A NUMERICAL METHOD FOR THE SOLUTION OF THE EIGENVALUE PROBLEM OF DAMPED VIBRATIONS

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1. Introduction

Let us consider the generalized eigenvalue problem of damped vibrating systems in the form of

$$(\mathbf{M}\omega^2 + \mathbf{D}\omega + \mathbf{C})\,\mathbf{z} = \mathbf{0} \tag{1}$$

where $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{C}$ are the matrices of masses, of damping and of spring constants, respectively; all the three of them are of n-th order. The $2\times n$ numerical values of the eigenvalue ω (multiplicity is also considered) and perhaps the corresponding eigenvectors $\mathbf{z}(\omega)$ have to be determined. To determine the eigenvalues ω, i.e. the roots of the polynomial of $2n$-th order:

$$P_{2n}(\omega) \equiv \det\,(\mathbf{M}\omega^2 + \mathbf{D}\omega + \mathbf{C}) = 0\;,$$

there are two well-known methods [1].

One of them reduces the problem to calculating the eigenvalues of the following $2n$-th order matrix:

$$\begin{bmatrix} -\mathbf{M}^{-1}\mathbf{D} & -\mathbf{M}^{-1}\mathbf{C} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}.$$

The other procedure assumes $2n+1$ different, arbitrary values for ω_i and calculates the corresponding values of the polynomial

$$P_{2n}(\omega_i) = \det\,(\mathbf{M}\omega_i^2 + \mathbf{D}\omega_i + \mathbf{C}) \qquad i = 1, 2, \ldots, 2n+1\,.$$

To these points an interpolation polynomial of $2n$-th order can be fitted and its zero points will give the eigenvalues desired. From a numerical point of view the disadvantage of the first method is to double the order of the matrix, while in the second procedure already the coefficients of the interpolation polynomial are containing errors and determining the roots means in itself serious numerical difficulties.

The essence of the method presented below is to reduce the problem (1) to two special eigenvalue problems of n-th order by solving an equation system of second degree with the number of unknowns being n^2.

2. Review of the procedure

Assuming the matrix $\mathbf{M}$ to be non-singular and multiplying Eq. (1) by $\mathbf{M}^{-1}$ from the left it gives:

$$(\mathbf{I}\omega^2 + \mathbf{A}\omega + \mathbf{B})\,\mathbf{z} = \mathbf{0} \tag{2}$$

where $\mathbf{A} = \mathbf{M}^{-1}\mathbf{D}$ and $\mathbf{B} = \mathbf{M}^{-1}\mathbf{C}$. In technical interpretations it is usual to have the matrices $\mathbf{M}$, $\mathbf{D}$, $\mathbf{C}$ as real symmetric and $\mathbf{M}$ being positive definite. To save the symmetry it is appropriate — instead of multiplying by $\mathbf{M}^{-1}$ — to apply $\mathbf{M} = \mathbf{L}\,\mathbf{L}^*$, i.e. Cholesky's decomposition (where $\mathbf{L}$ is a lower triangular matrix and $\mathbf{L}^*$ its transpose) and to multiply Eq. (1) by $\mathbf{L}^{-1}$ from the left. Introduce the notations $\mathbf{L}^*\mathbf{z} = \mathbf{y}$ hence $\mathbf{z} = \mathbf{L}^{*-1}\mathbf{y}$. Thereby the problem in Eq. (1) is transformed again into (2), where

$$\mathbf{A} = \mathbf{L}^{-1}\,\mathbf{D}\,\mathbf{L}^{*-1}, \qquad \mathbf{B} = \mathbf{L}^{-1}\,\mathbf{C}\,\mathbf{L}^{*-1}$$

and $\mathbf{y}$ stands for $\mathbf{z}$.

The further derivations are restricted to the eigenvalue problem of Eq. (2). Attempting the decomposition:

$$\mathbf{I}\omega^2 + \mathbf{A}\omega + \mathbf{B} = (\mathbf{I}\omega - \mathbf{X})\,(\mathbf{I}\omega - \mathbf{Y}) \tag{3}$$

performing the operation at the right-hand-side, and comparing the corresponding coefficient matrices of ω of the same power on both sides, the resulting matrix equations are

$$\mathbf{X} + \mathbf{Y} = -\mathbf{A}\,; \quad \text{and} \quad \mathbf{X}\,\mathbf{Y} = \mathbf{B}. \tag{4}$$

Let the matrix $\mathbf{Y}$ be expressed from the first one and substituted into the second to get:

$$\mathbf{X}^2 + \mathbf{X}\,\mathbf{A} + \mathbf{B} = 0 \tag{5}$$

i.e. a quadratic equation system of n^2 unknowns for $\mathbf{X}$ as the variable, n being the order of the quadratic matrices.

Having Eq. (5) been solved for the matrix $\mathbf{X}$, then from Eq. (4) we obtain the matrix:

$$\mathbf{Y} = -\mathbf{A} - \mathbf{X}$$

and in conformity with Eq. (3) the problem in Eq. (2) has been reduced to the form

$$(\mathbf{I}\omega - \mathbf{X})(\mathbf{I}\omega - \mathbf{Y})\mathbf{z} = 0. \qquad (6)$$

The relationship:

$$\det(\mathbf{I}\omega - \mathbf{X})(\mathbf{I}\omega - \mathbf{Y}) = \det(\mathbf{I}\omega - \mathbf{X}) \cdot \det(\mathbf{I}\omega - \mathbf{Y})$$

being valid, solution of Eq. (6) is equivalent to that of two special eigenvalue problems of n-th order.

