\dot{\boldsymbol{\varphi}}_J = 0 \tag{9}$$

where $J : \{j \mid \varphi_j = 0\}$ applied to designate the submatrices and subvectors corresponding to this index set.

Relations (7) to (9) are equivalent to a pair of dual quadratic programs [3], namely:

$$\text{Q1: } \min \left\{ \frac{1}{2}\dot{\mathbf{s}}^*\mathbf{F}\dot{\mathbf{s}} + \dot{\mathbf{t}}^*\dot{\mathbf{s}} \mid \mathbf{G}^*\dot{\mathbf{s}} + \dot{\mathbf{q}} = \mathbf{0}, \quad \dot{\boldsymbol{\varphi}}_J = \mathbf{N}_J^*\dot{\mathbf{s}} \leqq \mathbf{0} \right\} \tag{10}$$

$$\text{Q2: } \min \left\{ \frac{1}{2}\dot{\mathbf{e}}^*\mathbf{F}^{-1}\dot{\mathbf{e}} - \dot{\mathbf{q}}^*\dot{\mathbf{u}} \mid \mathbf{G}\dot{\mathbf{u}} + \dot{\mathbf{e}} + \dot{\mathbf{p}} + \dot{\mathbf{t}} = \mathbf{0}, \quad \dot{\mathbf{p}} = \mathbf{N}^J\dot{\boldsymbol{\lambda}}_J, \quad \dot{\boldsymbol{\lambda}}_J \geqq \mathbf{0} \right\} \tag{11}$$

($\dot{\mathbf{e}}$ denotes the generalized elastic strain rates: $\dot{\mathbf{e}} = \mathbf{F}\dot{\mathbf{s}}$). (10) and (11) express theorems analogous to those of linear elasticity but involving energy rates and sign constraint variables [3].

To examine the existence of a solution to problems (7) to (9), on the basis of theorems on quadratic programming [4] it is sufficient to analyze only one of the dual quadratic programs, for example program Q1. According to a theorem of quadratic programming, Q1 always has a unique optimum solution if its constraint set has a feasible solution because matrix $\mathbf{F}$ is positive definite. On the other hand, if its constraint set has no feasible solution it can be shown that the structure has reached a collapse state. Thus, at a given point of the loading history, either the response of the structure to given load rates is uniquely determined or there is no solution at all. This latter case can be characterized by the following theorem:

If the constraint set of Q1 has no feasible solution, then under constant load ($\dot{\mathbf{q}} = \mathbf{0}, \dot{\mathbf{t}} = \mathbf{0}$) the stress rates and the plastic potential rates are zero ($\dot{\mathbf{s}} = \mathbf{0}, \dot{\boldsymbol{\varphi}} = \mathbf{0}$) and a collapse mechanism develops, i.e. there is a nontrivial solution ($\dot{\mathbf{u}} = \mathbf{0}, \dot{\boldsymbol{\lambda}} = \mathbf{0}$) to the compatibility equation (8), and the plastic deformations are increasing without limit.

Proof: If $\dot{\mathbf{q}} = \mathbf{0}$ and $\dot{\mathbf{t}} = \mathbf{0}$ then

$$\begin{bmatrix} \mathbf{G}^* \\ \mathbf{N}_J^* \end{bmatrix} \dot{\mathbf{s}} + \begin{bmatrix} \mathbf{0} \\ \dot{\boldsymbol{\varphi}}_J \end{bmatrix} = \mathbf{0} \tag{12}$$

and

$$\mathbf{G}\dot{\mathbf{u}} + \mathbf{F}\dot{\mathbf{s}} + \mathbf{N}^J\dot{\boldsymbol{\lambda}}_J = \mathbf{0}. \tag{13}$$

Multiplying (13) by $\dot{\mathbf{s}}^*$ one obtains:

$$\dot{\mathbf{s}}^*\mathbf{G}\dot{\mathbf{u}} + \dot{\mathbf{s}}^*\mathbf{F}\dot{\mathbf{s}} + \dot{\mathbf{s}}^*\mathbf{N}^J\dot{\boldsymbol{\lambda}}_J = 0 \tag{14}$$

which yields

$$\dot{\mathbf{s}}^*\mathbf{F}\dot{\mathbf{s}} = 0 \tag{15}$$

because

$$\dot{\mathbf{s}}^*\mathbf{G} = \mathbf{0}^* \quad \text{and} \quad \dot{\mathbf{s}}^*\mathbf{N}^J\dot{\boldsymbol{\lambda}}_J = \dot{\boldsymbol{\varphi}}^{*J}\dot{\boldsymbol{\lambda}}_J = 0.$$

As $\mathbf{F}$ is positive definite $\dot{\mathbf{s}}$ cannot be but zero and so is $\dot{\boldsymbol{\varphi}}_J$.

