THE BECK STABILITY PROBLEM FOR VISCO-ELASTIC BARS

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1. Introduction

As early as in 1952, ZIEGLER [8] pointed to the likelihood of wrong results from examining the stability of the equilibrium of non-conservative mechanical systems on the basis of the static stability criterion. At the same time, stability analysis of either conservative or non-conservative systems, based on the kinetic stability criterion, leads to correct results. Stability analysis of conservative mechanical systems by either method yields identical results.

Static stability criterion states critical load to be the least one where the original (trivial) equilibrium is coexistent with other (non-trivial) equilibria.

According to the kinetic stability criterion, critical load is the least one where motion due to a sufficiently small, arbitrary disturbance is not restricted to an arbitrary small region about the equilibrium of the balanced system. To support the above, the so-called BECK stability problem will be quoted ([2], [1]). Based on ZIEGLER's idea [8], BECK [2] treated the following problem:

Let us consider a slender beam of constant cross section, straight axis, of a homogeneous, isotropic, elastic material, clamped on one end. Compressive force **P** applied at the free bar end is of constant magnitude throughout the motion, but its direction always follows that of the tangent to the free bar end, hence it is a so-called "follower" force (Fig. 1). Magnitude of the critical load is sought for, omitting the dead load of the column. Because of the non-conservativity of the follower load, the static stability criterion yields a wrong

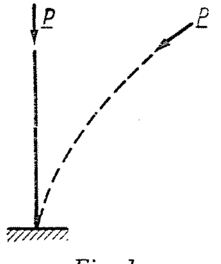

Fig. 1

result $P_{\text{crit}} \to \infty$; inexistence of a non-trivial equilibrium excludes the existence of a finite critical force.

BECK approximated the bar by a continuous model of linear elastic material; hence ignored the effect of viscosity. The kinetic stability criterion has led for the critical load to:

$$P_{\text{crit}} = 20.05 \frac{EJ_{\text{min}}}{l^2} \tag{1}$$

where EJ_{min} is the minimum flexural rigidity of the bar, and l its length.

To the author's knowledge, there is no similar formula available for the critical load of a visco-elastic continuous model. To help this deficiency, he attempted to approximate the critical load P_{crit} by the finite model analysis of a bar compressed by a follower load, involving the effect of viscosity.

Thus, in the following, the Beck stability problem will be considered in its finitized form, complemented by the assumption of the Kelvin-Voigt visco-elastic material model of the bar material, hence by the examination of the viscosity effect. According to the vibration terminology, the effect of viscosity will simply be termed damping.

The idea of assuming the problem is due to BOSZNAY [3].

A simple, finitized variant of the problem for two degrees of freedom has already been discussed by ZIEGLER [8]. In this case the stability analysis has been reduced to the solution of a fourth-degree algebraic equation, hence it can be treated even by analytic means i.e. by closed formulae.

Now, it will be demonstrated that the approximation by this model of two degrees of freedom is not close enough. (Notice that there are several possibilities of finitizing. For instance, OVERRATH [4] solved the Beck problem according to the SZABÓ—ROLLER general theory of bar systems [7]. The bar was approximated by elastic elements of two degrees of freedom each, and the effect of viscosity was ignored. This model offered a close approximation

$$P_{\text{crit}} = 20.19 \frac{EJ_{\text{min}}}{l^2}$$

with as few as two elements.)

ZIEGLER also examined the damping effect on his model of two degrees of freedom. He demonstrated that if the load applied on the elastic system was a non-conservative one, then the damping proportional to velocity (i.e. viscosity) might entrain instability; even a very small damping force might significantly alter the critical load.

The author's computations on models of several degrees of freedom supported Ziegler's statement on the instabilizing effect of damping (viscosity).

2. Motion equations

Let us consider the introductory Beck stability problem for a rectilinear, prismatic bar of a homogeneous, isotropic, visco-elastic material.

