UNDAMPED, FREE VIBRATIONS OF BAR STRUCTURES

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1. Introduction

Application of equations deduced in [1] for the analysis of small displacements of bar structures in the dynamic analysis at slight completions will be presented. In the following, only undamped, free vibrations of bar structures will be considered.

2. Concepts, notations

The bar structure is considered to consist of bars of constant rigidity and straight axis. The bar elements join in nodes at positions indicated in a co-ordinate system x, y, z valid to the entire bar structure. Besides, to any bar element j, k a proper co-ordinate system $\xi_{j,k}$, $\eta_{j,k}$, $\zeta_{j,k}$ will be assigned (Fig. 1).

Node displacements will be indicated in a global co-ordinate system, and the internal forces in a proper system with six-dimensional vectors u_j and $s_{j,k}$.

In static analyses, for a load vector q_j given in the global co-ordinate system, in conformity with deductions in [1], equilibrium and kinetic nodal equations can be written in the form (zeroing the kinetic load):

$$\begin{bmatrix} & \mathbf{G}^* \\ \mathbf{G} & \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{s} \end{bmatrix} + \begin{bmatrix} \mathbf{q} \\ 0 \end{bmatrix} = 0, \tag{1}$$

$\mathbf{G}$ and $\mathbf{F}$ being matrices including geometric and flexibility characteristics, respectively.

Solution of this matrix equation can be arrived at, if the displacement method is used, from the equation

$$-\mathbf{G}^* \mathbf{F}^{-1} \mathbf{G} \mathbf{u} + \mathbf{q} = 0 \tag{2}$$

expressible also as

$$-\mathbf{K} \mathbf{u} + \mathbf{q} = 0 \tag{3}$$

$\mathbf{K}$ being the stiffness matrix of the structure.

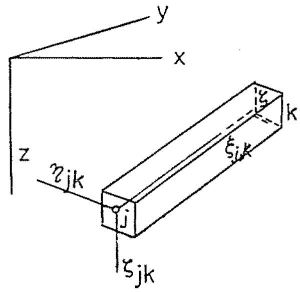

Fig. 1

3. Establishment of the matrix differential equation for the undamped free vibrations of the bar structure

For a structure other than at rest, according to the Newtonian law, Eq. (3) has to be completed by mass forces

$$-\mathbf{K}\,\mathbf{u} + \mathbf{q} = \mathbf{M}\,\ddot{\mathbf{u}}, \tag{4}$$

M being the mass matrix of the structure. For free vibration $q = 0$, hence the matrix differential equation sought for will be of the form:

$$\mathbf{M}\,\ddot{\mathbf{u}} + \mathbf{K}\,\mathbf{u} = 0\,. \tag{5}$$

Mass matrix **M**, similarly to the stiffness matrix, will be obtained by producing and combining mass matrices of each bar. Production of mass matrices will be examined below.

3.1 *Mass matrix for a bar element*

It is known from the literature that the motion in the co-ordinate system ξ, η, ζ of a body with gravity point S due to forces acting at j can be described by a mass matrix of the build-up (Fig.2):

$$\begin{bmatrix} m_j & 0 & 0 & 0 & m_j r_{j\zeta} & -m_j r_{j\eta} \\ 0 & m_j & 0 & -m_j r_{j\zeta} & 0 & m_j r_{j\xi} \\ 0 & 0 & m_j & m_j r_{j\eta} & -m_j r_{j\xi} & 0 \\ 0 & -m_j r_{j\xi} & m_j r_{j\eta} & J_\xi & -J_{\xi\eta} & -J_{\xi\zeta} \\ m_j r_{j\zeta} & 0 & -m_j r_{j\xi} & -J_{\xi\eta} & J_\eta & -J_{\eta\zeta} \\ -m_j r_{j\eta} & m_j r_{j\xi} & 0 & -J_{\xi\zeta} & -J_{\eta\zeta} & J_\zeta \end{bmatrix}$$

where m_j is the body mass; J_ξ, J_η, J_ζ and $J_{\xi\eta}, J_{\xi\zeta}, J_{\eta\zeta}$ being inertia moments for axes and for planes, respectively.

