NUMERICAL SOLUTION OF THE INTEGRAL EQUATION OF A HINGED BRIDGE STRUCTURE

By

K. Árvay

Department of Civil Engineering Mechanics, Technical University, Budapest

Received June 21, 1973

Presented by Prof. Dr. S. Kaliszky

1. Scope

In a previous paper [1], mechanical analysis of a hinged bridge structure has been discussed. Forces developing in hinges of precast main girders have been determined by an integral equation system. Functions of the horizontal and vertical components of the resulting hinge force have been obtained as a series of eigenfunctions of the integral equations.

In present study the hinge forces between the main girders of a skew bridge are determined by numerically solving the integral equation system. Simpler cases, important for practical reasons, and possible simplifications are discussed.

Stipulations for the tested bridge structures are identical with those in [1].

1.1 *Symbols*

Beside definitions used in the text and in the figures, the following symbols occur:

E modulus of elasticity
G modulus of elasticity in shear
I inertia moment of bending
I_c inertia moment of torsion
I_t inertia moment of warping rigidity
L span of the girder
m number of intermediate hinges
n number of main girders
z ordinate along the axis of the girder
α angle characteristic for the obliquity of the bridge
$\varepsilon_{\max}$ $|\varepsilon_k| < \varepsilon_{\max}$ [ε_k see under Eq. (8)]
maximum error value of the numerical solution.

2. Numerical solution of the integral equation

Several factors might necessitate a numerical solution of the integral equation, for instance, if the core function [1] of the integral equation can be determined only with difficulties or not at all, because of the form of the precast main girder. Provided the hinge connecting the main girders is not continuous, the numerical solution yields a more exact result than the assumption of a continuously distributed hinge force. The solution of the integral equation obtained in a closed form can neither be used if the transfer of the hinge force is bound to conditions. In practice this occurs when the horizontal component of the hinge force is taken into account but the hinge can only transmit the compressive force between the girders, while in case of a tensile force, there is no connection any more. Here the integral equation can only be solved numerically as discussed below. In other instances a choice can be made between the numerical and the direct solution [1].

In [1], the function of the hinge force components was obtained from an integral equation system. One integral equation of this system was written in the following form:

$$y(z) = \int_0^L K_i(z,\xi)\, p_i(\xi)\, d\xi - \int_0^L K_j(z,\xi)\, p_{i-1}(\xi)\, d\xi + \\ + \int_0^L K_l(z,\xi)\left[h_i(\xi) - h_{i-1}(\xi)\right] d\xi + y_0(z). \tag{1}$$

Instead of the unknown displacement $y(z)$ and force functions $p(z)$, $h(z)$ in the above equations, their approximating values are sought at equidistant points on the hinge line:

$$\Delta z = \frac{L}{m+1}. \tag{2}$$

By means of procedures of numerical integration, terms in Eq. (1) can be approximated by the following sum:

$$\int_0^L K_i(z,\xi)\, p_i(\xi)\, d\xi \approx \sum_{K=0}^{m+1} C_k K_i(z, z_k)\, p(z_k). \tag{3}$$

The values of the coefficients $C_0, C_1, \ldots, C_{m+1}$ were determined on hand of the trapezoid rule, taking into account boundary conditions of the unknown functions [2]. Thus:

$$C_0 = C_{m+1} = 0 \quad \text{and} \quad C_1 = C_2 = \ldots = C_m = \frac{L}{m+1}. \tag{4}$$

Substituting into Eq. (1) the ordinates of division points spaced apart by Δz on the hinge line and replacing the integral by (3), the integral equation can be substituted by a linear equation system. When substituting the ordinates of points on the hinge line, the obliquity of the bridge must also be taken into consideration. Therefore the ordinate z_k of the hinge line changes by $d_i = 0.5 b_i \operatorname{tg} \alpha$, and depending on the obliquity of the bridge and on the left or right side hinge line, Eq. (1) has to be written with girder axis values of:

$$z_k = k \cdot \Delta z \pm d_i \quad (k = 1, 2, \ldots, m). \tag{5}$$

Before writing the equation system let us introduce notations:

$$\begin{aligned} C_j p_i(z_j) &= P_{ij} \\ C_j h_i(z_j) &= H_{ij}. \end{aligned} \tag{6}$$

The expression (6) stands for the hinge force developing as a force concentrated at hinge points (Fig. 1) considered also as the resultant of the distributed hinge force along interval Δz.

