# ANALYSIS OF CIRCULAR ARC SHAPED DECK BRIDGES BY THE METHOD OF LARGE FINITE ELEMENTS

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## 1. Stating the problem

Bridge design often is concerned with bridges of arc ground plan over point supports. Because of vehicle loads in different positions, the design involves determination and examination of various influence surfaces, these can, however, be produced only by some numerical solution of the plate differential equation. Development of a method likely to simply determine various influence surfaces and stress diagrams of deck bridges over arched ground plan by means of a medium capacity computer has been attempted.

Our starting assumptions will be general enough to solve most of the practical problems, at the same time permit exploitation of computing advantages arising from the features of this structure type.

Be the tested structure a thin plate of homogeneous, elastic material over a ring segment ground plan, supported on both ends and along intermediate radii at equal angular distances by point-like or linear supports entraining arbitrary displacement constraints, and affected by an arbitrary system of vertical loads. Along the radii of supports, intermediate cross beams of identical design and end cross beams of a different design may be applied. Plate thickness is arbitrarily variable in radial direction, while in annular direction, it may be identically variable within each span (Fig. 1). A method and procedure convenient for the computation of stress diagrams, stress and strain influence surfaces had to be elaborated.

The most convenient method for taking the indicated stipulations of the problem into account was felt to be the displacement method of large finite elements, hence the solution will be based on this method.

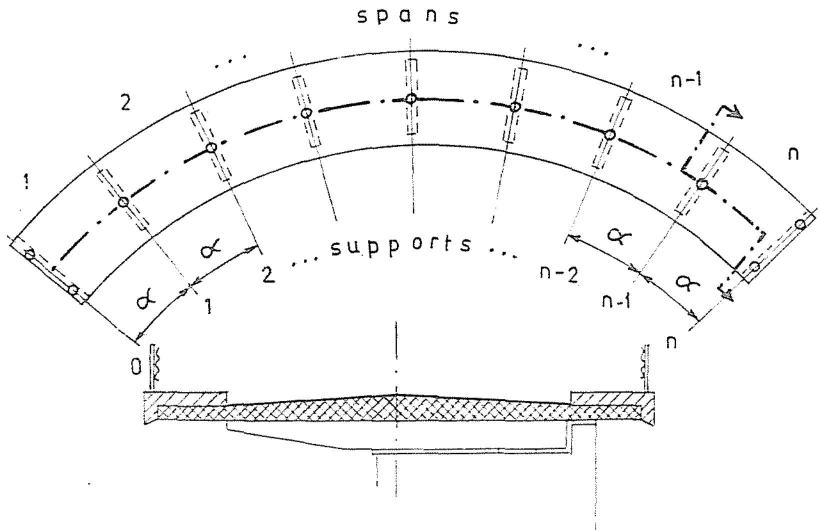


*Fig. 1*

## 2. Method of large finite elements

Similarly to the method of finite strips [2, 3], this method is a variety of the finite elements method, offering considerable computation advantages for special problems. It is especially useful for solving structural problems where the usual methods would require rather many unknowns for a given accuracy, or where deformational and stress discontinuities caused by internal constraints and stiffness jumps would inhibit the use of common methods [4].

This method is based on the division of the structure into possibly few, large elements along lines containing the deformation constraints and stiffness jumps, and establishing the compatibility equations separately for the structure as a whole, on the basis of connection conditions of these "finite elements" and those of the internal strains inside the elements on the basis of edge displacements and loads. This provides partly for the fact that nothing but an equation system of a reduced number of unknowns has to be solved, containing the displacements of connected edges, and partly for the possibility that deformation constraints of different types can be taken directly into consideration. In case of elements of the same type and boundary conditions, identity of "eigenstiffness matrices" of edge displacements for all elements means a great ease. For elements connected only at two opposite edges, the reduced compatibility matrix of the entire structure will be of a hyper-continuant type, permitting further essential simplifications

Authors of this method, A. Ghali and K. J. Bathe combined it to the method of finite differences and applied to the analysis of straight-edge plates and discs [4, 5].

### 3. Decomposition of the arched deck bridge to finite elements

Let us decompose the entire structure to as many $n$ identical plate elements numbered $1, 2, \ldots, n$, as there are spans, to two identical end cross beams marked 0 and $n$, and $n-1$ intermediate cross beams numbered $1, 2, \ldots, n-1$, along the connection lines of cross beams (or in their lack, assuming cross beams of zero rigidity). Elements join at intersection lines marked $0.1, 1.1, 1.2, \ldots, n,n$.