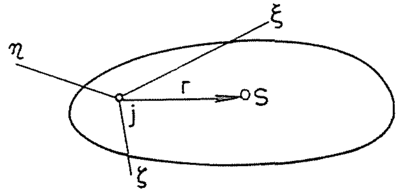

Fig. 2

Reducing, however, the half mass of a bar to the node j at the bar end simplifies the matrix significantly in the proper co-ordinate system:

$$\mathbf{M}_{j,k}^{j(\xi,\eta,\zeta)} = \begin{bmatrix} m_j & & & & & \\ & m_j & & & & m_j \dfrac{l}{4} \\ & & m_j & & -m_j \dfrac{l}{4} & \\ & & & J_\xi & & \\ & & -m_j \dfrac{l}{4} & & J_\eta & \\ & m_j \dfrac{l}{4} & & & & J_\zeta \end{bmatrix}$$

where $m_j = \mathbf{A}_j\, l_{j,k}/2$ while inertia moments can be computed in different ways, depending on the cross-section. Also the mass matrix at the bar end k can be written in a similar form. Mass matrices will occur in the equilibrium equations, therefore they are required to be transformed to the co-ordinate system x, y, z using matrices $T_{j,k}$ expressing the correlation of the two co-ordinate systems; such as:

$$\mathbf{M}_{j,k}^{j(x,y,z)} = \mathbf{T}_{j,k}^{*}\, \mathbf{M}_{j,k}^{j(\xi,\eta,\zeta)}\, \mathbf{T}_{j,k}. \tag{6}$$

(Since the mass matrix does not occur but in the co-ordinate system x, y, z, in the following the co-ordinate system will not be indicated.) With the approximating assumption that in the equilibrium analysis of individual nodes, only masses reduced to the proper node are effective, the mass matrix of the entire bar will be of the build-up:

$$\begin{bmatrix} \mathbf{M}_{j,k}^{j} & \\ & \mathbf{M}_{j,k}^{k} \end{bmatrix}. \tag{7}$$

It should be noticed that closer determinations have been published [2] for the bar mass matrix based on energy considerations.

3.2 *Dynamic equation of the structure*

Dynamic equations for the undamped vibration of a bar element will be arrived at by applying the Newtonian law.

Equilibrium equations can be written in the form, according to [1]:

$$\begin{bmatrix} \mathbf{T}^*_{j,k}\mathbf{B}^*_{j,k} \\ -\mathbf{T}_{j,k} \end{bmatrix} \begin{bmatrix} \mathbf{s}_{j,k} \end{bmatrix} + \begin{bmatrix} \mathbf{q}_j \\ \mathbf{q}_k \end{bmatrix} = \begin{bmatrix} \mathbf{M}^j_{j,k} & \\ & \mathbf{M}^k_{j,k} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_j \\ \ddot{\mathbf{u}}_k \end{bmatrix}, \tag{8}$$

$\mathbf{B}^*_{j,k}$ being the transposed of the so-called transfer matrix.

Kinematic equations are identical to those in static analyses.

In possession of the dynamic equations of bar elements, the entity of equations — equilibrium and kinematic equations in separate groups — makes up the dynamic equation of the entire bar system.

After reducing by the connecting matrix taking joints between bar ends into consideration, dynamic equations of the bar system are obtained.

Separating the part corresponding to the boundary conditions, the dynamic matrix equation of the bar system can be written in the form (omitting the kinematic load):

$$[\mathbf{M}][\ddot{\mathbf{u}}] - [\mathbf{G}^*][\mathbf{s}] = [\mathbf{q}] \tag{9}$$

$$\begin{bmatrix} \mathbf{G} & \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{s} \end{bmatrix} = 0 .$$

Solving it according to the displacement method, expressing s from the kinetic equation, and introducing $\mathbf{K} = \mathbf{G}^* \mathbf{F}^{-1} \mathbf{G}$ (zeroing $\mathbf{q}$), we obtain the matrix differential equation:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = 0.$$

In static analyses it is usual for hinged bar structures to calculate rotations of the hinged bar end later, after having determined the displacements required for the solution, and the internal forces. Thus, the stiffness matrix will be obtained by "compiling" stiffness matrices of bars restrained rigidly at one end, and hinged at the other. This method is not valid to dynamic analyses, namely the effect of masses belonging to the hinged end cannot be omitted in calculating the frequencies.