The proof of the second part is based on Gale's theorem for linear inequalities which states [5]:

For a given $p \cdot n$ matrix $\mathbf{A}$ and a given vector $\mathbf{c} \in R^p$ either

I. $\mathbf{A}\mathbf{x} \leqq \mathbf{c}$ has a solution $\mathbf{x} \in R^n$ or (16)

II. $\mathbf{A}^*\mathbf{y} = \mathbf{0}$, $\mathbf{c}^*\mathbf{y} = -1$, $\mathbf{y} \geqq \mathbf{0}$ has a solution $\mathbf{y} \in R^p$ but never both. (17)

In the present case, system I is:

$$\mathbf{G}^*\dot{\mathbf{s}} = -\dot{\mathbf{q}} \tag{18}$$

$$\mathbf{N}_J^*\dot{\mathbf{s}} \leqq \mathbf{0}$$

that is

$$\begin{bmatrix} \mathbf{G}^* \\ -\mathbf{G}^* \\ \mathbf{N}_J^* \end{bmatrix} \dot{\mathbf{s}} \leqq \begin{bmatrix} -\dot{\mathbf{q}} \\ \dot{\mathbf{q}} \\ \mathbf{0} \end{bmatrix}$$

system II is:

$$[\mathbf{G}\ -\mathbf{G}\ \mathbf{N}^J] \begin{bmatrix} \dot{\mathbf{u}}^+ \\ \dot{\mathbf{u}}^- \\ \dot{\boldsymbol{\lambda}}_J \end{bmatrix} = \mathbf{0}, \quad [-\dot{\mathbf{q}}^*\dot{\mathbf{q}}^*\mathbf{0}^*] \begin{bmatrix} \dot{\mathbf{u}}^+ \\ \dot{\mathbf{u}}^- \\ \dot{\boldsymbol{\lambda}}_J \end{bmatrix} = -1, \begin{bmatrix} \dot{\mathbf{u}}^+ \\ \dot{\mathbf{u}}^- \\ \dot{\boldsymbol{\lambda}}_J \end{bmatrix} \geqq \mathbf{0} \tag{19}$$

and introducing $\dot{\mathbf{u}} = \dot{\mathbf{u}}^+ - \dot{\mathbf{u}}^-$:

$$[\mathbf{G}\ \mathbf{N}^J] \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\boldsymbol{\lambda}}_J \end{bmatrix} = \mathbf{0}, \quad \dot{\mathbf{q}}^*\dot{\mathbf{u}} = 1, \quad \dot{\boldsymbol{\lambda}}_J \geqq \mathbf{0} \tag{20}$$

which completes the proof of the theorem.

3. The application of Wolfe's short algorithm for the solution of the incremental elasto-plastic problem

The relations of section **2** governing the incremental elasto-plastic response of a structure can be transformed into simpler quadratic programming and equivalent linear complementarity problems by eliminating the unconstraint variables $\dot{\mathbf{u}}, \dot{\mathbf{s}}$ and having only the sign constraint plastic potential and plastic multiplier vectors $\dot{\boldsymbol{\varphi}}$ and $\dot{\boldsymbol{\lambda}}$ as unknowns. These problems can be solved by WOLFE's short algorithm. Grouping the incremental relations:

$$\begin{bmatrix} \cdot & \mathbf{G}^* & \cdot \\ \mathbf{G} & \mathbf{F} & \mathbf{N}^J \\ \cdot & \mathbf{N}_J^* & \cdot \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{s}} \\ \dot{\boldsymbol{\lambda}}_J \end{bmatrix} + \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{t}} \\ -\dot{\boldsymbol{\varphi}}_J \end{bmatrix} = \mathbf{0}, \qquad \begin{matrix} \dot{\boldsymbol{\lambda}}_J \geqq \mathbf{0}, \quad \dot{\boldsymbol{\varphi}}_J \leqq \mathbf{0}, \\ \dot{\boldsymbol{\lambda}}^{*J}\dot{\boldsymbol{\varphi}}_J = 0. \end{matrix} \tag{21}$$

