ANALYSIS OF PLASTIC LOAD CAPACITY OF PLANE FRAMEWORKS BY KINEMATIC LOADING

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Introduction

The advent of computer technique opened new possibilities for the analysis of frameworks. The book by Dr. SZABÓ and Dr. ROLLER [1] interprets the wide-ranging analysis of that particular type of structures in an accordingly modern way. This paper is aimed at showing the advantages of extending the basic equations in [1] to the computer analysis of the ultimate load capacity of frameworks.

Analyses based on the first-order theory assume ideal elasto-plastic material of the structure with a one-parameter load. The whole course of loading is to be followed to define the plastic collapse load. In discussing and illustrating the principle, the yield condition is applied on the simple case of bending alone but not without considering the possibility of generalization.

This method has the great advantage to require the set-up of the stiffness matrix of the whole structure and the Cholesky's decomposition only once. The analysis of the structure, under the changing equilibrium and compatibility conditions due to the developing plastic hinges, is reduced to the original structure without modifying the stiffness matrix.

1. Basic equation and its solution

The basic equation of frameworks is the hypermatrix equation [1]:

$$\left[\begin{array}{c|c} & \mathbf{G}^* \\ \hline \mathbf{G} & \mathbf{F} \end{array}\right]\left[\begin{array}{c} \mathbf{u} \\ \hline \mathbf{s} \end{array}\right]+\left[\begin{array}{c} \mathbf{q} \\ \hline \mathbf{t} \end{array}\right]=0 \tag{1}$$

relating the external nodal forces $\mathbf{q}$ and the kinematic bar loads $\mathbf{t}$ as well as the nodal displacements $\mathbf{u}$ and internal bar forces $\mathbf{s}$. All the characteristics of the material and the cross-sections are included in the flexibility matrix $\mathbf{F}$ while the matrices $\mathbf{G}$ and $\mathbf{G}^*$ contain the geometry of the structure. In the followings, statically indetermined structures will be dealt with where matrix $\mathbf{G}$ has a structure of a standing rectangle.

Vectors and matrices in Eq. (1) understood in the global reference system x,y,z of the whole structure and in the local reference system ξ,η,ζ of the bar elements, can be written, as seen in Fig. 1:

$$\mathbf{s}_{jk}^{(\xi,\eta,\zeta)} = \begin{bmatrix} P_{jk,\xi} \\ P_{jk,\eta} \\ M_{jk,\zeta} \end{bmatrix} \quad \text{and} \quad \mathbf{t}_{jk}^{(\xi,\eta,\zeta)} = \begin{bmatrix} \Delta v_{jk,\xi} \\ \Delta v_{jk,\eta} \\ \Delta \varphi_{jk,\zeta} \end{bmatrix}, \tag{1a}$$

further

$$\mathbf{q}_j^{(x,y,z)} = \begin{bmatrix} R_{j,x} \\ R_{j,y} \\ N_{j,z} \end{bmatrix} \quad \text{and} \quad \mathbf{u}_j^{(x,y,z)} = \begin{bmatrix} v_{j,x} \\ v_{j,y} \\ \varphi_{j,z} \end{bmatrix}. \tag{1b}$$

The flexibility matrix:

$$\mathbf{F}_{jk} = \begin{bmatrix} \dfrac{l}{EA} & & \\ & \dfrac{l^3}{3EJ_\zeta} & \dfrac{l^2}{2EJ_\zeta} \\ & \dfrac{l^2}{2EJ_\zeta} & \dfrac{l}{EJ_\zeta} \end{bmatrix}, \tag{1c}$$

where A is the cross-sectional area of the bar, $l = l_{jk}$ its length, J_ζ the moment of inertia and E the modulus of elasticity.