4. Solution of the matrix differential equation

4.1 *The solution function*

The matrix differential equation is of the form

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = 0$$

with initial conditions

$$\mathbf{u}(t_0) = \mathbf{u}_{t0}$$
$$\dot{\mathbf{u}}(t_0) = \dot{\mathbf{u}}_{t0}\,.$$

From the theory of differential equations it is known that the solution can be sought for in the form:

$$\mathbf{u} = \mathbf{v}e^{\mu t}\,. \tag{10}$$

After substitution:

$$(\mu^2\,\mathbb{M} + \mathbb{K})\,\mathbf{v} = 0 \tag{11}$$

possessing a non-trivial solution for

$$\det(\mu^2\,\mathbb{M} + \mathbb{K}) = 0 \tag{12}$$

μ^2 values can be determined as roots of a characteristic polynomial, but the solution can also be reduced to a matrix eigenvalue problem. Introducing

$$\mu^2 = \frac{1}{\lambda}$$

we obtain

$$-\mathbb{K}^{-1}\,\mathbb{M}\,\mathbf{v} = \lambda\mathbf{v} \tag{13}$$

and substituting

$$\mathbb{A} = -\mathbb{K}^{-1}\,\mathbb{M}$$

leads to

$$\mathbb{A}\,\mathbf{v} = \lambda\,\mathbf{v}, \tag{14}$$

a non-symmetric, real eigenvalue problem. It results in n negative real eigenvalues and n linearly independent solution vectors $\mathbf{v}$. All these help to a solution meeting initial conditions (for $t_0 = 0$) in the form:

$$\mathbf{u} = \sum_{r=1}^{n} \mathbf{v}_r\,\mathbf{v}_r\,\mathbb{M}\left(\mathbf{u}_0\cos\alpha_r t + \frac{1}{\alpha_r}\,\dot{\mathbf{u}}_0\sin\alpha_r t\right), \tag{15}$$

α_r being the r-th natural frequency. Notice that the case of undamped free vibration can be treated as a symmetric eigenvalue problem, with details published in [3].

4.2 *Solution of the eigenvalue problem*

In analysing the matrix differential equation, the most important problem is to solve the eigenvalue of the non-symmetric matrix. In our computations, the QR transformation, considered in [4] to be the method of the best numeric stability, has been applied.

In solving the eigenvalue problem, the matrix has first been brought to Hessenberg form by Gaussian elimination and selection of principal elements in a finite number of steps, then it has been transformed into a triangular matrix by means of similarity transformations using an orthogonal matrix set.

In the main diagonal of the triangular matrix, the eigenvalues follow each other in the order of absolute values. Transformation to triangular form is facilitated by the gradual lessening of quotients of successive eigenvalues of vibrational problems, hence the greatest eigenvalue is the first to appear permitting to reduce the matrix order in course of computations, and to introduce an efficient method to accelerate the convergence.

Let us notice here that — as against item 4.1 — $\mathbf{M}^{-1}$ may be used instead of $\mathbf{K}^{-1}$ for throughout multiplication. In this case the eigenvalues appear first below, and the convergence is still accelerated.

In knowledge of the Hessenberg form, the eigenvectors are easy to determine by iteration, using the eigenvalues. Constants of the general solution of the matrix differential equation to meet initial conditions are simple to determine, while variations vs. time of the displacement function can be followed by means of a drawing machine.