and multiplying the first two rows by (2):

$$\begin{bmatrix} \mathbf{E} & \cdot & \mathbf{Y}^* & \mathbf{N}^J \\ \cdot & \mathbf{E} & \mathbf{Z} & \mathbf{N}^J \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{s}} \\ \dot{\boldsymbol{\lambda}}_J \end{bmatrix} + \begin{bmatrix} \mathbf{X}\dot{\mathbf{q}} + \mathbf{Y}^*\dot{\mathbf{t}} \\ \mathbf{Y}\dot{\mathbf{p}} + \mathbf{Z}\ \dot{\mathbf{t}} \end{bmatrix} = \mathbf{0}. \tag{22}$$

Expressing $\dot{\mathbf{s}}$ from the second row:

$$\dot{\mathbf{s}} = -\mathbf{Z}\ \dot{\mathbf{N}}^J\ \boldsymbol{\lambda}_J - \mathbf{Y}\dot{\mathbf{q}} - \mathbf{Z}\dot{\mathbf{t}} = \dot{\mathbf{s}}^P + \dot{\mathbf{s}}^e. \tag{23}$$

Substituting $\dot{\mathbf{s}}$ into the third row results in the desired linear complementarity problem:

$$\text{LC:}\quad \mathbf{N}_J^*\mathbf{Z}\ \mathbf{N}^J\dot{\boldsymbol{\lambda}}_J - \mathbf{N}_J^*\dot{\mathbf{s}}^e + \dot{\boldsymbol{\varphi}}_J = \mathbf{0}, \quad \dot{\boldsymbol{\lambda}}_J \geqq \mathbf{0}, \quad \dot{\boldsymbol{\varphi}}_J \leqq \mathbf{0}, \quad \dot{\boldsymbol{\lambda}}^{*J}\dot{\boldsymbol{\varphi}}_J = 0 \tag{24}$$

This linear complementarity problem is equivalent to a pair of dual quadratic programming problems, namely:

$$\text{K1:}\quad \min\left\{\frac{1}{2}\dot{\boldsymbol{\lambda}}^{*J}\mathbf{N}_J^*\ \mathbf{Z}\ \dot{\mathbf{N}}^J\boldsymbol{\lambda}_J - \dot{\boldsymbol{\lambda}}^{*J}\mathbf{N}_J^*\dot{\mathbf{s}}^e \mid \dot{\boldsymbol{\lambda}}_J \geqq 0\right\} \tag{25}$$

$$\text{K2:}\quad \min\left\{\frac{1}{2}\dot{\boldsymbol{\lambda}}^{*J}\mathbf{N}_J^*\mathbf{Z}\mathbf{N}^J\dot{\boldsymbol{\lambda}}_J \mid -\mathbf{N}_J^*\mathbf{Z}\mathbf{N}^J\dot{\boldsymbol{\lambda}}_J + \mathbf{N}_J^*\dot{\mathbf{s}}^e \leqq \mathbf{0}, \quad \dot{\boldsymbol{\lambda}}_J \geqq \mathbf{0}\right\}. \tag{26}$$

Wolfe's quadratic programming algorithm based on the simplex method of linear programming is suitable for solving quadratic programs of the type [6]:

$$\min\left\{\frac{1}{2}\mathbf{x}^*\mathbf{C}\mathbf{x} + \mathbf{p}^*\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geqq \mathbf{0}\right\}. \tag{27}$$

The algorithm exists in a short and in a long form. The short form is applicable for problems of positive definite $\mathbf{C}$ or positive semidefinite $\mathbf{C}$ and $\mathbf{p} = \mathbf{0}$.

According to the statements of the previous section, the incremental elasto-plastic problem has a unique solution if the structure is not in a collapse state. It follows that in these cases the matrix $\mathbf{N}_J^*\,\mathbf{Z}\,\mathbf{N}^J$ must be positive definite so the short algorithm is suitable for solving problem (25). In the state of collapse, by choosing $\dot{\mathbf{t}} = \mathbf{0}$ and $\dot{\mathbf{q}} = \mathbf{0}$, the short algorithm is again applicable because, though $\mathbf{N}_J^*\mathbf{Z}\mathbf{N}^J$ becomes semidefinite, $\dot{\mathbf{s}}^e = \mathbf{0}$ holds. To study, however, the possible continuations of the loading history from the collapse state on, the short form is no longer sufficient.