Deformations due to shear force and normal force are neglected; the problem is solved according to the second-order theory. Model in Fig. 2 is the Beck stability problem in its finitized form, besides this model takes also the viscosity effect, i.e. damping into consideration. The finitized model is

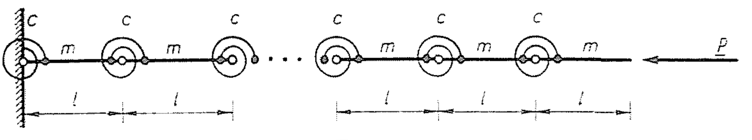

Fig. 2

a chain of n rigid members, each of length l and mass m. Hence, the total bar length is nl, and the mass nm. Elasticity of the original bar is simulated by springs in cylinder hinges supplying a return moment proportional to the relative rotation of the members. Each hinge has a springs constant $c = EJ_{\min}/l$, $EJ_{\min}$ being flexural rigidity of the bar cross section. Hinges also contain a device — although not indicated — supplying a damping moment proportional to the relative angular velocity of the members; moment damping coefficient is d for each hinge. Force $\mathbf{P}$ applied on the free chain end is of a constant value throughout the motion, and of the same direction as the last link, hence it is a so-called *follower* force.

Let us determine what is the least — critical — force likely to entrain kinetic instability.

Motion equations of the system will assume a slight in-plane motion and take the transversal displacement of bar element end points $y_0 = 0$, y_i; $i = 1, \ldots, n$ as co-ordinates (see Fig. 3).

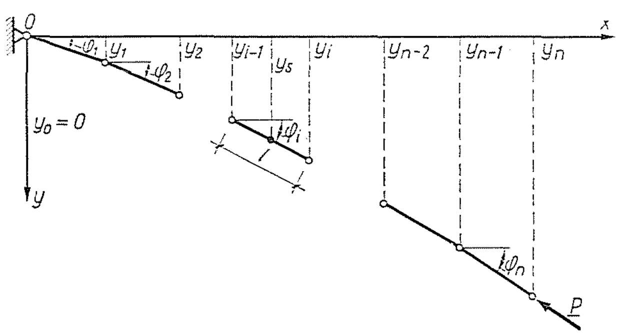

Fig. 3

Motion energy of the bar chain:

$$T = \frac{1}{2}\dot{\mathbf{y}}^*\mathbf{M}\dot{\mathbf{y}}$$

where elements of vector $\dot{\mathbf{y}}$ are velocities $\dot{y}_i$, $i = 1, \ldots, n$ and

$$\mathbf{M} = \frac{m}{6}\begin{bmatrix} 4 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 2 \end{bmatrix} \tag{2}$$

is the so-called mass matrix, always positive definite because the motion energy is also positive (except for the case at rest $\dot{\mathbf{y}} = 0$).

Potential energy of return moments supplied by hinge springs and proportional to the relative rotation of bar elements:

$$V = \frac{1}{2}\mathbf{y}^*\tilde{\mathbf{C}}\mathbf{y}$$

where elements of vector $\mathbf{y}$ are co-ordinates y_i; $i = 1, \ldots, n$ and

$$\tilde{\mathbf{C}} = \frac{c}{l^2}\begin{bmatrix} 6 & -4 & 1 & & & \\ -4 & 6 & -4 & 1 & & \\ 1 & -4 & 6 & -4 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & 1 & -4 & 6 & -4 & 1 \\ & & 1 & -4 & 5 & -2 \\ & & & 1 & -2 & 1 \end{bmatrix} \tag{3}$$

the so-called spring matrix, always positive definite since the potential energy is also positive.

Effect of damping moments supplied by the spring device and proportional to the relative angular velocity of bar elements is expressed by the dissipation function

$$F = \frac{1}{2}\dot{\mathbf{y}}^*\mathbf{D}\dot{\mathbf{y}}$$

where the so-called damping matrix

$$\mathbf{D} = \frac{d}{l^2}\begin{bmatrix} 6 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 5 & -2 \\ & & & & 1 & -2 & 1 \end{bmatrix} \tag{4}$$

is positive definite because of the viscous damping force; d is the damping coefficient for each hinge moment.