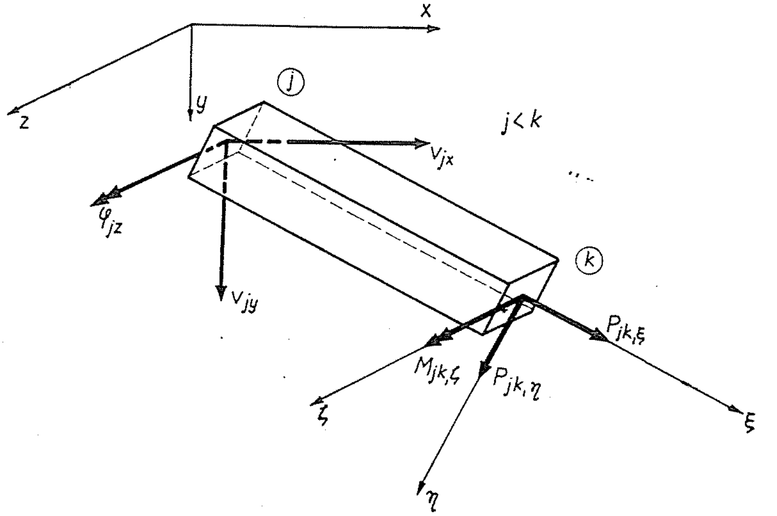

Fig. 1

To solve Eq. (1) the displacement method was applied by the equations:

$$\mathbf{u} = (\mathbf{G}^* \mathbf{F}^{-1} \mathbf{G})^{-1} (\mathbf{q} - \mathbf{G}^* \mathbf{F}^{-1} \mathbf{t}) = \mathbf{K}^{-1} \mathbf{a}$$

and

$$\mathbf{s} = -\mathbf{F}^{-1}(\mathbf{G}\,\mathbf{u} + \mathbf{t}) . \tag{2}$$

The coefficient matrix, i.e. the global stiffness matrix of the structure is a symmetric hyper-band-matrix:

$$\mathbf{K} = \mathbf{G}^* \mathbf{F}^{-1} \mathbf{G} \tag{2a}$$

compiled of the elementary stiffness matrices of the bars. The connections between the single bar elements are considered by the diadic reduction of the individual stiffness matrices of the bars.

2. Analysis of the collapse load parameter

The further discussion of the yield condition will be concerned with bending effects alone but the principle of analysis could be extended for combined effects. The yield condition in a cross-section k:

$$^{\pm}M_k = {}^{\pm}M_{kt} \tag{3}$$

where M_k and M_{kt} are the actual and ultimate values of the bending moment, respectively, at cross-section k, true to sign.

An ideal plastic hinge will only develop for an ultimate bending moment. Up to then, the cross-section behaves elastically. After a plastic hinge has developed in a cross-section, the load increment will be carried by the structure, less indetermined statically, until the next yield of a cross-section, so that finally the whole structure or part of it becomes unstable. From this process it would follow theoretically that in every step of computation the stiffness matrix $\mathbf{K}$ of the total structure would change and the stepwise solution would consist in decomposing $\mathbf{K}$ in every step, a rather inefficient method. The method to be presented leaves the matrix $\mathbf{K}$ unaltered so that the computation as a whole affects the original structure. Matrix $\mathbf{K}$ has to be transformed only once and stored in decomposed form. The computation involves only free vectors.

The method is based on the following principle: Let the structure be acted upon by basic combined loads q_0 and t_0 from Eq. (1). Solving the equation by the displacement method yields the stress vector:

$$\mathbf{s}_0 = -\mathbf{F}^{-1} [\mathbf{G}\,\mathbf{K}^{-1}(\mathbf{q}_0 - \mathbf{G}^* \mathbf{F}^{-1} \mathbf{t}_0) + \mathbf{t}_0] . \tag{4}$$

The comparison of the ultimate and actual values of the bending moments gives the first load parameter:

$$\alpha_1 = \left| \frac{M_{ti}}{M_{i0}} \right|_{\min} .$$

Multiplying the loads by α_1:

$$\mathbf{q}_1 = \alpha_1 \mathbf{q}_0 \quad \text{and} \quad \mathbf{t}_1 = \alpha_1 \mathbf{t}_0 , \tag{5}$$

results in the first plastic hinge at the i-th cross-section, where the bending moment M_{i0} has the value of M_{ti}. Now at the plastic hinge in the original structure a relative rotation will be inserted as kinematic load $\mathbf{t}_2'$ to be increased by the same parameter as, and applied together with, the original load; no moment increment develops at the plastified section. (The kinematic load $\mathbf{t}_2'$ will be defined in item 3.) The load can be increased until the next section yields. Then the load is composed of