5. Examples

Example 1

Let us examine by means of the presented algorithm the plane natural vibrations of a beam restrained at both ends. Eigenvalues of this simple problem can be verified by values obtained from the continuum model, and various forms of eigenvectors (principal nodes) are known.

$$A = 7 \cdot 10^{-3}\,\mathrm{m}^2$$

$$J = 9.8 \cdot 10^{-5}\,\mathrm{m}^4$$

$$E = 2.1 \cdot 10^7\,\mathrm{Mp/m}^2$$

$$\gamma = 7.85\,\mathrm{Mp/m}^3$$

The bar has been divided into equal parts numbered from 2 to 8. (Eigenvalue problem of max. 21 size, Fig. 3.)

The highest eigenvalue belonging to longitudinal vibrations (the lowest natural frequency) as a function of the number of divisions is shown in Fig. 4, both for our mass matrix and the more exact mass matrix suggested by PRZEMIENIECKI.

Also eigenvalues corresponding to flexural vibrations have been computed on the basis of the two kinds of mass models, outcomes being rather similar and intercepting the actual eigenvalue.

First three principal modes of the longitudinal vibration, and 6th, 7th and 8th ones of the flexural vibration are shown in Fig. 5. (Eigenvector element of the maximum absolute value being 1.0.)

Example 2

Four frameworks shown in Fig. 6 are of identical size but bars are differently connected. Natural frequencies of the structures have been computed by considering each actual bar (an eigenvalue problem of max. 9 size). The mass matrix of the structure has been determined by means of the mass matrix of Przemieniecki.

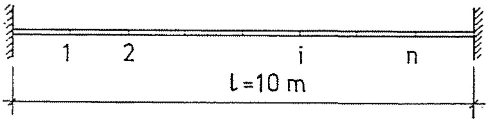

Fig. 3

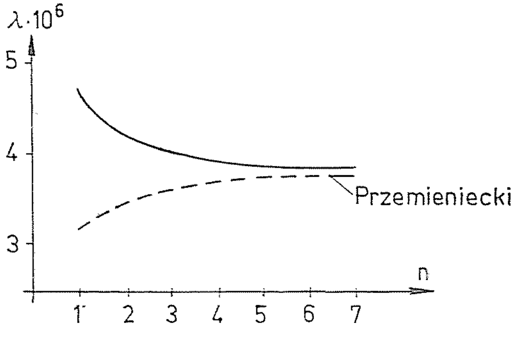

Fig. 4

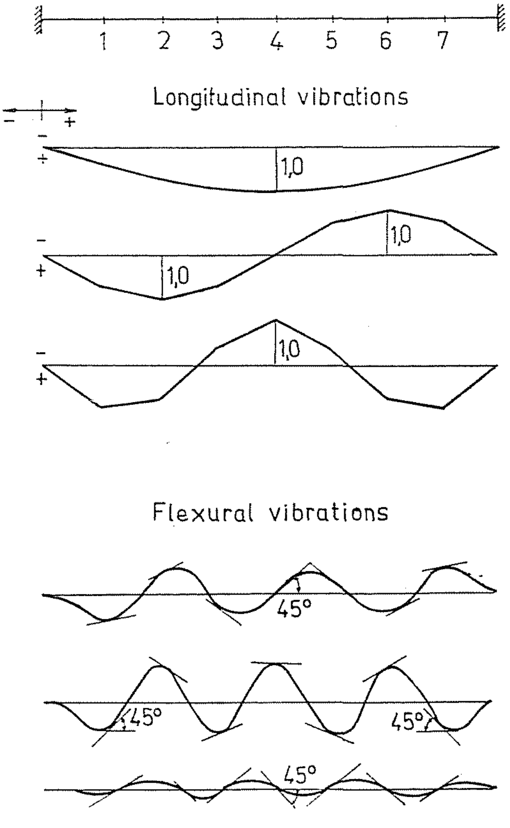

Fig. 5

Framework data:

$$A = 10^{-3}\ \mathrm{m}^2$$
$$J = 1.5 \cdot 10^{-2}\ \mathrm{m}^4$$
$$E = 2.1 \cdot 10^6\ \mathrm{Mp/m}^2$$
$$\gamma = 2\ \mathrm{Mp/m}^3$$