Effect of the "follower" force $\mathbf{P}$ applied at the free chain end is expressed by

$$\mathbf{Q} = \mathbf{R}\mathbf{y}$$

where elements of vector $\mathbf{Q}$ numbering n are generalized forces and a quadratic matrix of order n:

$$\mathbf{R} = \frac{P}{l}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ \cdot & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & -1 & 2 & -1 \\ & & & 0 & 0 \end{bmatrix}.$$

Substituting into the Lagrange motion equation

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\mathbf{y}}}\right) + \frac{dF}{d\dot{\mathbf{y}}} + \frac{dV}{d\mathbf{y}} = \mathbf{Q}$$

we obtain

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{D}\dot{\mathbf{y}} + \tilde{\mathbf{C}}\mathbf{y} = \mathbf{R}\mathbf{y}.$$

Rearranging:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{D}\dot{\mathbf{y}} + \mathbf{C}\mathbf{y} = \mathbf{0} \tag{5}$$

where

$$\mathbf{C} = \tilde{\mathbf{C}} - \mathbf{R} =$$

$$= \frac{c}{l^2} \begin{bmatrix} 6 - \frac{2Pl}{c}, & -4 + \frac{Pl}{c}, & 1 & & & & \\ -4 + \frac{Pl}{c}, & 6 - \frac{2Pl}{c}, & -4 + \frac{Pl}{c}, & 1 & & & \\ 1, & -4 + \frac{Pl}{c}, & 6 - \frac{2Pl}{c}, & -4 + \frac{Pl}{c}, & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & 1 & -4 + \frac{Pl}{c}, & 6 - \frac{2Pl}{c}, & -4 + \frac{Pl}{c}, & 1 & \\ & & 1 & -4 + \frac{Pl}{c}, & 5 - \frac{2Pl}{c}, & -2 + \frac{Pl}{c} & \\ & & & 1 & -2 & 1 & \end{bmatrix} \quad (6)$$

a quadratic and — because of the non-conservativity of the follower force — non-symmetric matrix of order n.

3. Mathematical analysis of kinetic stability

Motion equations of the finitized variant of Beck's stability problem have been written as matrix equation (5), where $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{C}$ are matrices detailed under (2), (4) and (6), respectively.

Let us consider in general the linear mechanic systems of n degrees of freedom, where the motion in the vicinity of the equilibrium configuration — given by a vector $\mathbf{y} = \mathbf{0}$ of n elements — is described by the homogeneous differential equation system (5), $\dot{\mathbf{y}}$ and $\ddot{\mathbf{y}}$ being first and second derivatives, respectively, of vector $\mathbf{y}(t)$ with respect to time.

Equilibrium of the mechanical system of vector $\mathbf{y} = \mathbf{0}$ is kinetically stable if, provided initial values

$$\mathbf{y}(0) = \mathbf{y}_0, \qquad \dot{\mathbf{y}}(0) = \dot{\mathbf{y}}_0 \tag{7}$$

have been chosen sufficiently small, solution $\mathbf{y}(t)$ of differential equation system (5) remains arbitrarily small for any $t > 0$.

If any initial condition (7) of the differential equations (5) has a bounded solution $\mathbf{y}(t)$, then (multiplying the initial conditions by a constant) the solution can be made arbitrarily small.Thus, for a bounded solution $\mathbf{y}(t)$ the equilibrium is kinetically stable.

Solution of (5) has to be found in the form:

$$\mathbf{y} = \mathbf{z}e^{\lambda t} \tag{8}$$

substituting it into the differential equation leads to the generalized eigenvalue problem

$$(\mathbf{M}\lambda^2 + \mathbf{D}\lambda + \mathbf{C})\mathbf{z} = \mathbf{0}\,. \tag{9}$$

Trivial solution $\mathbf{z} = \mathbf{0}$ of the homogeneous linear algebraic equation system (9) corresponds to the equilibrium state $\mathbf{y} = \mathbf{0}$ of the mechanical system — origin of the term "trivial equilibrium".

A non-trivial solution exists if and only if λ is root of the generalized characteristic equation

$$|\,\mathbf{M}\lambda^2 + \mathbf{D}\lambda + \mathbf{C}\,| = 0 \tag{10}$$

an algebraic equation at most of $2n$ order in λ, and exactly of $2n$ order for $|\,\mathbf{M}\,| \neq 0$.