$$\mathbf{q}_2 = \mathbf{q}_1 + \alpha_2 \mathbf{q}_1 \quad \text{and} \quad \mathbf{t}_2 = \mathbf{t}_1 + \alpha_2 \mathbf{t}_1 + \alpha_2 \mathbf{t}_2'$$

where

$$\alpha_2 = \left| \frac{M_{tj} - M_j}{\Delta M_j} \right|_{\min} , \quad \text{but} \quad |\Delta M_j| > \varepsilon . \tag{6}$$

Here M_j and ΔM_j are bending moments at cross-section j caused by load $\mathbf{q}_1$, and by the combined effect of $\mathbf{q}_1$ and the relative rotation $\mathbf{t}_2'$, respectively. ε is a small positive number depending on the accuracy of the computer. Of course, ΔM_j is zero at the plastified section exactly due to $\mathbf{t}_2'$!
Accordingly, in step n:

$$\begin{aligned} \mathbf{q}_n &= \mathbf{q}_{n-1} + \alpha_n \mathbf{q}_{n-1} = (1 + \alpha_n) \ \mathbf{q}_{n-1} = \\ &= (1 + \alpha_n)(\mathbf{q}_{n-2} + \alpha_{n-1} \mathbf{q}_{n-2}) = (1 + \alpha_{n-1})(1 + \alpha_n) \mathbf{q}_{n-2} = \\ &= \ldots = \alpha_1 (1 + \alpha_2)(1 + \alpha_3) \ldots (1 + \alpha_{n-1})(1 + \alpha_n) \mathbf{q}_0 . \end{aligned}$$

The analysis can be continued in a similar way. The ultimate load on the structure becoming partially or totally unstable by developing r plastic hinges is:

$$\mathbf{q}_t = \mathbf{q}_r = \alpha \mathbf{q}_0 \quad \text{and} \quad \mathbf{t}_t = \alpha \mathbf{t}_0 \tag{7}$$

where

$$\alpha = \alpha_1 (1 + \alpha_2)(1 + \alpha_3) \ldots (1 + \alpha_{r-1})(1 + \alpha_r) .$$

So far no recovery (unloading) of the plastic hinge has been mentioned. In the analysis this case will be indicated by changing the sign of the kinematic load simulating the plastic hinge. In this case the ultimate value of the bending moment at that section will be reduced according to the recovery degree and the previous step of the analysis repeated. In a similar "rearrangement" of plastic hinges, further moment corrections are applied until all the yield sections are returned to plastic hinges.

Considering as load the relative rotation simulating plastic hinges permits to analyse throughout the original structure without affecting its stiffness matrix. Thus, matrix **K** transformed for solving the equation system is put in an external store and only the free vectors corresponding to each load case have to be dealt with.

3. Relative rotations simulating the hinge

As to the determination of relative rotations simulating plastic hinges, this is going on as follows:
As an intermediate step of the analysis let us know the existing plastic hinge at cross-section $k-1$ with the corresponding load $\mathbf{q}_{k-1}$, as well as $\mathbf{t}_{k-1}$ combined of the original kinematic load and the kinematic load excluding moment increments at $k-1$ (simulating the plastic hinge). Load required to plastify cross-section k is combined of:

$$\mathbf{q}_k = (1+\alpha_k)\mathbf{q}_{k-1}$$

and

$$\mathbf{t}_k = (1+\alpha_k)\mathbf{t}_{k-1} + \alpha_k\,\mathbf{t}'_k\,. \tag{8}$$

Before obtaining α_k from Eq. (6) let us define kinematic load $\mathbf{t}'_k$, containing all the relative rotations which, applied together with the original loads, prevent moment increments from developing in the plastified sections. This condition can be expressed as:

$$\mathbf{S}\cdot\mathbf{t}'_k + \mathbf{s}_{k-1} = 0\,. \tag{9}$$

It is appropriate to take stresses $\mathbf{s}_{k-1}$ in (9) from the solution of the previous step for loads $\mathbf{q}_{k-1}$ and $\mathbf{t}_{k-1}$:

$$\mathbf{s}_{k-1} = -\mathbf{F}^{-1}\,[\mathbf{G}\,\mathbf{K}^{-1}(\mathbf{q}_{k-1} - \mathbf{G}^*\,\mathbf{F}^{-1}\,\mathbf{t}_{k-1}) + \mathbf{t}_{k-1}]\,. \tag{9a}$$