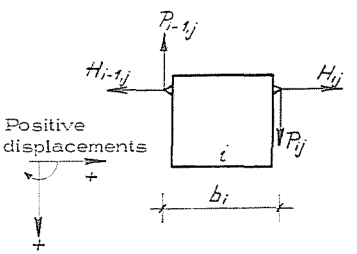

Fig. 1

Now the integral equation (1) can be substituted by the following linear equation system:

$$\begin{aligned} y(z_k) = \sum_{j=1}^{m} K_i(z_k, z_j) P_{ij} - \sum_{n=1}^{m} K_j(z_k, z_j) P_{i-1,j} + \\ + \sum_{j=i}^{m} K_l(z_k, z_j) [H_{i,j} - H_{i-1,j}] + y_0(z_k). \end{aligned} \tag{7}$$

Writing the above relationship for values $k = 1, 2, \ldots, m$, this equation system substitutes integral equation (1). Determination of the coefficients in (7) and the equation system expressing the compatibility conditions of the bridge structure will be discussed in the next item.

The exact function value, obtained by solving the integral equation, is examined for the approximation error involved in values $p(z_i)$ and $h(z_i)$, determined from the equation system based on (7).

The analysis was carried out according to [2]. It results in the error value only if Eq. (1) can be written and solved. In this case a limit value can be given for the difference of

$$p_i(z_k) - \frac{P_{ik}}{\Delta z} = \varepsilon_k \tag{8}$$

or for the maximum of differences between the results from (1) and (7).

The upper limit ($\varepsilon_{\max}$) for ε in (8) can be calculated. Instead of the rather long relationship, only its valuation is given here. For a given span L and spacing, the error decreases with increasing girder rigidity. It is especially sensitive to the torsion rigidity value: if the torsion rigidity is near the minimum value [1] derived of the eigenvalue of the integral equation, the approximation error is considerably increasing. The error depends also on the distribution of the loads. For given L, $\varepsilon_{\max}$ decreases quadratically with the increase of the number of spacings, for identical spacing Δz it increases nearly quadratically with the span. The effect of width b is unimportant, provided the torsion rigidity is adequate. The obliquity of the bridge is not too important for the error value, provided other data are identical. Examining actual cases, for Δz of about the girder width, the two methods yield values for midspan deflections due to unit force differing by less than 2%.

3. General formulation of the linear equation system

Top view of the bridge in Fig. 2 indicates points where the unknown function values of integral equation (1) are to be determined by the approximating term (7). With regard to (6), this problem can be solved both by applying the mathematic method outlined above, and by using a new mathematical model: the main girders are connected by point-like in-plane hinges at given, equal spacings. Testing the structure by the force method, applying a structure deprived of hinges as primary beam, writing the zero relative displacements at the hinges, Eqs (7) are obtained. In this formulation the explicit determination of the Green-functions can be omitted, because the coefficients $K(z_i, z_k)$ i.e. horizontal and vertical displacements due to bending and torsion resp., of each girder in the marked points of Fig. 2 can be calculated with well-known methods.

In Fig. 3, one section of two adjacent girders is plotted, marking some hinge points. This figure shows that in case of a skew bridge forces P_{ij} and H_{ij}

belong to different cross-sections of two adjacent girders. Therefore displacements of point (ij) have also to be determined in different cross-sections of ordinate z for adjacent girders.

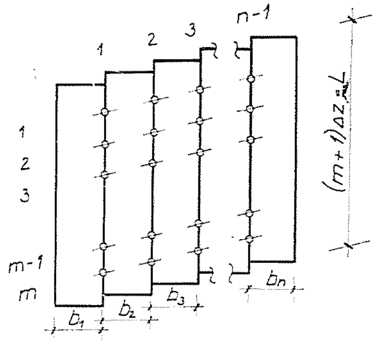

Fig. 2

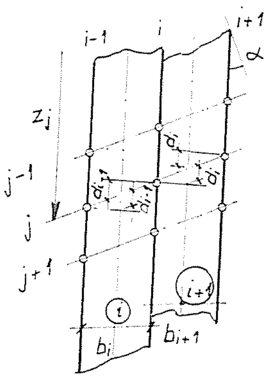

Fig. 3

To determine the unknown hinge forces, displacements at every point have to be calculated from the unit forces acting at any hinge point. A force causes displacement only of the girder it acts at, displacements originating from one force are summed in one vector, then these vectors are assembled in a coefficient matrix, where in the subscript the serial numbers of the hinge line, first of the displacement, and then of the acting force are written (Fig. 2). Where the serial number of the girder cannot be established from the two subscripts, it is given as superscript. The signs of displacements and forces are given in Fig. 1.