Provided all roots λ_j, $j = 1, \ldots, 2n$ are real and distinct, general solution of (5) may be written as linear combination of $2n$ functions

$$\mathbf{y}_j(t) = \mathbf{z}_j e^{\lambda_j t} \tag{11}$$

where $\mathbf{z}_j$ is the generalized eigenvector belonging to the generalized eigenvalue λ_j. Constants c_j, $j = 1, \ldots, 2n$ in the general solution

$$\mathbf{y}(t) = \sum_{j=1}^{2n} c_j \mathbf{z}_j e^{\lambda_j t} \tag{12}$$

can be determined from initial conditions (7). In the occurrence of complex generalized eigenvalues where nevertheless all roots are distinct, they may be included as conjugated pairs (since matrices $\mathbf{M}$, $\mathbf{D}$, $\mathbf{C}$ are real); it may be written $\lambda = \alpha \pm i\beta$. Also the corresponding generalized eigenvectors are conjugated complex pairs of vectors $\mathbf{u} \pm i\mathbf{v}$ and, according to (8), the general solution contains terms of the form:

$$\mathbf{z}e^{\lambda t} + \bar{\mathbf{z}}e^{\bar{\lambda} t} = e^{\alpha t}[(\mathbf{u} \pm i\mathbf{v}) \cos \beta t - (\mathbf{v} \mp i\mathbf{u}) \sin \beta t]. \tag{13}$$

Since real and imaginary parts of complex solutions are each a solution of (5), complex solution (13) can be replaced in the general solution by two, linear independent real solutions of the form:

$$\mathbf{a}e^{\alpha t} \cos \beta t \qquad \text{and} \qquad \mathbf{b}e^{\alpha t} \sin \beta t. \tag{14}$$

Kinetic instability arises if there is at least one finite initial condition leading to unbounded motion according to the general solution (12). This is only possible if at least one among the $2n$ linear independent solutions constituting the general solution is unbounded. According to (8) and (14), if there exists at least one generalized eigenvalue with a positive real part, then the equilibrium of the mechanical system is kinetically unstable.

If some of the generalized eigenvalues are equal roots, there are two cases. Either is

$$\mathbf{K}(\lambda) = \mathbf{M}\lambda^2 + \mathbf{D}\lambda + \mathbf{C} \tag{15}$$

a simple λ-matrix, i.e. for each root λ of multiplicity r, exactly r linear independent generalized eigenvectors $\mathbf{z}$ can be determined, then the general solution consists exclusively of solutions of the form (8) or (14). Or is $\mathbf{K}(\lambda)$ no simple λ-matrix, then the general solution will be found as linear combination of the exponential function multiplied by a polynomial with the variable t:

$$\mathbf{p}(t)e^{\lambda t}. \tag{16}$$

Except for $\lambda = 0$ or $\alpha = 0$ (i.e. where λ is a pure imaginary root), functions (16) and (8) have the same asymptotic properties (if e.g. $\lim_{t\to\infty} \mathbf{z}e^{\lambda t} = \mathbf{0}$ then also $\lim_{t\to\infty} \mathbf{p}(t)e^{\lambda t} = \mathbf{0}$).

As a conclusion:

1. If all the generalized eigenvalues $\lambda_j = \alpha_j + i\beta_j$ have negative real parts, i.e.:

$$\alpha_j < 0, \quad j = 1, \ldots, 2n .$$

the equilibrium of the mechanical system is *kinetically* stable.

2. If there exists at least one generalized eigenvalue λ_j with the real part:

$$\alpha_j > 0$$

then the equilibrium of the system is *kinetically unstable.*

3. If the roots include no generalized eigenvalue with positive real part but at least one generalized eigenvalue with zero real part exists, then the problem of kinetic stability has to be decided from further information. Hence, this is the *critical case.*

In conformity with the above, in critical cases the problem of kinetic stability may be decided by examining matrices $\mathbf{K}(\lambda_j)$ where λ_j are generalized eigenvalues with zero real part. If the degeneracy of any such matrix $\mathbf{K}(\lambda_j)$ (i.e. the number of linear independent generalized eigenvectors belonging to each pure imaginary generalized eigenvalue λ_j) equals the multiplicity of root λ_j, the system is *kinetically stable;* if there exists a pure imaginary generalized eigenvalue λ_j with a degeneracy of $\mathbf{K}(\lambda_j)$ less than the multiplicity of root λ_j, the system's equilibrium is *kinetically unstable.*

4. Algorithm of the computation

Let us assume the availability of a numerical method suiting solution of generalized eigenvalue problem (9). As seen from the detailed expression (6) of matrix $\mathbb{C}$, its elements are functions of the value P of follower force $\mathbb{P}$, hence also generalized eigenvalues are functions of value P. The least value P (critical force) is sought for, where kinetic instability may occur.