The coefficient matrix is:

$$\mathbf{S} = [s_{jm}] = \mathrm{e}_j\,\mathbf{F}^{-1}\,\mathbf{G}\,\mathbf{K}^{-1}\,\mathbf{G}^*\,\mathbf{F}^{-1}\,\mathrm{e}_m\,, \tag{9b}$$

where s_{jm} denotes the bending moment due to a singular unit relative rotation inserted at the plastic hinge m supposed at plastic hinge j in the original structure. The principle applied here is obviously that of the displacement method and factors s_{jm} are the unit factors of the displacement method. From Eq. (9b) it is clear that while deriving the coefficient matrix $\mathbf{S}$ the stiffness matrix $\mathbf{K}$ is unaffected and only the free vectors corresponding to unit load are manipulated. The operations assigned in Eq. (9a) to produce vector $\mathbf{s}_{k-1}$ can be omitted since

$$\mathbf{s}_{k-1} = \mathbf{m}_{t,k-1},$$

i.e., in the yield cross-sections only the ultimate moment can develop, a known quantity.

Analysing Eq. (9) the coefficients and free vectors turn out to be easy to define, but the development of each new plastic hinge requires another set-up of the equation:

$$\mathbf{t}'_k = -\mathbf{S}^{-1}\,\mathbf{s}_{k-1}.$$

Now, plastic hinges are to be examined for recovery. If the corresponding elements of $\mathbf{t}_k$ and $\mathbf{t}_{k-1}$ change sign then the step is repeated after reducing the corresponding $\mathbf{s}_{k-1}$ element in proportion to the recovery. The equation system is not too big, the first step contains a single unknown, and others add for each step, as many as there are new plastic hinges, or may be subtracted in case of recovery. Nevertheless the basic equation (1) — involving time-consuming inversion of its larger-size coefficient matrix — will be solved only once and that is the great advantage of the method.

4. Examples

a) All bars of the plane framework in Fig. 2 have a cross-section area $A = 0.00691\ \mathrm{m}^2$ and a moment of inertia $J_\xi = 0.000098\ \mathrm{m}^4$. The ultimate moment: $|M_t| = 18.5$ Mpm is the same for all bars. All the horizontal loads

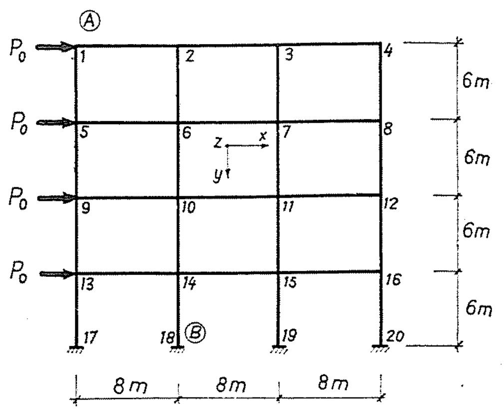

Fig. 2

are of the same value. The value of the ultimate load and the collapse mechanism are sought for.

The analysis follows the above principles throughout the loading procedure. Fig. 3 shows some intermediate stages of the plastification and the collapse mechanism. The plastic hinges are numbered according to the sequence of development. At the 18-th step, for $P_t = 6.16667$ Mp the structure became unstable. This appeared by $\mathbf{S}$ becoming singular. The collapse mechanism shows a symmetry due to total symmetry of the structure and its loading, however, the symmetric plastic hinges did not develop simultaneously because of considering the bar-length changes due to normal forces. It is clearly seen, the two plastic hinges developed at two bar ends in two interim nodes — after step 12 — while the other two joining bars are still rigidly connected. In step 13, the third bar end yields, transforming the whole node into a plastic hinge due to the relative rotation of the joining elements. In Fig. 4 the values of the absolute rotation and horizontal displacements φ_{Az} and e_A of node A, resp., are plotted vs. load variation. Fig. 5 shows relative rotation v_B at section B, the first plastic hinge of the structure, vs. load increase. The plastic hinge is seen not to recover throughout the process, hence, the relative rotation to steadily increase.