The vertical and horizontal displacements due to unit vertical forces are summed up in matrices $\mathbf{A}_{ij}$ and $\mathbf{V}_{ij}$, respectively; the vertical and horizontal ones due to horizontal forces in matrices $\mathbf{U}_{ij}$ and $\mathbf{B}_{ij}$, respectively.

The a_{ji} etc. elements of these matrices correspond to the values $K(z_i, z_j)$ of the core functions of the integral equation system. The sign convention is shown in Fig. 1. These matrices are of $m \cdot m$ size. At a difference from usual coefficient matrices, the matrices with different subscripts are not symmetrically established, because, due to the obliquity, interchange of the coefficient subscripts does not mean the exchange of cross-sections. If no stipulations are made concerning the form of the girders, these four coefficient matrices are in general not identical.

The unknown vertical and horizontal forces arising at each hinge point on the hinge line of order i are denominated by vectors $\mathbf{P}_i$ and $\mathbf{H}_i$, respectively. Summing up the vertical and the horizontal displacement at the hinge points due to external forces in vectors $\mathbf{A}_{i,0}$ and $\mathbf{B}_{i,0}$, resp., connecting conditions vertically of hinge points on line of order i are formulated in the following matrix equation:

$$\begin{aligned} &-\mathbf{A}_{i,i-1}\mathbf{P}_{i-1} + (\mathbf{A}^{i}_{i,i} + \mathbf{A}^{i+1}_{i,i})\mathbf{P}_i - \mathbf{A}_{i,i+1}\mathbf{P}_{i+1} - \mathbf{U}_{i,i-1}\mathbf{H}_{i-1} + \\ &+ (\mathbf{U}^{i}_{i,i} - \mathbf{U}^{i+1}_{i,i})\mathbf{H}_i + \mathbf{U}_{i,i+1}\mathbf{H}_{i+1} = \mathbf{A}^{i+1}_{i,0} - \mathbf{A}^{i}_{i,0}\,. \end{aligned} \tag{9}$$

And horizontally:

$$\begin{aligned} &\mathbf{V}_{i,i-1}\mathbf{P}_{i-1} + (\mathbf{V}^{i}_{i,i} - \mathbf{V}^{i+1}_{i,i})\mathbf{P}_i - \mathbf{V}_{i,i+1}\mathbf{P}_{i+1} - \mathbf{B}_{i,i-1}\mathbf{H}_{i-1} + \\ &+ (\mathbf{B}^{i}_{i,i} + \mathbf{B}^{i+1}_{i,i})\mathbf{H}_i - \mathbf{B}_{i,i+1}\mathbf{H}_{i+1} = \mathbf{B}^{i+1}_{i,0} - \mathbf{B}^{i}_{i,0}\,. \end{aligned} \tag{10}$$

According to the above two equations, a matrix equation system formulated for $i = 2, \ldots n - 1$ and taking into consideration $\mathbf{P}_0 = \mathbf{H}_0 = \mathbf{P}_n = \mathbf{H}_n = 0$ yields the unknown hinge forces. It is useful to write the equation system as the following hypermatrix equation, where the individual blocs are hyper-continuants of size $(n - 1) \cdot (n - 1)$ consisting of identically lettered matrices as coefficients:

$$\begin{bmatrix} \mathbf{A} & \mathbf{U} \\ \mathbf{V} & \mathbf{B} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{P} \\ \mathbf{H} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_0 \\ \mathbf{B}_0 \end{bmatrix} = 0\,. \tag{11}$$

This equation system substitutes the initial integral equation system [1]. There $2(n - 1)$ integral equations gave $2(n - 1)$ unknown functions $p(z)$ and $h(z)$. Here the hypermatrix equation (11) contains $2(n - 1)$ unknown vectors and each vector is of size m. In solving the above equation three cases have to be considered, according to the design of the hinge:

a) The hinge can transmit any horizontal force. In this instance, Eq. (11) has to be solved according to the known method.