According to the above, the critical force can be numerically determined by means of the following algorithm. Start computation at a small P (e.g. $P = 0$) where the system is in a kinetically stable equilibrium. The P value will be increased step by step by equal increments ΔP until P causes the equilibrium to become kinetically unstable. The computation is continued with the mid-point of interval $\langle P - \Delta P, P \rangle$. Depending on whether for a load $P - \frac{\Delta P}{2}$ the system is in kinetically stable or unstable equilibrium, the critical load is element of interval $\left(P - \frac{\Delta P}{2}, P \right\rangle$ or $\left\langle P - \Delta P, P - \frac{\Delta P}{2} \right)$. Continuing this halving procedure throughout for the interval including the critical load, then after k steps the interval containing the critical load will be of length $2^{-k} \Delta P$. Theoretically, this interval length can be arbitrarily restricted by properly increasing the number k of steps.

Obviously, in computing the critical force, the most of difficulty is due to the numerical solution of the generalized eigenvalue problem (9).

According to a current method, the generalized eigenvalue problem can be reduced to the solution of the special eigenvalue problem of a hypermatrix of order $2n$:

$$\begin{bmatrix} -\mathbb{M}^{-1}\mathbb{D} & -\mathbb{M}^{-1}\mathbb{C} \\ \mathbb{E} & \mathbb{0} \end{bmatrix}$$

where $\mathbb{E}$ is the unit matrix of order n.

From computation aspects, the obvious difficulty of this method consists in doubling the order of the matrix. Therefore the author has developed a new procedure for solving generalized eigenvalue problem (9), consisting essentially in reducing the generalized eigenvalue problem to the solution of the special eigenvalue problem of two matrices, of order n each, via solution of a quadratic algebraic equation system. This permits to determine the generalized eigenvalues and generalized eigenvectors at a higher accuracy — as exemplified in [6] — than by reducing the problem to the solution of the special eigenvalue problem of the matrix of order $2n$.

This procedure [6] has been applied for the generalized eigenvalue problem (9) involved in the Beck stability problem.

5. Conclusions

Let us consider a bar of straight axis, constant square cross section, made of a homogeneous, isotropic, first linear elastic, then linear visco-elastic material with the following geometry: bar length $L = 1$ m, cross section area $F = 2\ \text{cm} \times 2\ \text{cm} = 0.0004\ \text{m}^2$, density $\varrho = 7850\ \text{kp/m}^3$ and modulus of elasticity $E = 21 \times 10^9\ \text{kp/m}^2$.

Applying result (1) by Beck for the elastic continuous model, the critical force is seen to be:

$$P_{\text{crit}} = 20.05 \frac{EJ_{\min}}{L^2} = 5614\ \text{kp}, \tag{17}$$

since

$$J_{\min} = ab^3/12 = \frac{4}{3}\, 10^{-8}\ \text{m}^4.$$

To have an idea of the accuracy of the approximation by a finite model, computations will first refer to an elastic (undamped) model; namely then the results for the critical force can be compared to Beck's formulae (17) equally for an elastic continuous model.

Variation of the critical force P_{crit} determined on the elastic finite model vs. number of freedoms, as well as the critical force from the continuous model have been plotted in top of Fig. 4. Critical force obtained from the finite model appears to approximate asymptotically, from below, the critical force obtained for the continuous model. It is also obvious that while the critical force for the case $n = 2$ is only about 50% of the continuous value, for the case $n = 15$ the deviation is reduced to below 7%. With a view on design safety prescriptions, it is advantageous to have the critical force calculated