Omitting the fact that plates do not join exactly the strength axis of cross beams, it can be stated that the displacement functions of cross beams have to coincide with those of the adjacent edges. Denoting the former by $u_0, u_1, \ldots, u_n$ and the latter by $u_{0,1}, u_{1,1}, u_{1,2}, \ldots, u_{n,n}$ in this order, joint conditions are:

$$u_0 = u_{0,1}\,; \quad u_1 = u_{1,1} = u_{1,2}\,; \quad \ldots\,; \quad u_n = u_{n,n}\,. \tag{1}$$

Indicating the direct loads on the cross beams by $l_0^0, l_1^0, \ldots, l_n^0$, the forces acting on plate element edges by $l_{0.1}, l_{1.1}, l_{1.2}, \ldots, l_{n,n}$ and total loads on the cross beams by $l_0, l_1, \ldots, l_n$ then these load functions are related as:

$$l_0 = l_0^0 - l_{0,1}; \; l_1 = l_1^0 - l_{1,1} - l_{1,2}; \ldots; l_n^0 = l_n - l_{n,n}\,. \tag{2}$$

Equation systems (1) and (2) permit to establish the reduced compatibility equation system of the system of large finite elements, after the stiffness relationships of the individual elements have been determined.

Determination of approximate "eigenstiffness" relationships of plate elements and then of cross beams by means of the method of finite differences will be presented below.

### 4. "Eigenstiffness" relationships of the plate element

Eigenstiffness relationships will be determined according to variational principles as usual in the method of finite elements [1].

Be the total deformation system of the plate elements — intermediating partitioning, convenient for subsequent steps — denoted by the generalized vector of left-hand (precedent) edge displacements, of internal displacements and of right-hand (subsequent) edge displacements, respectively:

$$\{\delta\} = \{u_l, w, u_r\}\,.$$

At the same time, be the generalized vector of the functions of left-hand edge loads, internal surface loads and right-hand edge loads, making up the loads on the plate element:

$$\{f\} = \{l_l, p, l_r\}\,.$$

According to the sign convention, positive forces do positive work if the corresponding (dual) displacements are positive (Fig. 2).

According to the principle of minimum potential energy, the relationship between force and displacement functions is given by the condition:

$$\Pi = \frac{1}{2} Q(\{\delta\}, \{\delta\}) - (\{f\}, \{\delta\}) = \text{minimum!} \tag{3}$$

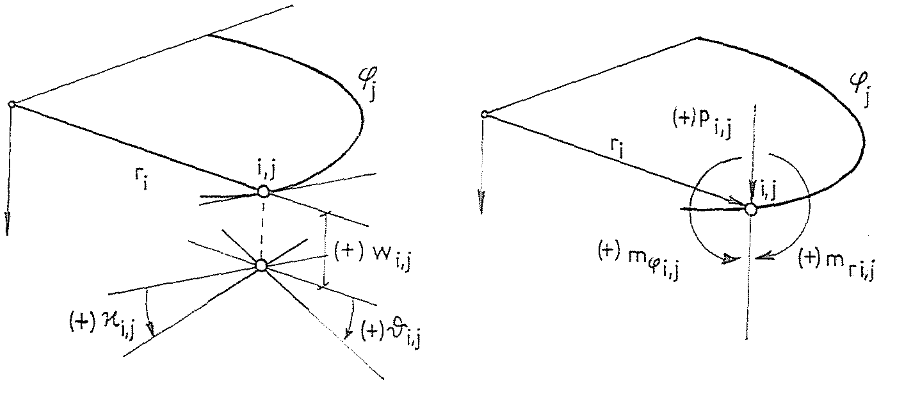


*Fig. 2*

First term in this expression, the internal elastic energy of the plate is a quadratic form of the displacement vector, the second one is the work of external forces along the deformations, as generalized scalar product of both vectors. Formally deriving (3) with respect of $\{\delta\}$ yields the stiffness relationship:

$$\frac{\partial}{\partial\{\delta\}}(\Pi) = Q(\{\delta\}) - \{f\} = 0. \tag{4}$$

$Q$ is a real, Hermitic transformation belonging to the quadratic form. Approximating the total deformation and load function system of infinite degrees of freedom of the surface element by populations of finite values each, namely by deformations related to the nodes of a network of finite differences, and a system of external forces concentrated at these nodes. Now, the condition of minimum potential energy can be written with real vectors, of course only as an approximation, rather than with abstract vectors:

$$\Pi \approx \frac{1}{2} \boldsymbol{\delta}^* \mathbf{Q} \boldsymbol{\delta} - \boldsymbol{\delta}^* \mathbf{f} = \text{minimum!} \tag{5}$$

$$\mathbf{Q}\boldsymbol{\delta} - \mathbf{f} = 0. \tag{6}$$

Be the assumed differential network a ring-radial one with a mesh of interval $\lambda_r$ radially and of $\Delta\varphi$ in annular direction. Introducing notations in Fig. 3, be the value system for inner point and arched boundary point deflections:

$$w_{i,j} \qquad 1 \leq i \leq m\,, \qquad 1 < j < k\,.$$

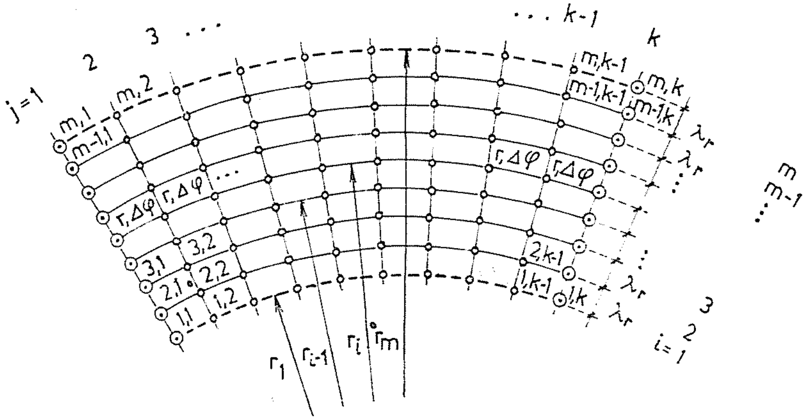


*Fig. 3*

Arched boundary point edge rotations:

$$\vartheta_{i,j} \qquad i = 1 \quad \text{and } m\,, \qquad 1 < j < k\,.$$

Let displacements $w_{i,j}$ and $\vartheta_{i,j}$ constitute the vector w of the internal deformations of the plate element:

$$w^* = [\vartheta_{1,2}, w_{1,2}, w_{2,2}, \ldots w_{m,2}, \vartheta_{m,2}, \vartheta_{1,3}, \ldots, \vartheta_{m,k-1}]\,.$$

Be the deflections and rotations at straight edge point:

$$w_{i,j} \text{ and } \varkappa_{i,j} \text{ respectively, for } 1 \leq i \leq m\,, \quad j = 1 \text{ and } k\,.$$

Radial slopes at the corner points:

$$\vartheta_{11}, \vartheta_{m1}, \vartheta_{1k} \quad \text{and} \quad \vartheta_{mk}\,.$$

Vectors $\mathbf{u}_l$ and $\mathbf{u}_r$ of left and right side edge displacements are composed of elements of the latter three displacement systems:

$$\mathbf{u}_l^* = [\vartheta_{11}, u_{11}, u_{21}, \ldots u_{m1}, \vartheta_{m1}, \varkappa_{11}, \varkappa_{21}, \ldots, \varkappa_{m1}]$$
$$\mathbf{u}_r^* = [\vartheta_{1k}, u_{1k}, u_{2k}, \ldots u_{mk}, \vartheta_{mk}, \varkappa_{1k}, \varkappa_{2k}, \ldots, \varkappa_{mk}]\,.$$

Loads at inner points and arched edge points of the plate element are:

$$p_{i,j} \quad \text{for } 1 \leqq i \leqq m\,, \qquad 1 < j < k \quad \text{and}$$
$$m_{ri,j} \quad \text{for } i = 1 \text{ and } m\,, \quad 1 < j < k\,, \quad \text{respectively}\,,$$

to be replaced by loads and moments concentrated at nodes of the difference network in case of distributed load and edge moment. (Direct loads on the boundary strip $\Delta\varphi r/2$ wide are considered as loads on the joining cross beams.) Each vector of internal loads is formed from these loads in the sequence of elements of the inner deformation vector:

$$\mathbf{p}^* = [m_{r12}, p_{12}, p_{22}, \dots p_{m2}, m_{rm2}, m_{rm3}, \dots m_{mk-1}]\,.$$

Connection forces acting at the connection line and concentrated at the nodes are:

$$q_{i,j} \text{ and } m_{\varphi i,j} \qquad 1 \leqq i \leqq m\,, \qquad j = 1 \text{ and } k\,;$$
$$m_{r11}, m_{rm1}, m_{r1k} \text{ and } m_{rmk}\,, \quad \text{respectively}\,.$$

Let them constitute the vectors of left and right side edge forces of the plate element in the sequence of the edge displacements:

$$\mathbf{l}_l^* = [m_{r11}, q_{11}, q_{21}, \dots, q_{m1}, m_{rm1}, m_{\varphi 11}, m_{\varphi 21}, \dots m_{\varphi m1}]$$
$$\mathbf{l}_r^* = [m_{r1k}, q_{1k}, q_{2k}, \dots, q_{mk}, m_{rmk}, m_{\varphi 1k}, m_{\varphi 2k}, \dots m_{\varphi mk}]\,.$$

Partitioning matrix $\mathbf{Q}$ with respect to vector components $\delta$ and $f$ results in the following hypermatrix equation:

$$\begin{bmatrix} \mathbf{C}_l & \mathbf{K}_l & \mathbf{L} \\ \mathbf{K}_l^* & \mathbf{A} & \mathbf{K}_r \\ \mathbf{L}^* & \mathbf{K}_r^* & \mathbf{C}_r \end{bmatrix} \begin{bmatrix} \mathbf{u}_l \\ \mathbf{w} \\ \mathbf{u}_r \end{bmatrix} = \begin{bmatrix} \mathbf{l}_l \\ \mathbf{p} \\ \mathbf{l}_r \end{bmatrix}. \tag{7}$$

Determination of the elements of matrix $\mathbf{Q}$ is started by approximately writing the potential energy to be minimized [6].