b) Effect of the horizontal force is neglected. In this case the equation reduces to the form $\mathbf{AP} + \mathbf{A}_0 = 0$.

c) The hinge can transmit a horizontal force of limited magnitude. Here those connecting places have to be found where the given load causes a disconnecting displacement. Equations and unknowns corresponding to these have to be cancelled from Eq. (11) i.e. eliminating these rows and columns of the coefficient matrix, the size of the coefficient matrix and the unknown vector $\mathbf{H}$ have to be reduced according to the number of disconnections. For the new solution all the places have to be re-examined, of course implementing the necessary corrections again, and so on. This labour-absorbing work necessitates in general the use of a digital computer where the steps can easily be programmed.

4. Special cases

Since in practical cases, such bridge structures are always prefabricated, main girders are identical, requiring fewer coefficient matrices to be computed. This statement is valid for all four blocs of the hypermatrix coefficient of Eq. (11). In case of identical subscripts, the coefficient matrices of Eqs (9) and (10) are symmetrical and there are two types for each bloc: value calculated for the girder on the left or the right side of the tested hinge line. Coefficient matrices with different non-zero subscripts are asymmetrical but related as $\mathbf{A}_{i,i-1} = \mathbf{A}^{*}_{i,i+1}$. A similar equality exists in the blocs $\mathbf{B}$, $\mathbf{U}$ and $\mathbf{V}$. Furthermore there exists an equality between the coefficient matrices of blocs $\mathbf{U}$ and $\mathbf{V}$ of identical position. Therefore only three coefficient matrices each of three blocs have to be determined for any number of main girders.

Further reductions are possible if $\alpha = 0°$ viz. for an orthogonal bridge. Here the main diagonals of both blocs $\mathbf{A}$ and $\mathbf{B}$ contain the sum of two identical matrices each: $\mathbf{A}^{i}_{i,i} + \mathbf{A}^{i+1}_{i,i} = 2\mathbf{A}_{i,i}$, while the two adjacent coefficient matrices are symmetrical and equal: $\mathbf{A}_{i,i-1} = \mathbf{A}_{i,i+1}$ etc. The elements in the main diagonals of blocs $\mathbf{U}$ and $\mathbf{V}$ will be zero and the other coefficient matrices identical. Thus, to determine blocs $\mathbf{A}$ and $\mathbf{B}$ it is sufficient to establish two symmetrical coefficient matrices each: $\mathbf{A}_{i,i}$ and $\mathbf{A}_{i,i-1}$; and $\mathbf{B}_{i}$ and $\mathbf{B}_{i,i-1}$, resp., while for blocs $\mathbf{U}$ and $\mathbf{V}$ a single one: $\mathbf{V}_{i,i-1}$. In this case these coefficients can also be formulated in the simple way by means of the Green-functions [3].

The calculation is much simplified by neglecting the horizontal forces at the hinges. This neglect causes no important deviation in girder stresses as the horizontal component of the hinge force is by many orders smaller than the vertical one. This ratio is easily established by the iterative solution of Eq. (11). Choosing $\mathbf{H}_1 = 0$, then in the first step $\mathbf{P}_1 = -\mathbf{A}^{-1}\mathbf{A}_0$. Upon substituting the value $\mathbf{H}_1 = -\mathbf{B}^{-1}(\mathbf{B}_0 + \mathbf{V}\mathbf{P}_1)$ is obtained. Because of forces $\mathbf{H}_1$ in the second step vector $\mathbf{P}_1$ has to be increased by vector $\Delta\mathbf{P}_1 = -\mathbf{A}^{-1}\mathbf{U}\,\mathbf{H}_1$. Continuing the relaxation, $\mathbf{H}_1$ must be corrected by vector $\Delta\mathbf{H}_1 = -\mathbf{B}^{-1}\mathbf{V}\Delta\mathbf{P}_1$ etc. Numerical analysis of the above steps with data of constructed bridge structures [1] has shown that element sizes in $\mathbf{H}_1$ are only 0.01 to 0.03 times those in $\mathbf{P}_1$ and vector $\Delta\mathbf{P}_1$ can be neglected against vector $\mathbf{P}_1$. Therefore the assumption $\mathbf{H} = 0$ is permitted for the practical requirements of accuracy.