4. Algorithm for the analysis of a loading history with piecewise constant load rates

The analysis of the elasto-plastic response of a structure to a given loading history is an initial value problem. For starting point one can use a state characterized by known vectors $\mathbf{q}_0, \mathbf{t}_0, \mathbf{u}_0, \mathbf{s}_0, \boldsymbol{\lambda}_0, \boldsymbol{\varphi}_0$ at a time τ_0. In most cases this state is the stress-free state of the structure. The rates $\dot{\mathbf{u}}_0, \dot{\mathbf{s}}_0, \dot{\boldsymbol{\lambda}}_0, \dot{\boldsymbol{\varphi}}_0$ at time τ_0 can be calculated on the basis of section 3.

Until the load rates do not change and the matrix $\mathbf{N}^{J0}$, of importance for the derivation of the plastic multiplier rates, is inaltered (i.e. up to another yielding) $\dot{\mathbf{u}}, \dot{\mathbf{s}}, \dot{\boldsymbol{\lambda}}, \dot{\boldsymbol{\varphi}}$ also remain constant.

Due to this fact the state variables $\mathbf{u}, \mathbf{s}, \boldsymbol{\lambda}, \boldsymbol{\varphi}$ at time $\tau_0 + \Delta\tau$ can be calculated by the simple linear expressions:

$$\mathbf{u} = \mathbf{u}_0 + \dot{\mathbf{u}}_0 \cdot \Delta\tau, \quad \mathbf{s} = \mathbf{s}_0 + \dot{\mathbf{s}}_0 \cdot \Delta\tau, \quad \boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \dot{\boldsymbol{\lambda}}_0 \cdot \Delta\tau,$$
$$\boldsymbol{\varphi} = \boldsymbol{\varphi}_0 + \dot{\boldsymbol{\varphi}}_0 \cdot \Delta\tau. \tag{28}$$

Of course, the validity of these expressions is limited in time because of the eventual change of the load rates and of matrix $\mathbf{N}^{J0}$. Let $\Delta\tau_{1\dot{q}}$ and $\Delta\tau_{1\dot{\varphi}}$ denote the increments of time until the load rates and matrix $\mathbf{N}^{J0}$ do not change. Then the time increment $\Delta\tau_1$ determining the validity of expressions (28) is:

$$\Delta\tau_1 = \min\{\Delta\tau_{1\dot{q}}, \Delta\tau_{1\dot{\varphi}}\},$$

that is the minimum determining the validity of the constant load rates and of matrix $\mathbf{N}^{J0}$. $\Delta\tau_{1\dot{q}}$ is at disposal as data and:

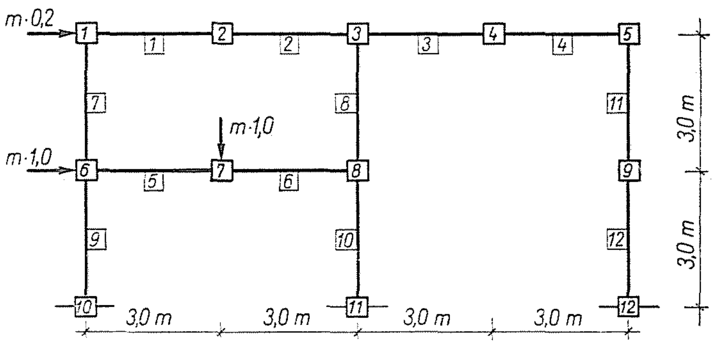
$$\Delta\tau_{1\dot{\varphi}} = \min\left\{-\frac{\varphi_{0i}}{\dot{\varphi}_{0i}} \,\middle|\, \dot{\varphi}_{0i} > 0, \quad \varphi_{0i} \neq 0\right\}.$$

The analysis starting from the new state determined by $\Delta\tau_1$ ($\tau_1 = \tau_0 + \Delta\tau_1$, $\mathbf{u}_1 = \mathbf{u}_0 + \dot{\mathbf{u}}_0 \Delta\tau_1$, etc.) can likewise be continued to the end of the loading history or up to the collapse.