The elastic deformation work of the tested plate is given by the integral (written in polar co-ordinates):

$$\frac{1}{2} Q(\{\delta\}, \{\delta\}) = \iint \left\{ \frac{1}{2} K \left( \frac{1}{r^2} \frac{\partial^2 w}{\partial \varphi^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial r^2} \right)^2 - \right.$$
$$- (1-\mu) \left[ K \cdot \frac{\partial^2 w}{\partial r^2} \cdot \left( \frac{1}{r^2} \frac{\partial^2 w}{\partial \varphi^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) - \right.$$
$$\left. \left. - K \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial w}{\partial \varphi} \right) \right]^2 \right] \right\} r\, d\varphi\, dr\,.$$

Let the potential derivatives in the integral be approximated by the difference quotients of elements in the finitized deformation vector $\delta$ of the deformation function $\{\delta\}$. On the basis of interpolating polynomials of the lowest degree, the following expressions are valid:

$$\left[\frac{\partial^2 w}{\partial \varphi^2}\right]_{i,j} \approx \frac{1}{\Delta\varphi^2}(-2w_{i,j}+w_{i,j-1}+w_{i,j+1}) \qquad \text{for} \quad 1<j<k\,,$$

$$\approx \frac{1}{\Delta\varphi^2}(-2w_{i,j}+2w_{i,j\pm1}\mp 2\varkappa_{i,j}\Delta\varphi\cdot r_i) \quad \text{for} \quad j=\begin{cases}1\\k\end{cases} \text{or}$$

$$\left[\frac{\partial^2 w}{\partial r^2}\right]_{i,j} \approx \frac{1}{\lambda_r^2}(-2w_{i,j}+w_{i-1,j}+w_{i+1,j}) \qquad \text{for} \quad 1<i<m\,,$$

$$\approx \frac{1}{\lambda_r^2}(-2w_{i,j}+2w_{i\pm1,j}\mp 2\vartheta_{i,j}\cdot\lambda_r) \quad \text{for} \quad i=\begin{cases}1\\m\end{cases} \text{or}$$

$$\left[\frac{\partial w}{\partial r}\right]_{i,j} \approx \frac{1}{2\lambda_r}(w_{i+1,j}-w_{i-1,j}) \qquad \text{for } 1<i<m\,,$$

$$\approx \vartheta_{i,j} \qquad \text{for } i=1 \text{ or } m\,.$$

These difference quotients will be considered as constant in the region surrounding points $i, j$ of half-strip width each, hence the integration results in the following two sums for the first two integrands of the expression

$$I_1=\sum_{i=1}^{m}\sum_{j=1}^{k}\frac{1}{2}K_{i,j}\left\{\frac{1}{r_i^2}\left[\frac{\Delta^2 w}{\Delta\varphi^2}\right]_{i,j}+\frac{1}{r_i}\left[\frac{\Delta w}{\Delta r}\right]_{i,j}+\left[\frac{\Delta^2 w}{\Delta r^2}\right]_{i,j}\right\}^2 F_{i,j}$$

$$I_2=\sum_{i=1}^{m}\sum_{j=1}^{k}-(1-\mu)K_{i,j}\left\{\left[\frac{\Delta^2 w}{\Delta r^2}\right]_{i,j}\cdot\left[\frac{1}{r_i^2}\left[\frac{\Delta^2 w}{\Delta\varphi^2}\right]_{i,j}+\frac{1}{r_i}\left[\frac{\Delta w}{\Delta r}\right]_{i,j}\right]\right\}F_{i,j}$$

where $K_{i,j}$ is the bending stiffness assumed to be constant also in the surrounding $F_{i,j}$ of point $i, j$ and

$$F_{i,j} = \Delta\varphi\cdot r_i\cdot\lambda_r \qquad \text{for} \quad 1<i<m \quad \text{and} \quad 1<j<k,$$

$$= 1/2\cdot\Delta\varphi\cdot r_i\cdot\lambda_r \qquad \text{for} \quad 1<i<m \quad \text{and} \quad j=1 \text{ or } k\,,$$

$$= 1/2\cdot\Delta\varphi\left(r_i\pm\frac{1}{4}\lambda_r\right)\lambda_r \quad \text{for} \quad i=\begin{cases}1 \text{ or}\\m\end{cases} \quad \text{and} \quad 1<j<k,$$

$$= 1/4\cdot\Delta\varphi\left(r_i\pm\frac{1}{4}\lambda_r\right)\lambda_r \quad \text{for} \quad i=\begin{cases}1 \text{ or}\\m\end{cases} \quad \text{and } j=1 \text{ or } k\,.$$