Let us illustrate the advantages of the method by the example. The frame shown has 20 nodes, among them the zero boundary node displacements are known. There are 16 nodes left, with three unknown displacement components at each. The half band-width of the symmetric stiffness matrix $\mathbf{K}$ of the

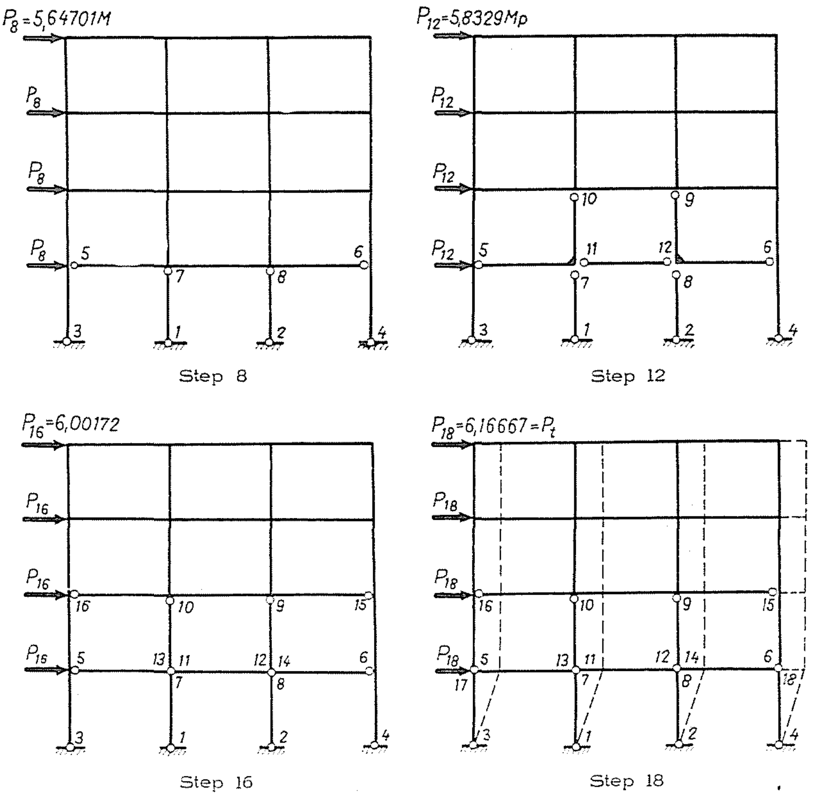

Fig. 3

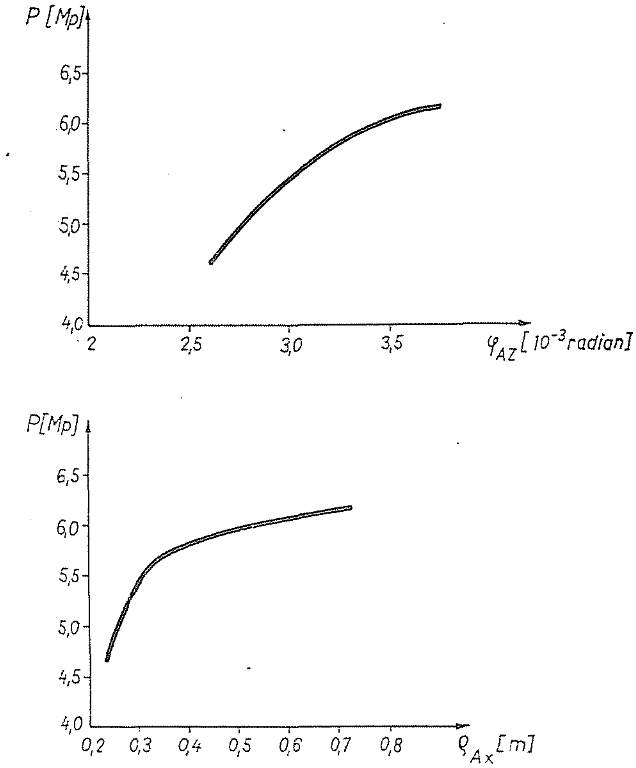

Fig. 4

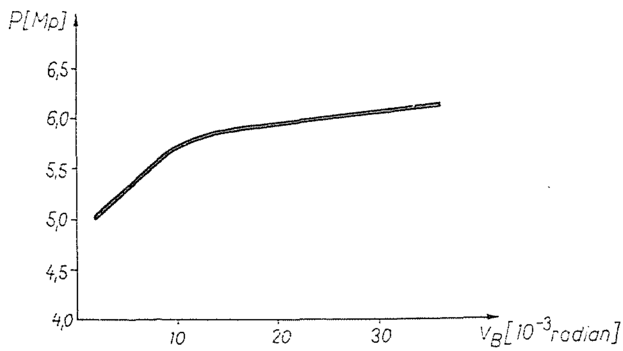

Fig. 5

complete structure is 15, the length of the band is 48. To determine the redundants there is an equation system of 48 unknowns to be solved. The structure has to have 18 plastic hinges developed before collapsing. That would mean in the traditional way to solve 18 times the equation system of 48 unknowns to obtain the step-wise changing structural behaviour. Instead of this time-consuming procedure, here the coefficient matrix — containing 48 rows — has to be decomposed only once. On the other hand, 18 equation systems increasing in size from 1 to 18 — thus, hardly beyond the half band-width of matrix **K** — will have to be solved.

Concerning the running time it has to be mentioned that just because what was discussed above there is no linear relationship between the number of plastic hinges at failure and the running time. Even for one and