5. Numerical example

For practical realization of the algorithm described in sections 4 and 5 a computer program has been developed using the ALGOL subset of the SIMULA 67 language on the CDC 3300 computer of the Hungarian Academy of Sciences.

For the inversion of symmetric matrices, algorithm 37 of Applied Statistics [8] was adopted and improved by storing only the upper half of the matrix in a one dimensional array. For Wolfe's short quadratic programming method, algorithm 59 of The Computer Journal [7] was applied completed



Element	Element properties		
	EF(Mp)	EJ (Mpm^2)	Mp (mMp)
1, 2	112 140	995.4	11.25
3, 4, 11, 12	83 160	642.6	7.50
5, 6	163 380	2627.1	20.00
7, 8	58 950	304.5	3.75
9, 10	145 110	2058.0	16.25

Fig. 1. Structure and loads

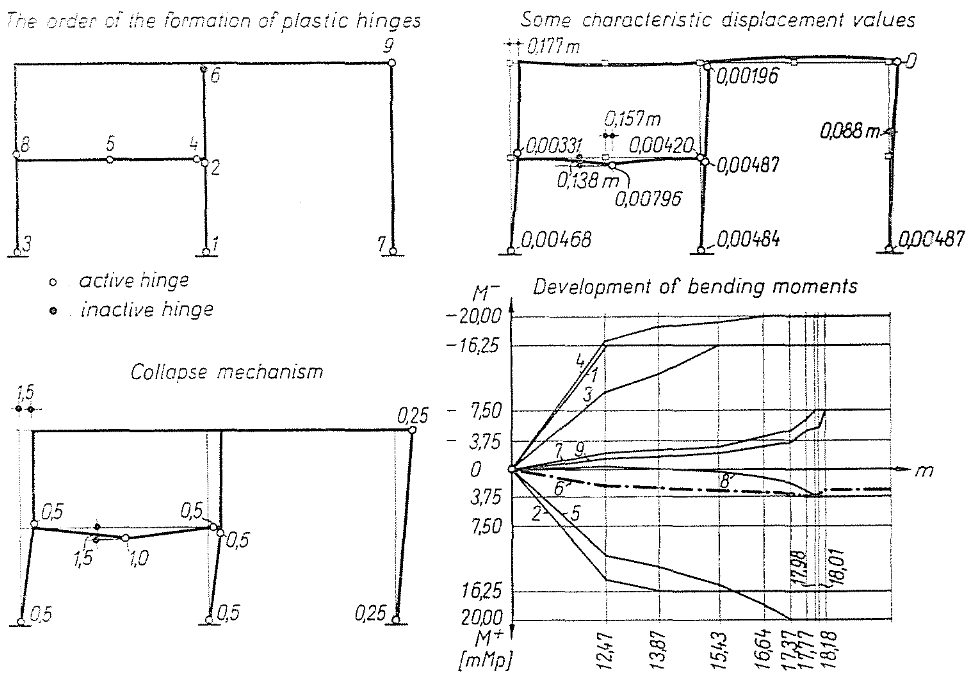


Fig. 2

by the necessary slight modification concerning the choice of the entering variable in the basis.

As an illustrative example, the analysis of the frame in Fig. 1 subject to proportional loading is presented requiring 161 s of computer running time.

The adopted yield condition is the classical plastic hinge assumption. Fig. 2 shows the sequence of the formation of the plastic hinges during the loading process, the displacements of the structure just before collapse, the collapse mechanism and the development of bending moment at cross sections where plastic hinges develop.

It is worth noting that in spite of proportional loading, hinge 6 formed at load parameter 17.77 becomes inactive at load parameter 18.01 illustrating the applicability of the method for cases of local plastic unloading.

Summary

Formulae are presented for the incremental elastic-plastic problem of linear elastic-perfect plastic finite degree of freedom systems. The conditions for the existence and uniqueness of a solution are outlined and a theorem is developed to characterize the structural response in a state of collapse. Using the known quadratic programming formulation to determine the plastic multiplier rates, it is shown that in both cases (collapse or not) Wolfe's short quadratic programming algorithm is applicable. A simple procedure is described for the analysis of the structural response under loading histories with piecewise constant load rates. Numerical computer analysis of a frame is given as an illustration.

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