For the third integrand, the difference quotient will be written for a secondary network of nodes shifted by half interval each in directions $r$ and $\varphi$ of the difference system:

$$\left[\frac{\partial}{\partial r}\left[\frac{1}{r}\cdot\frac{\partial w}{\partial\varphi}\right]\right]_{i+0.5,\,j+0.5}=\frac{1}{\lambda_r(r_i+0.5\lambda_r)\,\Delta\varphi}\cdot \qquad 1\leq i<m$$

$$\cdot\,[w_{i,j}+w_{i+1,j+1}-w_{i+1,j}-w_{i,j+1}]. \qquad 1\leq j<k.$$

Assuming constant torsion and plate torsional stiffness within surface elements confined by primary nodes, integration leads to the sum:

$$I_3=\sum_{i=1}^{m-1}\sum_{j=1}^{k}(1-\mu)K_{i+0.5,\,j+0.5}\left[\frac{\Delta}{\Delta r}\left(\frac{1}{r}\cdot\frac{\Delta w}{\Delta\varphi}\right)\right]^2_{i+0.5,j+0.5}\lambda_r\cdot(r_i+0.5\lambda_r)\,\Delta\varphi$$

where

$$K_{i+0.5,j+0.5}=\frac{K_{i,j}+K_{i+1,j+1}+K_{i,j+1}+K_{i+1,j}}{4}.$$

In sums $I_1$, $I_2$ and $I_3$, elements of vectors $\mathbf{u}_l$, $\mathbf{w}$ and $\mathbf{u}_r$ are equally contained.

The condition (5) of minimum potential energy can be written by means of the deduced sums as:

$$I_1+I_2+I_3-\mathbf{l}_l^*\,\mathbf{u}_l-\mathbf{p}^*\,\mathbf{w}-\mathbf{l}_r^*\,\mathbf{u}_r=\min!$$

To simplify writing, let us renumber elements in $\boldsymbol{\delta}$ and $f$ in the natural sequence of listing, denoting them as $\delta_1,\ldots,d_\nu,\ldots,d_N$ and $f_1,\ldots,f_\nu,\ldots f_N$ $N=(k+2)(m+2)-4$. Minimum condition is met if the partial derivate of $\Pi$ with respect to any deformation element is just zero.

$$\frac{\partial\Pi}{\partial\delta_1}=\frac{\partial I_1}{\partial\delta_1}+\frac{\partial I_2}{\partial\delta_1}+\frac{\partial I_3}{\partial\delta_1}-f_1=0$$
$$\vdots$$
$$\frac{\partial\Pi}{\partial\delta_\nu}=\frac{\partial I_1}{\partial\delta_\nu}+\frac{\partial I_2}{\partial\delta_\nu}+\frac{\partial I_3}{\partial\delta_\nu}-f_\nu=0$$
$$\vdots$$
$$\frac{\partial\Pi}{\partial\delta_N}=\frac{\partial I_1}{\delta\partial_N}+\frac{\partial I_2}{\partial\delta_N}+\frac{\partial I_3}{\partial\delta_N}-f_N=0.$$

All equations will be linear difference equations each, the equation system results in the "eigenstiffness" equation system of the plate element:

$$\mathbf{Q}\,\boldsymbol{\delta}=\mathbf{f}.$$

Element of matrix **Q** in position $\mu, \nu$ will be given by

$$Q_{\mu,\nu} = \frac{\partial^2 \Pi}{\partial \delta_\mu \, \partial \delta_\nu} \, .$$

In conformity with the identity between mixed derivates, $Q_{\mu,\nu}$ equals element $Q_{\nu,\mu}$ in transposed position. Thus, **Q** is symmetrical, as follows otherwise from the real, Hermitic nature of transformation $Q(\;)$.

Remind that to determine matrix **Q** it is useless to write total sums $I_1$, $I_2$ and $I_3$ but only terms containing both variables $\delta_\mu$ and $\delta_\nu$ corresponding to the position of matrix elements $Q_{\mu,\nu}$ to be determined.

In order to determine all elements of one row of **Q** in a single step, in fact, operator weights of the difference operator assumed in view of the corresponding point environment have to be established. Deduction of difference operators corresponding to various boundary conditions of arched plates — such as that of the free edge along the arched edge — has been presented by BERGFELDER in his study on difference equations [6]. His operators — combined with "transient" operators of the radial edge and the corner points — are suitable for computer writing matrix **Q**.

Maximum number of non-zero operator weights of the operator understood at point $i, j$ is 13, the farthest elements of non-zero operator weight occur to the right and to the left, upwards and downwards of point $i, j$, at two intervals' distance. Hence, if the plate element is wider than two intervals in direction $\varphi$, then difference equations understood at deformations $\mathbf{u}_l$ do not contain elements $\mathbf{u}_r$ and vice versa. Excluding the practically irrelevant case where $k \leq 2$ it can be stated that in the partitioned form of **Q** (7):

$$\mathbf{L} = \mathbf{L}^* = 0 \, .$$

Hence, the stiffness matrix is:

$$\begin{bmatrix} \mathbf{C}_l & \mathbf{K}_l & 0 \\ \mathbf{K}_l^* & \mathbf{A} & \mathbf{K}_r^* \\ 0 & \mathbf{K}_r & \mathbf{C}_r \end{bmatrix} \begin{bmatrix} \mathbf{u}_l \\ \mathbf{w} \\ \mathbf{u}_r \end{bmatrix} = \begin{bmatrix} \mathbf{l}_l \\ \mathbf{p} \\ \mathbf{l}_r \end{bmatrix} . \tag{7a}$$