the same problem the kind of the plastic collapse mechanism depends on the stiffness conditions. If in the example e. g. the columns at any level had been assumed of a lower stiffness, local instability would bring about failure as soon as for 8 plastic hinges, much reducing the running time.

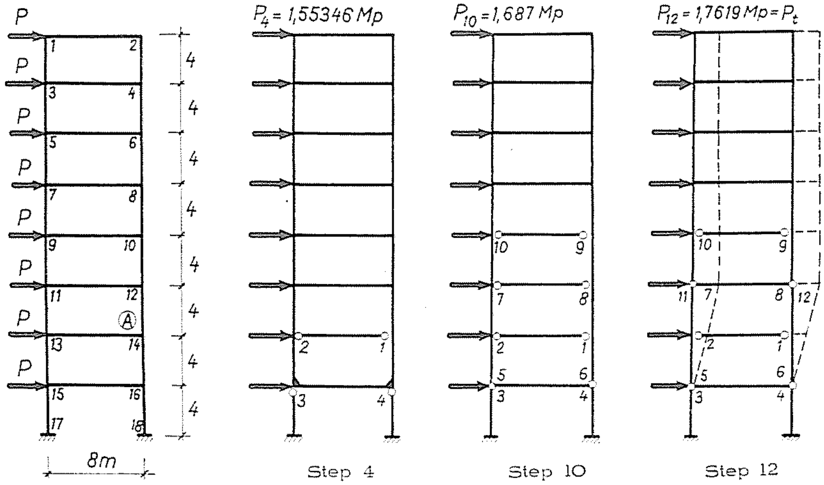

Fig. 6

b) The frame bars in Fig. 6 have the same properties as those in the former example. There are shown some intermediate stages of the analysis and the collapse mechanism. Here the half band-width of matrix **K** is 9, the length of the band is 48. Fig. 7 shows the relative rotation of the first plastic hinge at section A as a function of load.

c) The collapse of the last framework is caused by its yield nodes failing to carry the moment-load. The bars of the structure in Fig. 8 have the same cross-sectional area and moment of inertia as before. The ultimate moment $|M_t| = 18.5$ Mpm. The structure collapses in 4 steps. At each step, several plastic hinges are developing. The collapse mechanisms in each step are shown in Fig. 9.

The illustrative examples were selected in an attempt to emphasize the main points of the method. Remind, however, that the algorithm and the program itself are valid for a wide range of structures and stiffness conditions.