5. Numerical example

With the above procedure, the transversal distribution diagram of a bridge consisting of $n = 8$ identical box-section main girders 1,00 m wide for the parapet main girder and the main girder near the midline was determined.

Starting data:

$G = 0.416 \cdot E$

$I = 0.02004$ m^4 $\qquad I_c = 0.0344$ m^4 $\qquad I_t = 0.000341$ m^6

The computation was carried out for $\alpha = 0°$, i.e. an orthogonal bridge, and for one with $\alpha = 30°$.

Hinge points were assumed at the eighth-points of the span ($m = 7$).

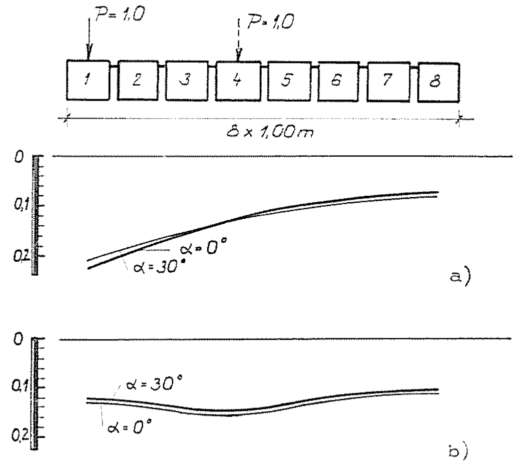

Fig. 4

The sketch of the bridge cross-section, the transversal distribution diagram of the parapet main girder and the intermediate main girder are plotted in Fig. 4. Computation assuming a continuous hinge line was also carried out with the integral equation system [1]: the obtained result closely agreed with the indicated results, the max. deviation being less than 2%.

Neglect of the horizontal hinge forces has no practical effect on the result. This neglect halved, however, the number of the unknowns in the equation system and the coefficient matrix could be established with only two formulae which could be written in a closed form.

6. Generalization of the procedure

The presented calculation procedure concerned bridge structures with hinged main girders. A case was discussed where the approximate numerical solution of the integral equation system applied for the problem in [1] had to be found. Now the possibilities to use the above procedure for other problems and its apparent advantages will be pointed out.

Several factors may impose an approximate solution, but the linear equation system substituting, and derived from, the integral equation system can be used also independently, replacing the model in [1] by a different one.

This method can be used not only to solve the given problem but also for other strength, stability and vibration problems using the integral equation systems [1]. The numerical method is advantageous not only in determining the unknown functions of the integral equations, but also in formulating the Green functions. Function values at each point can be calculated at a sufficient accuracy using the known methods of statics if their writing in a closed form is impossible or difficult.

An advantage of the presented method applied in other problems is to formulate the maximum deviation between the function obtained from the integral equation and the approximate value from the substituting linear equation system. This error formula gives the error limit of the approximate solution satisfying the connecting conditions replacing continuous hinges. The accuracy of this error limit is not influenced by the fact that the Green function of the problem is only given with values calculated at random points.

This approximate method can be used not only to determine unknown hinge forces but also for stability and vibration analyses. To determine the eigenvalues and eigenfunctions of the integral equations, the approximate method can be used. In this way the maximum deviation of the obtained result from the eigenvalue or the eigenfunction of the integral equation can be well estimated. In some instances even the actual formulation of the integral equations can be omitted.

Summary

Determination of hinge forces of skew bridge structures with hinged main girders is described in cases where — instead of the function of these hinge forces — its approximate values at given points are determined.

The hypermatrix equation yielding the approximate solution of the integral equation system has been determined. Determination method of this equation system was treated independently of the formulation of the integral equation system, and simple practical cases have been considered. The limit of error for results obtained by the approximate linear equation system was examined.

Finally, the above method has been generalized and a numerical example presented.

References

1. ÁRVAY, K.: Design of Bridge Structures Comprising Hinged Main Girders. Periodica Polytechnica M.E. Vol. 17. No. 2. 1973.
2. KANTOROVICH—KRILOW: Approximate Methods of Higher Analysis.* Akadémiai Kiadó, Budapest, 1953.
3. ÁRVAY, K.: Computation of Bridge Structures with Hinged Main Girders.* Thesis for C. Sc. Budapest, 1968.

* In Hungarian

Sen. Ass. Dr. Kálmán ÁRVAY, 1111 Budapest, Műegyetem rkp. 3, Hungary