Minormatrix **A** is the matrix of the difference equation system of the plate element rigidly restrained at both ends. Since, however, restraint causes kinematic redundancy in the structure, **A** must be regular and invertible. Making use of the inverse of **A**:

$$\mathbf{w} = \mathbf{A}^{-1} (\mathbf{p} - \mathbf{K}_l^* \mathbf{u}_l - \mathbf{K}_r^* \mathbf{u}_r) \tag{8}$$

or, from (7a) and (8):

$$\mathbf{l}_l = (\mathbf{C}_l - \mathbf{K}_l \mathbf{A}^{-1} \mathbf{K}_l^*) \mathbf{u}_l - \mathbf{K}_l \mathbf{A}^{-1} \mathbf{K}_r^* \mathbf{u}_r + \mathbf{K}_l \mathbf{A}^{-1} \mathbf{p}$$
$$\mathbf{l}_r = (\mathbf{C}_r - \mathbf{K}_r \mathbf{A}^{-1} \mathbf{K}_r^*) \mathbf{u}_r - \mathbf{K}_r \mathbf{A}^{-1} \mathbf{K}_l^* \mathbf{u}_l + \mathbf{K}_r \mathbf{A}^{-1} \mathbf{p}. \qquad (9a,b)$$

(8) delivers internal point displacements if edge displacements are known, while (9a, b) is an integer part of the reduced compatibility matrix.

## 5. "Eigenstiffness" matrix of cross beams

Again, the eigenstiffness matrix of cross beams is written by minimizing the total potential energy:

$$\Pi = \int_L \frac{1}{2}\left[B\left(\frac{d^2w}{dr^2}\right)^2 + D\left(\frac{d\varkappa}{dr}\right)^2\right] dr - \int_L [q \cdot w + m_\varphi \cdot \varkappa]\, dr -$$
$$- m_{r1}\vartheta_1 - m_{rm}\vartheta_m = \text{minimum}!$$

where $L$ is the cross beam length; $B$ its bending stiffness; $D$ the torsional stiffness, $q$ and $m_\varphi$ the vertical load and the distributed torque, $w$ and $\varkappa$ are the vertical displacement and the angle of rotation; $m_{r1}$, $m_{rm}$ and $\vartheta_1$, $\vartheta_m$ being bending moments and radial slopes at the end points, respectively.

Without describing particulars of finitization steps, the following stiffness relationship can be written as difference equation system of the cross beam:

$$\mathbf{C}_i \mathbf{u}_i = \mathbf{l}_i \qquad (10)$$

where $\mathbf{C}_i$ is the stiffness matrix of the cross beam, $\mathbf{u}_i$ and $\mathbf{l}_i$ being displacement and load vectors in the order of edge displacements and edge forces of the plate element.

For structures without edge beam, the stiffness matrix $\mathbf{C}_i$ will be zero.

Let us notice here that eigenstiffness matrices $\mathbf{Q}$ and $\mathbf{C}_i$ as well as the reduced eigenstiffness matrix

$$\left[\begin{array}{c:c} \mathbf{C}_l - \mathbf{K}_l \mathbf{A}^{-1} \mathbf{K}_l^* & -\mathbf{K}_l \mathbf{A}^{-1} \mathbf{K}_r^* \\ \hdashline -\mathbf{K}_r \mathbf{A}^{-1} \mathbf{K}_l^* & \mathbf{C}_r - \mathbf{K}_r \mathbf{A}^{-1} \mathbf{K}_r^* \end{array}\right] \qquad (11)$$

composed of coefficients of (9a, b) are singular, physically meaning that rigid-body-like motions of the elements can be interpreted without loads.

## 6. Reduced compatibility equation system

There being three different types of finite elements, in establishing the eigenstiffness relationships, only establishment of the coefficient matrix of end cross beams

$$\mathbf{C}_l \cdot u_i = l_i\,, \qquad (i = 0 \text{ and } n) \tag{13a}$$

of intermediate cross beams

$$\mathbf{C}_k \cdot u_i = l_i\,, \qquad (0 < i < n) \tag{13b}$$

and determination of the coefficients of matrix equations

$$(\mathbf{C}_l - \mathbf{K}_l \mathbf{A}^{-1} \mathbf{K}_l^*)\,\mathbf{u}_{i-1} - \mathbf{K}_l \mathbf{A}^{-1} \mathbf{K}_r^*\,\mathbf{u}_i + \mathbf{K}_l \mathbf{A}^{-1} \mathbf{p}_i = \mathbf{l}_{i-1,i} \qquad (0 < i < n) \tag{13c}$$

$$(\mathbf{C}_r - \mathbf{K}_r \mathbf{A}_{-}^{1} \mathbf{K}_l^*)\,\mathbf{u}_i - \mathbf{K}_r \mathbf{A}^{-1} \mathbf{K}_l\,\mathbf{u}_{i-1} + \mathbf{K}_r \mathbf{A}^{-1} \mathbf{p}_i = \mathbf{l}_{i,i} \tag{13d}$$

is needed for all plate elements by substituting

$$\mathbf{u}_l = \mathbf{u}_{i-1}\,,\; u_r = u_i\,,\; l_l = l_{i-1,i}\,,\; l_r = l_{i,i}\,.$$

Writing these equations for every beam and plate element and substituting them into the set of equations (1, 2), we obtain the reduced compatibility equation system of the structure.