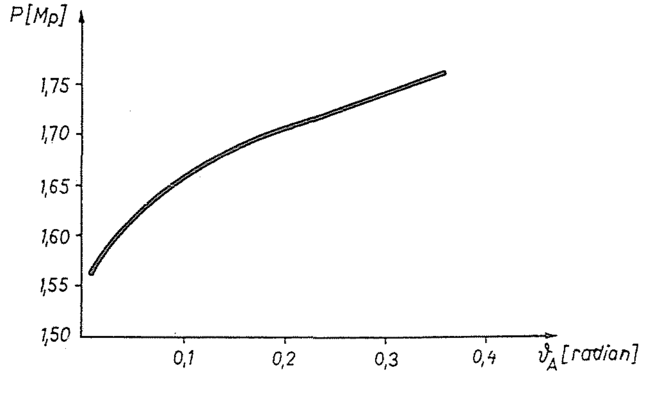

Fig. 7

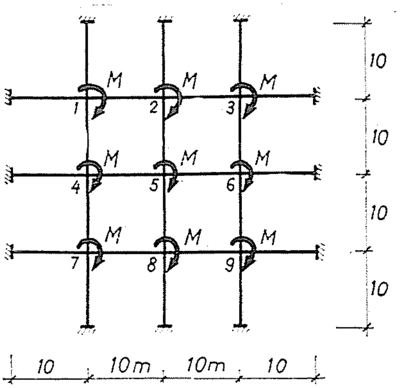

Fig. 8

As for the further development of the method it has to be noted that more research work is planned in the field of combined stresses to develop a method of load bearing analysis involving kinematic loads simulating plastic constraints.

Summary

A computer method for the plastic analysis of plane frameworks of an ideal elasto-plastic material is described. The ultimate value of the one-parameter load has been determined by analysing the whole loading process. The method is advantageous by analysing the original structure for step-wise changing equilibrium and compatibility conditions — due to the developing plastic hinges — without changing its original stiffness matrix and simulating the plastic hinges by kinematic loads. The yield condition was only applied for the simple case of bending but the principle itself can be extended for combined stresses.

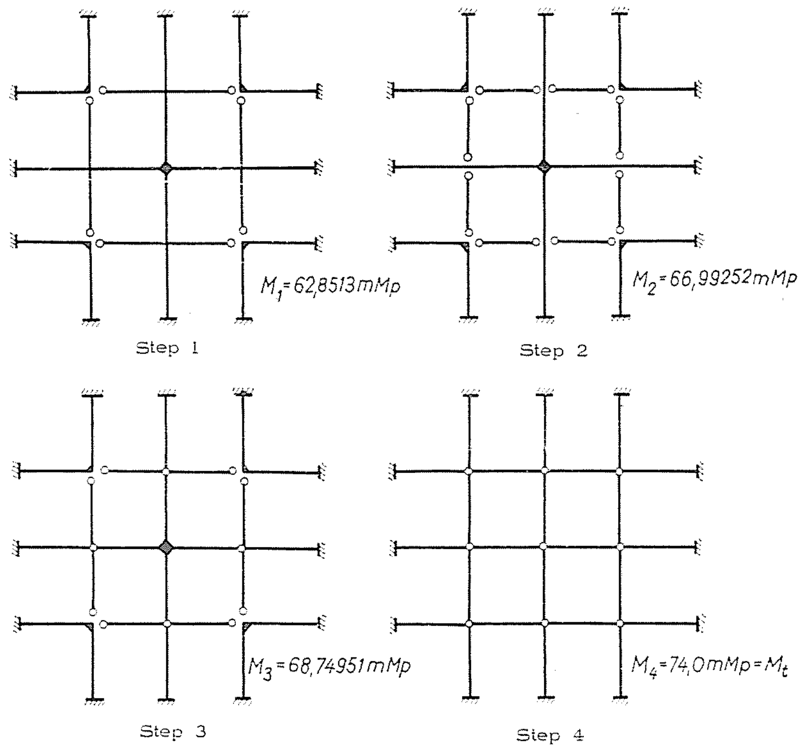

Fig. 9

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