The numerical efficiency of the procedure introduced above depends on the solution possibilities of Eq. (5), therefore the further discussions will be restricted to this subject.

3. Solution of the matrix equation (5)

Let the i-th rows of matrices $\mathbf{X}$ and $\mathbf{B}$ denoted by $\mathbf{x}_i^*$ and $\mathbf{b}_i^*$, respectively. If Eq. (5) is written in scalar form, it is easy to see the equivalency to the equation:

$$\begin{bmatrix} \mathbf{X}^* + \mathbf{A}^* & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{X}^* + \mathbf{A}^* & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & & \mathbf{X}^* + \mathbf{A}^* \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_n \end{bmatrix} = \mathbf{0}. \qquad (7)$$

Introducing the notation:

$$\mathbf{A} \otimes \mathbf{B} = [\mathbf{A} b_{ij}]$$

for direct products and defining vectors as

$$\xi = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{bmatrix} \qquad \beta = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_n \end{bmatrix}$$

Eq. (6) can be written in the form of usual non-linear equation systems as

$$\mathbf{f}(\xi) \equiv [(\mathbf{X} + \mathbf{A})^* \otimes \mathbf{I}]\,\xi + \beta = \mathbf{0}. \qquad (8)$$

This equation system, since being of second order, could be considered as fairly special, although the number of the variables is n^2. There are several methods known as approximate solutions of non-linear equation systems. [2], [3]

As one of the best known, the generalized Newton—Raphson iteration method

$$\xi^{(i+1)} = \xi^{(i)} - \mathbf{J}^{-1}\,[\mathbf{f}(\xi^{(i)})]\,\mathbf{f}(\xi^{(i)}) \tag{9}$$

is going to be considered. For the present case the Jacobian function-matrix $\mathbf{J}\,[\mathbf{f}(\xi)]$ takes the hypermatrix form:

$$\mathbf{J} = \begin{bmatrix} \mathbf{X}^* + \mathbf{A}^* + x_{11}\mathbf{I} & x_{12}\mathbf{I} & \dots & x_{1n}\mathbf{I} \\ x_{21}\mathbf{I} & \mathbf{X}^* + \mathbf{A}^* + x_{22}\mathbf{I} & \dots & x_{2n}\mathbf{I} \\ \vdots & \vdots & & \vdots \\ x_{n1}\mathbf{I} & x_{n2}\mathbf{I} & \dots & \mathbf{X}^* + \mathbf{A}^* + x_{nn}\mathbf{I} \end{bmatrix}$$

or using the direct product in the form of:

$$\mathbf{J} = (\mathbf{X} + \mathbf{A})^* \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{X}.$$

4. Example

Let the eigenvalue problem in Eq. (2) be solved in case of the coefficient matrices:

$$\mathbf{A} = \begin{bmatrix} -20 & -8 & 0 & 3 \\ -11 & -15 & 1 & 3 \\ -5 & -4 & -17 & 13 \\ 2 & 5 & 9 & -13 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 32 & 15 & -44 & 35 \\ -4 & -6 & -60 & 58 \\ 36 & 49 & 64 & -75 \\ 9 & 12 & 35 & -40 \end{bmatrix}.$$

To solve matrix Eq. (5) the iteration method shown in Eq. (9) has been applied with $\xi^{(0)} = \mathbf{0}$ as initial value. For each step of iteration the values of

$$\| \xi_i - \xi_{i-1} \| \tag{10}$$

were calculated, where the norm of the vector means the sum of the absolute values of the elements. The convergence is fast as it appears from Table 1.

Table 1

Iter. steps	Norm
1	$2.2318 \cdot 10^{1}$
2	$5.9944 \cdot 10^{0}$
3	$4.6622 \cdot 10^{-1}$
4	$6.7397 \cdot 10^{-3}$
5	$1.4294 \cdot 10^{-6}$
6	$1.4291 \cdot 10^{-9}$

Taking the values obtained in the sixth iteration for final results in matrices:

$$\mathbf{X} = \begin{bmatrix} 2.0794 & 0.9206 & -2.0794 & 1.9206 \\ 0.5238 & 0.4762 & -2.5238 & 2.4762 \\ -0.3810 & 2.3810 & 2.3810 & -2.6190 \\ -0.0635 & 0.0635 & 1.0635 & -1.9365 \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} 17.9206 & 7.0794 & 2.0794 & -4.9206 \\ 10.4762 & 14.5238 & 1.5238 & -5.4762 \\ 5.3810 & 1.6190 & 14.6190 & -10.3810 \\ -1.9365 & -5.0635 & -10.0635 & 14.9365 \end{bmatrix}.$$

(The results are shown up to four decimals.) The eigenvalues of matrices $\mathbf{X}$ and $\mathbf{Y}$ obtained by the double-step QR algorithm [4] give the eigenvalues of the original problem with an accuracy of 8 digits:

$$\begin{aligned} \omega_{x1} &= -1 & \qquad \omega_{y1} &= 4 \\ \omega_{x2} &= 2 & \omega_{y2} &= 8 \\ \omega_{x3} &= 1+2i & \omega_{y3} &= 18 \\ \omega_{x4} &= 1-2i & \omega_{y4} &= 32\,. \end{aligned}$$

Summary

A method has been introduced to show how to reduce the eigenvalue problem of damped vibrating systems to two special eigenvalue problems of n-th order by the approximate solution of a quadratic equation system with n^2 unknowns. The program of the numerical example had been run on a computer ODRA—1204 at the Faculty of Civil Engineering, Technical University, Budapest.

References

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