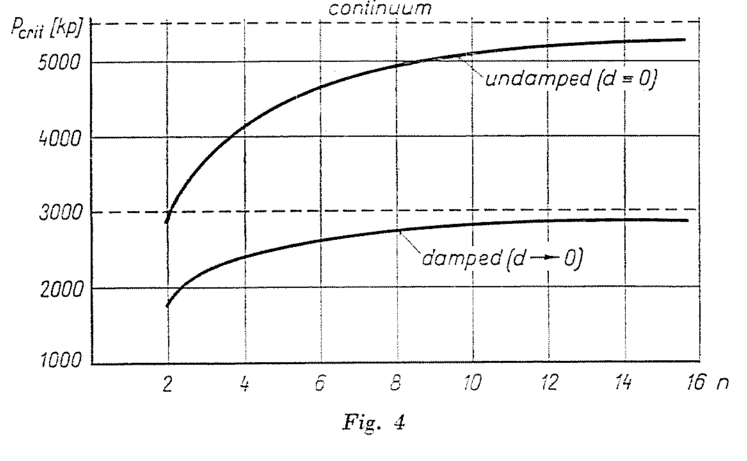

Fig. 4

from a finite model tending to the critical force calculated from the continuous model from below.

In the case of an undamped, finite model $\mathbf{D} = \mathbf{0}$ may be written, causing generalized eigenvalue problem (9) to become:

$$(\mathbf{M}\lambda^2 + \mathbf{C})\mathbf{z} = \mathbf{0} \tag{18}$$

i.e., with the notation

$$\lambda^2 = -\varkappa$$

special eigenvalue problem of matrix $\mathbf{M}^{-1}\,\mathbf{C}$ of order n, of real (constant) elements, that is, however, non-symmetric, because of the non-conservativity of the follower force, and it can be solved by directly applying e.g. the Francis double-step QR algorithm [5].

Since matrix $\mathbf{M}^{-1}\mathbf{C}$ is a real one, complex eigenvalues $\varkappa$ occur always in conjugated pairs. In knowledge of eigenvalues of matrix $\mathbf{M}^{-1}\mathbf{C}$, the generalized eigenvalues sought for are delivered by:

$$\lambda = \pm\sqrt{-\varkappa}$$

Consequently, if matrix $\mathbf{M}^{-1}\mathbf{C}$ has at least one negative real or complex eigenvalue $\varkappa$, then there exists a generalized eigenvalue λ that is either positive real or has a positive real part. Accordingly, — in conformity with item **3** — the mechanical system is in kinetically unstable equilibrium. The equilibrium of the system *can* only be stable if all eigenvalues $\varkappa$ of matrix $\mathbf{M}^{-1}\mathbf{C}$ are non-negative real numbers; namely then all generalized eigenvalues λ are pure imaginary (taking $\lambda = 0$ as such), at the same time this is the critical case according to item **3**: if $\mathbf{M}^{-1}\mathbf{C}$ is a matrix of simple structure then the equilibrium is kinetically stable, else it is unstable. Let us remark that if all eigenvalues $\varkappa$ are different (and so are all λ) then $\mathbf{M}^{-1}\mathbf{C}$ is a matrix of simple structure.

Variation of generalized eigenvalues λ performing the transition from kinetically stable to unstable equilibrium versus follower force P (for $n = 3$ and $n = 9$ degrees of freedom) has been plotted in Fig. 6 separately for real part $R_e(\lambda)$ and imaginary part $J_m(\lambda)$. The different pure imaginary generalized eigenvalues are seen to converge with the increase of force P (stable range) to become pure imaginary twice generalized eigenvalues (critical case). Since for an infinitesimal increase of force P, generalized eigenvalues become a conjugated complex pair with a non-zero real part (unstable range), the critical case is the critical force, no further analysis to determine the structure of matrix $\mathbf{M}^{-1}\mathbf{C}$ is needed.

Computations made on undamped and damped finite models differ by requiring the solution of problem (9) rather than (18). The difference between both eigenvalue problems is not of numerical importance alone. No ring-off motion is possible in the undamped case; stability refers to the oscillating motion.