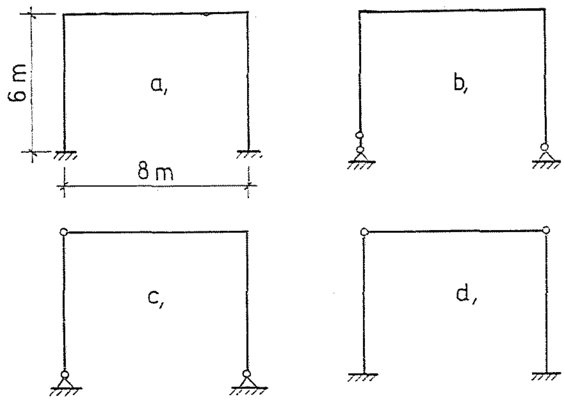

Fig. 6

The obtained natural frequencies have been tabulated below. Frequencies of less rigid structures are obviously lower than those of stiff ones.

Column *dl* contains eigenvalues computed for a *d*-type structure completed by nodes assumed at mid-bar.

Frequency No.	*a*	*b*	*c*	*d*	*dl*
1	91.2	40.7	25.9	61.8	61.5
2	160.0	158.6	150.0	139.6	134.0
3	227.5	223.7	218.0	216.5	211.0
4	297.5	240.0	238.2	297.5	286.0
5	425.0	371.0	331.0	318.0	292.0
6	662.0	439.0	386.0	591.0	446.0
7	—	752.0	541.0	635.0	496.0
8	—	956.0	916.0	751.0	610.0
9	—	—	1070.0	—	694.0

Fig. 7 presents modes belonging to the first two eigenvalues of the four different structures. For the *d*-type structure, two independent vibration groups are seen to have formed, appearing in our computations by the Hessenberg form detached into two separate parts.

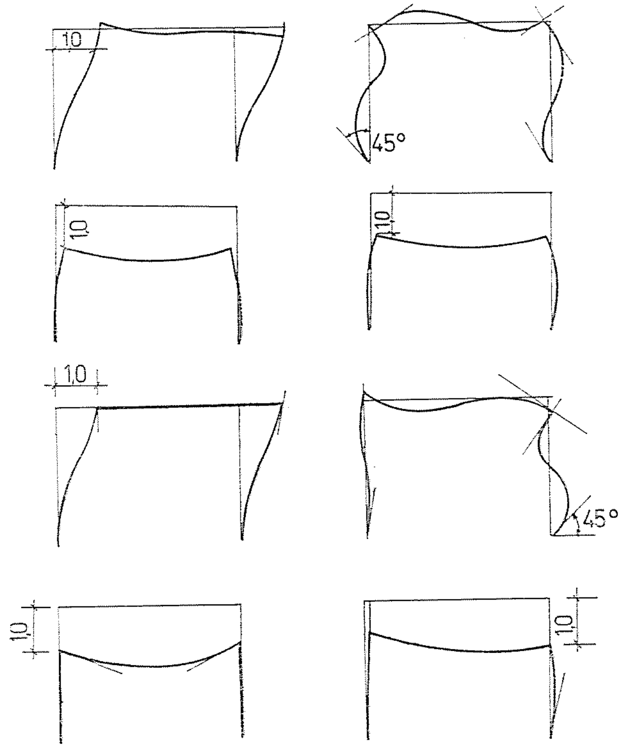

Fig. 7

Summary

In connection with the undamped free vibrations of bar structures, possibility of deriving a matrix differential equation for vibration motion from the matrix equation of bar structures has been examined. Similarly to static analyses, dynamic equations of a bar have been written first, by means of the stiffness and mass matrices of the bar, then the matrix differential equation of the bar structures has been established in conformity with joints and boundary conditions. Solution of these bar structure equations has been reduced to the problem of non-symmetric matrix eigenvalue problem, applying the QR transformation to compute eigenvalues and eigenvectors. A program has been developed in ALGOL—60 language and run on an ODRA—1204 computer.

Computation observations and outcomes have been illustrated on examples.

References

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