Introducing simplified notations:

$$\begin{aligned}
\mathbf{M}_l &= \mathbf{C}_l + \mathbf{C}_e - \mathbf{K}_l \mathbf{A}^{-1} \mathbf{K}_l^* \\
\mathbf{M}_r &= \mathbf{C}_r + \mathbf{C}_e - \mathbf{K}_r \mathbf{A}^{-1} \mathbf{K}_r^* \\
\mathbf{M} &= \mathbf{C}_l + \mathbf{C}_r + \mathbf{C}_k - \mathbf{K}_l \mathbf{A}^{-1} \mathbf{K}_l^* - \mathbf{K}_r \mathbf{A}^{-1} \mathbf{K}_r^* \\
\mathbf{N} &= \mathbf{K}_l \mathbf{A}^{-1} \mathbf{K}_r^* \\
\mathbf{l}_0 &= \mathbf{l}_0^0 - \mathbf{K}_l \mathbf{A}^{-1} \mathbf{p}_1 \\
\mathbf{l}_i &= \mathbf{l}_i^0 - \mathbf{K}_r \mathbf{A}^{-1} \mathbf{p}_i - \mathbf{K}_l \mathbf{A}^{-1} \mathbf{p}_{i+1} \\
\mathbf{l}_n &= \mathbf{l}_n^0 - \mathbf{K}_r \mathbf{A}^{-1} \mathbf{p}_n
\end{aligned}$$

the reduced compatibility equation system will be:

$$\begin{bmatrix}
\mathbf{M}_l & -\mathbf{N} & & & & \\
-\mathbf{N}^* & \mathbf{M} & -\mathbf{N} & & & \\
 & \ddots & \ddots & \ddots & & \\
 & & -\mathbf{N}^* & \mathbf{M} & -\mathbf{N} & \\
 & & & \ddots & \ddots & \ddots \\
 & & & -\mathbf{N}^* & \mathbf{M} & -\mathbf{N} \\
 & & & & -\mathbf{N}^* & \mathbf{M}_r
\end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_i \\ \vdots \\ \mathbf{u}_{n-1} \\ \mathbf{u}_n \end{bmatrix} = \begin{bmatrix} \mathbf{l}_0 \\ \mathbf{l}_1 \\ \vdots \\ \mathbf{l}_i \\ \vdots \\ \mathbf{l}_{n-1} \\ \mathbf{l}_n \end{bmatrix} \tag{14a}$$

$$\mathbf{C} \cdot \mathbf{u} = \mathbf{l}. \tag{14b}$$

The hypermatrix equation permits to directly take into account the following loads and displacement constraints of different character:

a) Group of loads $\mathbf{p}_i$ acting at inner and arched edge points of plate surfaces can be involved in the hypervector elements $\mathbf{l}_i$ and $\mathbf{l}_{i+1}$ in the right-hand side hypervector of the matrix equation, by means of terms $-\mathbf{K}_l\mathbf{A}^{-1}\mathbf{p}_i$ and $-\mathbf{K}_r\mathbf{A}^{-1}\mathbf{p}_i$.

b) Edge moments acting at nodes of arched elements inside the support lines can be accounted for in the same manner.

c) Forces acting along the support lines can be directly reckoned with in elements of the respective vector $\mathbf{l}_i^0$ of the load hypervector.

d) Moments concentrated at nodal points of cross beams (or of the joint line) can be considered in the same term,

e) and so can be bending moments acting at cross beam end points.

Among duals of the enumerated load types, the following displacements can be directly specified:

c′) vertical displacement of arbitrary nodes along the supporting line (or the cross beams),

d′) rotations in direction $\varphi$ of the same nodes, and

e′) cross beam end plate rotations.

Applicability of this method does not suffer from the fact that displacements type a′) and b′) cannot be directly taken into consideration, since in our case deformation constraints are encountered only along the joint lines of the elements.

On the other hand, application of the method permits to meet any type of deformation conditions, without modifying the reduced compatibility matrix. This would, however, be outside the scope of this paper.

## 7. Regularizing the reduced compatibility matrix by taking supports into consideration

In writing eigenstiffness correlations and joint conditions, no kind of external deformation constraints were reckoned with, resulting in the singularity of matrices (7, 10, 11). Since the reduced compatibility matrix $\mathbb{C}$ is still devoid of support deformation constraints, this one is also singular. For regularizing, at least as many external displacements constraints as needed for the structure to be statically determined, and effects of rigid or elastic supports have to be considered.