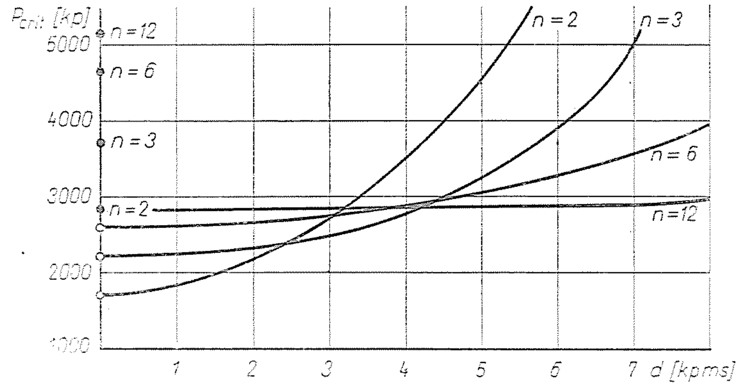

Fig. 5

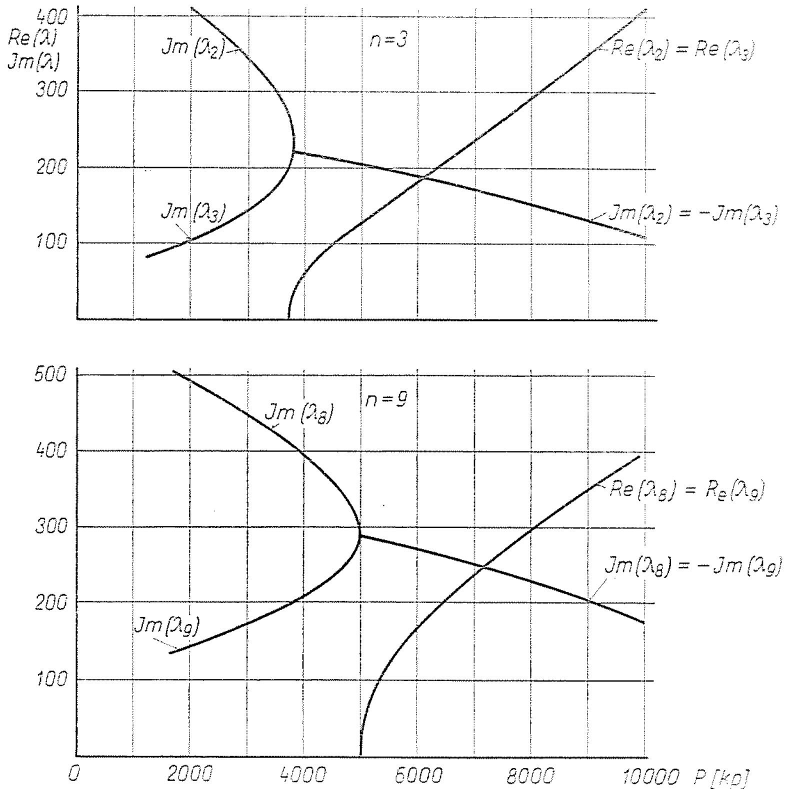

Fig. 6

Variation of critical force P_{crit} vs. damping coefficient d has been plotted in Fig. 5 for various degrees n of freedom, exhibiting the inequality expressing the instabilizing effect of small viscous damping:

$$\lim_{d \to 0} P_{\text{crit}}(d) < P_{\text{crit}}(0)$$

supporting for a system of more than $n = 2$ degrees of freedom Ziegler's statement on the instabilizing effect of the damping force deduced from the analysis of a system of two degrees of freedom. $P_{\text{crit}}(0)$ values are accompanied by the corresponding n values in Fig. 5.

The possibility appears from Fig. 5 that, if $n \to \infty$ then $P_{\text{crit}}(d) \to$ const. for all positive d values. Is it so then in spite of the instabilizing effect of viscosity, for models of high degrees of freedom, critical force P_{crit} is likely to little depend on the damping coefficient $d > 0$, a question worth of further consideration.

The instabilizing effect of an even slight viscous damping d appears from Fig. 4, indicating — for the sake of comparison — the critical force vs. the degrees of freedom, and this separately for the elastic (undamped, $d = 0$) case, and for a slight damping coefficient $d \to 0$.

Fig. 4 demonstrated critical force values — computed from a finite model taking the damping effect into consideration — to tend asymptotically, from below, to a limiting value with increasing degrees of freedom, assumed to equal the critical force that would result from the analysis of a visco-elastic continuous model.

In reality there is no elastic bar, a slight viscosity is always present. Therefore this critical load — extrapolated from damped model analyses — yields a closer, safer approximation of the exact value than formula (1) by Beck.

Summary

A finitized variant of the Beck stability problem has been discussed, completed with the assumption of the Kelvin—Voigt visco-elastic material model for the bar material, hence involving the effect of viscous damping.

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