Effect of rigid supports is simple to be taken into consideration either by:

1. Zeroing values of displacements numbered $\alpha, \beta, \ldots, \lambda$ inhibited by the support, by cancelling the corresponding rows and columns $\alpha, \beta, \ldots, \lambda$ of

the compatibility matrix. The resulting non-singular matrix of lower order contains also the supporting conditions;

2. zeroing the corresponding columns numbered $\alpha, \beta, \ldots, \lambda$ and substituting 1 for main diagonal elements $C_{\alpha\alpha}, C_{\beta\beta}, \ldots, C_{\lambda\lambda}$ resulting in a non-singular equation system of the same size as the original one the solution of which contains the reaction dynams among elements of the deformation vector; or by

3. applying the method of considering the elastic deformation constraints, involving the least of change. Increasing diagonal elements $C_{\alpha\alpha}, C_{\beta\beta}, \ldots, C_{\lambda\lambda}$ is essentially equivalent to take into consideration elastic deformation constraints realized at corresponding displacements $u_{\alpha}, u_{\beta}, \ldots, u_{\lambda}$. Since bedding stiffness is proportional to the increase of diagonal elements, replacing $C_{\alpha\alpha}, C_{\beta\beta}, \ldots, C_{\lambda\lambda}$ by sufficiently great fictive diagonal elements results in practically stiff deformation constraints [7].

Methods 1 and 3 are advantageous by maintaining the symmetry of the reduced compatibility matrix in course of modification.

## 8. Analysis of influence surfaces

Computational advantages of this system mostly appear in computing influence surfaces of stress and strain.

From the principle of commutability it follows that any influence surface is identical to a special deflection diagram belonging to a load of dynam or kinematic character. Determining influence values of each plate field along boundaries $\mathbf{u}_l$ and $\mathbf{u}_r$ influence surface values of internal points will be, in conformity with (8):

$$\mathbf{w}^{\eta} = -\mathbf{A}^{-1}\,\mathbf{K}_l^*\,\mathbf{u}_l^{\eta} - \mathbf{A}^{-1}\,\mathbf{K}_r^*\,\mathbf{u}_r^{\eta}$$

or, in case of plate elements subject to the load producing the deformed surface:

$$\mathbf{w}^{\eta} = \mathbf{w}^{\eta_0} - \mathbf{A}^{-1}\,\mathbf{K}_l^*\,\mathbf{u}_l^{\eta} - \mathbf{A}^{-1}\,\mathbf{K}_r^*\,\mathbf{u}_r^{\eta}$$

$\mathbf{w}^{\eta 0}$ being the influence values on the primary beam plate rigidly fixed along its joint lines.

Assuming variables of force or displacement character in the reduced compatibility equation system, the set of equations for determining vectors $\mathbf{u}_l$ and $\mathbf{u}_r$ of the following types of influence surfaces can directly be written:

- reaction force influence surfaces of point supports;
- displacement influence surfaces of joint lines;
- reaction and displacement influence surfaces of elastic displacement constraints.

With the intermediary of multipliers $K_l A^{-1}$ and $K_r A^{-1}$ in the load hypervector, the set of equations of vectors $\mathbf{u}_l$ and $\mathbf{u}_r$ belonging to the displacement influence lines of inner points of each plate element can be written.

Since in analysis, nodal displacement differences are involved in expressing the first and higher derivatives of the deformation area, the kinematic strain influence surfaces cannot directly be produced. On the other hand, approximation by the difference quotients themselves permits to determine strain influence surfaces at the same accuracy as by the method of finite differences, such as:

Expressing the tested stress by partial derivatives of the deformation function, the derivatives will be approximated by difference quotients understood at the reference point of the influence surface sought for. Thereby the tested stress has been approximated as a linear combination of nodal displacements understood at and around the reference point. Obviously, the stress influence surface will be a similar linear combination of the influence surfaces of corresponding nodal displacements. Applying the operator weights of the difference operator abstracted from the linear combination as loads at the reference point of the influence surface and at the corresponding environmental nodal points, then, in conformity with the principle of interchangeability, this load will result in a deformation diagram identical to the approximate stress influence surface. Thus, it is useless to determine the superimposed displacement influence surfaces one by one.

Since the difference method fails in demonstrating singularity of stress at the reference point, in the environment of singularity, calculated and exact influence surface values greatly differ. This fact has a rather theoretical significance, namely in design practice, no loads concentrated to a degree to require an overdue accuracy of influence values around the singular point have to be reckoned with.

## Summary

The presented method of analysis lends itself to the determination of stresses, especially of stress influence surfaces of deck bridges over circular arc floor plan, by means of a medium-size computer.

The displacement method of large finite elements has been applied, combining computing advantages of the methods of finite differences and of finite elements.

Essentially, the method consists in decomposing the tested structure into elements of a size permitting to determine distribution of internal strains and stresses on the available computer, taking direct loads and joint conditions into consideration. Thus, stress-strain relationships of the entire structure will be given by the solution of the reduced compatibility equation written for the connection of large finite elements, size of this set of equations being but a fraction of the set of difference equations for the entire structure, raising no computer problems for most of practical cases.

The method of large finite elements, illustrated here on the example of a special structure, is equally convenient to the analysis of large or composed surface structures, in particular, plates and discs of zig-zagged boundary conditions.

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